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A critical exploration of student teacher's choice and use of representations in a challenging environment

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Abstract
The teaching and learning of mathematics in economically challenged environments is demanding. Thus, the teachers who teach in these contexts should have a sound pedagogical content knowledge to make a success of their teaching thereof. International community in mathematics education agrees that multiple representations play a pivotal role in making mathematics accessible to learners. However, the challenge might be the feasibility of such a use in African contexts where a student teacher learns to teach mathematics in a crowded classroom of learners (n=80) with significantly varied age differences (ages 9-15). Against this background, therefore the purpose of the study was to explore the choice and use of mathematical representations in a grade 4 class in Lesotho. Data was collected through lesson observations and reflection sessions. Knowledge quartet typology was used as a data analysis approach. The findings of the study indicate that the use of fake money in the lesson improved participation and helped to scaffold learners' strategies of addition and subtraction. It is therefore recommended that the prospective teachers be equipped with skills to choose and use multiple representations.

Keywords: content knowledge, knowledge quartet, mathematical representations, multiple representations, pedagogical content knowledge

INTRODUCTION

Samsuddin and Retnawati (2018) provide an interesting analogy that describes the role of "representation" in mathematics. According to these authors, if the "real world" is considered a land and mathematics, another land (meaning–two different lands, which are strange to each other) and these two lands happen to be separated by a river, then a bridge would be required to connect the two lands. In this instance, Samsuddin and Retnawati (2018) regard the role of representation to be that of a "bridge" that connects "the abstract mathematics concept with daily life context" (p. 2). In line with this, therefore, Matteson (2006) regards representations as the "key elements" for those who wish to comprehend and express mathematical ideas appropriately and conveniently. Although representations make it easy for mathematical concepts and ideas to be understood (Dalton, 2017; Shaw et al., 2001), many learners find the use of representations to be challenging since they are expected to learn to use them whilst at the same time, having to learn other things using them (Kilpatrick et al., 2001). Duval (2006), however, stresses the significance of using representations since it is only through them that learners can better access abstract mathematical concepts. To be able to provide meaningful teaching, Shulman (1986) argues that teachers must have at hand, a veritable armamentarium of alternative forms of representation since there are no single most powerful forms (of representation).

Ball et al. (2008), on the other hand, argue that teachers need to have a rich specialized content knowledge (SCK) so that they can be able to choose and use "appropriate representations" that will in turn, make content comprehensible to learners. This is because some representations are more powerful than others in affording learners' access into mathematical concepts. In line with this therefore the study reported in this paper aimed to explore the student teacher's choice and use of

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Contribution to the literature

- The paper serves as guide for selecting and using representations to teach basic mathematics.
- The study uses knowledge quartet framework as a guide for integrating representations into the teaching of mathematics to provide meaningful learning.
- The study empowers the student teachers with knowledge on how to improvise to teach mathematics in an economically challenged environment which is constituted by overcrowded classrooms and lack of resources.

mathematical representations to scaffold learning in a grade 4 class in Lesotho.

In this article, we argue that efforts need to be made to provide quality and meaningful teaching of mathematical concepts. To achieve this, teachers' pedagogical content knowledge (PCK) needs to be strengthened for them to provide better teaching through selecting tools that would afford learners access into mathematical concepts. Therefore, teachers' ability to choose and use suitable tools in teaching certain mathematical concepts needs to be nurtured, especially, for the student teachers whose higher education training does not encompass certain complicated teaching aspects such as dealing with crowded classes of learners (n=80) of varied ages in one class (ages 9-15), etc.

The implications of the study are therefore situated in debates on innovation, sustainability and inclusivity in the teaching and learning of mathematics as highlighted by the results of some international benchmark assessments, such as *trends in international mathematics and science study*. Based on all this, therefore, the study was guided by the following questions:

1. How does a student teacher choose and use multiple representations to scaffold grade 4 mathematics learning?
2. What strengths and weaknesses are revealed in a student teacher's choice and use of multiple representation in teaching grade 4 mathematics?

Representations

A corpus of literature outlines the roles of representations in mathematics teaching and learning. According to Mitchell et al. (2014), representations aid learners' understanding and assist in making sense of the mathematical tasks and concepts. Representations have proven to be the excellent tools for facilitating student learning, enabling students to manage and express their thinking and to make mental models of their mathematical ideas (Schwarz et al., 1994). Furthermore, research shows that representations are useful in terms of helping students to understand the abstract mathematical concepts (Kang & Liu, 2018). They also assist to communicate the mathematical ideas and concepts in a flexible manner (Mainali, 2021). Moleko and Mosimege (2021) emphasize a point that the use of multiple representations in the teaching of mathematics enhances understanding. Way back in the 1960s, Bruner (1966) identified three modes of representations namely enactive representation (hands on/action-based); iconic representation (visuals/image-based); and symbolic representation (abstract and language-based).

According to Bruner (1966), modes of representation are the means through which information is stored and encoded in a leaner's cognitive structures. Teachers who teach mathematics at the primary school level must be aware of the role representations play in enhancing learners' understanding of mathematics.

Enactive representations involve tasks that call for action (hands on activities) on the part of learners. In many Lesotho primary schools, teachers use concrete objects such as counters, matches, sticks, and stones in mathematics lessons to assist learners to do addition, subtraction, multiplication, and division calculations. These physical objects could be valuable especially in the early years of schooling because learners at this stage are not yet conversant with the four basic mathematical operations namely addition, subtraction, multiplication, and division. Therefore, the models play an important role in helping learners to physically see why, for instance this sentence is true, 13+5=18.

Iconic representations are visuals that both learners and a teacher can refer to in class to facilitate effective learning and teaching of certain mathematical concepts. Iconic representations are resources that act as scaffolds for learners' indecisive thinking and strategies of mathematical operations. Examples of iconic representations commonly used at primary school level might include number-line, number square, number fan, number track, place value cards, multiplication square, and multiplication array. While most of these representations are commercial, student teachers must be taught during their training ways of improvising and constructing these resources using recycled cardboard, plastic, etc. Given the economical state of Lesotho, it is logical to assume that many primary schools cannot afford commercial teaching aids. But irrespective of financial constraints of the country learners in all schools have an educational right to be taught mathematics well. The use of iconic representations is meant to assist learners' mathematical strategies to be strong and independent, and once this stage is reached then an iconic representation at hand can be removed.

The three modes of representations are not only hierarchical in nature but also intertwined (Rowland et al., 2009). Symbolic representation being at the highest level and the most abstract representation compared to the other two (enactive and iconic). This is the most adaptable form of representation than actions and images, which have a fixed relation to that which they represent. Symbols are flexible in that they can be manipulated, ordered, and classified and as such the learner is not constrained by actions or images. In the symbolic stage, knowledge is stored primarily as words, mathematical symbols, or even in other symbolic systems (Bruner, 1966).

Rowland et al. (2009) have cited an example of an empty number line to substantiate their point that the three modes of representations cannot only be used hierarchically but also in an intertwined fashion. They argue that an empty number line is an iconic representation by its nature but as learners make 'hops' or 'jumps' on it, they are using it in an inactive way. Yet the operations demonstrated by such hops and the answer reached are symbolic in nature. During their training courses, student teachers are expected to acquire and develop a web of connected representations for various mathematical concepts which they can draw on in lessons to help learners understand mathematics.

Theoretical Orientations

The ability to represent mathematical ideas in various and flexible ways is an important component of PCK. Shulman (1986, p. 9) argues that "since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation." It then follows that both experienced and novice primary school teachers' PCK of mathematics can be made manifest through their ability to select and use multiple representations in lessons to make abstract mathematical concepts accessible to learners.

Pedagogical Content Knowledge and Representations

The ability of any teacher to translate his/her knowledge of the subject matter into something comprehensible to learners is an important part of that teacher's PCK. When working with experienced teachers in Australia, Loughran et al. (2004) found that teachers' knowledge of their practice (teaching) is tacit. They found that although teachers find it challenging to provide reasons for teaching certain concepts in particular ways, in general, teachers commonly share activities, teaching styles, and insightful thoughts of how best to teach. Loughran et al. (2004, p. 374) take a view that researching teachers' PCK requires working at both an individual and collective level because as they put it "PCK resides in the body of science teachers as a whole while still carrying important individual diversity and idiosyncratic specialized teaching and learning practices." When exploring the notion of mathematics teachers' knowledge resource, Rowland et al. (2009, p. 14) concur that within a school context, teacher's knowledge resource is "both individual–what each teacher knows, and collective–what is accessible by reference to colleagues." In class a teacher draws from his/her own content knowledge (CK) but in situations where teachers work cooperatively in school, they plan lessons together and talk about methods of delivery in class. In that way each teacher gets necessary assistance from colleagues. When working with teachers in South Africa Rollnick et al. (2008) argue that the ability to choose an appropriate representation and use it effectively in lessons reflects that teacher's PCK. Shulman (1986, 1987) too takes a view that multiple representations that teachers use in lessons are the central part of each individual teacher's PCK. It then follows that being able to choose and use representations efficiently in lessons is an important component of any teacher's PCK.

Specialized Content Knowledge

The notion of SCK is understood as the type of knowledge possessed by teachers in that discipline (Ball et al., 2008). Ball et al. (2008) argue that specialized mathematical knowledge is a particular knowledge possessed by mathematics teachers alone. For example, history teachers do not have that knowledge. Nurses do not have that knowledge and engineers do not have that knowledge as well. An example of that knowledge is teaching learners how to calculate the following: $\frac{1}{3}+\frac{2}{5}$. Mathematics teachers and those who learn to become mathematics teachers must have this type of knowledge. The use of multiple representations in teaching is a critical part of specialized mathematical knowledge. When chosen carefully and used effectively, representations have a potential of making mathematics concepts understandable to learners. In this study, the student teacher is beginning to display the possession of the specialized mathematics knowledge by trying to choose and use some representations in her grade 4 class. However, it might be hard to argue that the use of 'fake money' was indeed effective especially in teaching decimal numbers. But it could be said that learners seemed to have enjoyed the activity of constructing fake notes and bringing them to class. The effort Thandi made to use the buying and selling context could be taken as a powerful starting point for building her special CK.

The Knowledge Quartet

In Thandi's (student teacher's) lesson, there were multiple activities taking place, however we needed a theoretical tool/lens through which we could focus on the teaching/learning of mathematics through representations. The 'knowledge quartet (KQ)' proved to be a useful tool in this case. Devised by Rowland et al. (2005), the KQ is a typology that emerged from a grounded approach to data analysis of primary mathematics teaching in the UK. The KQ identifies the way the student teacher's mathematical knowledge impacts on a mathematics lesson along four dimensions namely, foundation, transformation, connection, and contingency. While all four dimensions are interconnected and all are useful in looking at mathematics teaching, the 'transformation' dimension lends itself particularly well to this study in that it focuses the eyes of the researchers on the choice and use of representations that student teachers make when teaching mathematics.

In 'foundation' dimension of the KQ, the teacher's background knowledge and beliefs regarding the meanings and descriptions on mathematical concepts and practices are manifested during teaching. A teacher's ontological position of mathematics and the rationale for teaching it at primary school level is noticeable and made manifest in mathematics lessons. According to Rowland et al. (2009), the codes for this dimension include among others: awareness of purpose, identifying errors, overt subject knowledge, use of mathematical terminology, use of textbook, reliance on mathematical procedures, and theoretical underpinning of pedagogy.

The 'contingency' dimension calls for a teacher to make sound decision during the lesson about learners' contributions. Unlike experienced teachers, student teachers lack the ability to take on and respond on the sport to unexpected learners' contributions in class. Hume and Berry (2011) distinguish the most limiting factor as student teachers' lack of classroom experience and experimentation. According to Rowland et al. (2009, p. 126), "there are times when the teacher is faced with an unexpected response to a question or an unexpected point within a discussion and so has to make a decision whether or not to explore the idea with the child." A teacher has to be always alerted for such moments and be ready to react appropriately to such unexpected situations during the teaching episode. It could be unfortunate if unexpected learners' contributions can pass unnoticed in a class by the teacher because unpacking such contributions might be of special benefit to that learner or as Rowland et al. (2009) put it, might suggest a particularly fruitful avenue of enquiry for others. They identify the key contributory codes in this dimension of the KQ as responding to children's ideas; use of opportunity; and deviation from agenda. In what follows, we address the ways in which methodological issues were attended to.

Common Thread Between Content Knowledge, Pedagogical Content Knowledge, and Quartet Knowledge

CK denotes the body of knowledge (e.g., facts, theories, principles, ideas, vocabulary, etc.), which mathematics teachers must have to be effective (Ball et al., 2008). This means that teachers must have a deep understanding of the subject (mathematics) they are teaching and its corresponding curriculum (subject CK). PCK on the other hand, refers to special knowledge that teachers must have, which is based on the way they relate their pedagogical knowledge (what they know about teaching) to their subject matter knowledge (what they know about what they teach) (Shulman, 1987). CK and PCK are some of the important components of teacher knowledge and they play a pivotal role in the development of subject (mathematics in this context) teaching. The KQ on the other hand, is a useful theoretical framework to be used for the analysis and development of mathematics teaching (Rowland, 2005). The knowledge and beliefs demonstrated in mathematics teaching can be realized in four dimensions of KQ namely, foundation, transformation, connection, and contingency. According to Rowland et al. (2009), the four dimensions play out well in mathematics lessons where student teachers learn to teach the subject. They argue that KQ is an analytical lens through which a trainee's CK and PCK might be noticed.

METHODOLOGICAL APPROACH

In this phenomenological case study, one primary school was selected as a study site using purposive sampling technique. The school was the biggest in Lesotho with an enrolment of 1,500 learners. To ensure rigor in phenomenological research, the contextual factors that may influence how the study is conducted need to be highlighted (Armour et al., 2009). Thus, the primary school where the study was conducted was under-resourced even though it was the biggest. Data were generated through a series of observations and focus group discussions. For this study, the data that is reported is the one that was generated during lesson observations. Thus, the observations served as ideal instruments to explicitly show how the representations were chosen and used. Subsequently, an observation guide was used which outlined the aspects of KQ typology to be observed. The content analysis technique was used for the purpose of data analysis. The analysis was "directed because the general themes were determined *a priori*". Being a qualitative case study that sought to explore a phenomenon using one student teacher as a "case" in her school setting, the findings will not be generalized. However, the findings will contribute to the body of knowledge by illuminating some ways in which learning could be scaffolded using representations.

Figure 1. Sample of the money learners constructed

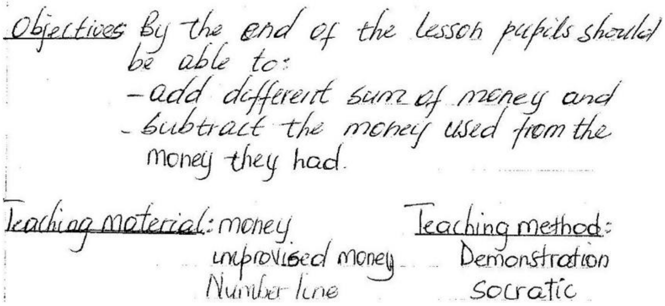
Figure 2. Pictures that Thandi placed on the wall

Thandi's Lesson

There were 80 (n=80) grade 4 learners in Thandi's class. The learners' ages varied from 9 to 15 years. Thandi's lesson synopsis is, as follows: At the beginning of the lesson, the cooperating teacher asks learners to stand and sing a song. This looks to be a good starter for the lesson. Thandi then asks learners to take out their money (on the previous day, Thandi had given learners homework in which they had to construct fake money and bring it to class in the next day). Learners take out their fake moneys made from paper. Thandi mentions that money is subtracted when used to buy items in a shop. She then places pictures of different fruit on the wall in front of learners. She asks six learners to come to the front of the class and give them fake money to buy items placed on the wall. She asks each learner to say how much money they have. Learners are then asked to buy items of their choice and say whether they have change or not, and if they do, to say how much change they had. This goes on till all chosen learners have used their money. Learners are then asked to go back to their seats. Thandi then distributed textbooks to learners. A pair of learners is asked to share a textbook. Learners are asked to turn to page 42 and do exercise 4 (a). The time for the lesson elapses and the lesson ends at this moment.

The picture in **Figure 1** is a sample of the money that learners had constructed.

The teacher placed representations of fruits items on the wall and asked the six (n=6) chosen learners to imagine being in a shop to purchase certain fruits. It is a common practice for learners in this area to buy fruits not only in supermarkets but also from the streets and at the school gates. Many people in Lesotho earn a living by selling fruits in various places including school


Objectives By the end of the lesson pupils should be able to:
- add different sum of money and
- subtract the money used from the money they had.

Teaching material: money, improvised money, Number line
Teaching method: Demonstration, socratic

Figure 3. Part of Thandi's lesson plan

surroundings. It is reasonable, to conclude that learners in Thandi's class are too familiar with buying fruit.

Figure 2 shows some of the pictures that Thandi placed on the wall.

The lesson was conducted over a period of two days for two consecutive lessons, each lesson lasting for 40 minutes. The lesson focused on addition and subtraction concepts whereby the money was used as a form of "representation" to facilitate the teaching and learning of these concepts. We (researchers) sat in the class to observe the lesson. We used the video recorder to record the lesson and generated some field notes. We spoke to the teacher and the learners before the commencement of the lesson, explaining what we will be doing during the lesson (i.e., record the lesson and take notes). We indicated to the learners that we were fascinated by the good work that their teacher was doing and therefore, we wanted to sit in her class to observe her session as we would also like to learn from her. We did that to make sure that the learners did not feel threatened by our presence and to also ensure that they fully participated in the lesson without any fear. The data which we collected we analyzed it at a later stage using content analysis technique. We created an opportunity to further engage with the teacher at the end of the lesson to give her a chance to reflect on her teaching, thus highlighting reasons for selecting the type of representation that she used and providing reasons for using it in the way she did.

FINDINGS AND DISCUSSIONS

The following sections provide analysis of the data drawn from the observed lesson as well as discussion of the findings. The discussion of the analysis of data is organized around the four dimensions of the KQ.

Foundational Knowledge

One of the key duties of a teacher is lesson planning. A lesson plan serves as a guide on how teaching should be carried out. This preparation for a lesson affords a teacher a chance to consult with various textbooks and decide on the method to use to teach concepts. Part of Thandi's lesson plan is shown in **Figure 3**.

According to the lesson plan, the lesson was meant to teach addition and subtraction within the context of 'improvised money'. The plan shows that Thandi had planned to use 'demonstration' and 'Socratic' methods of teaching. This implies she would demonstrate how to do some calculations within the context of fake money and that would be followed by question and answer session. She was also intending to use a number line to teach addition and subtraction. Thandi introduced the notion of 'decomposition'. The following excerpt shows the introductory part of her lesson:

> Thandi: Yes, when we decompose a number we break it into pieces, ha ke re (is that so)?
>
> Pupils (chorus): E-ea 'm'e (yes madam).
>
> Thandi: If you break it up into pieces, we just take out any numbers that can add up to fifty, ha ke re (is that so)? So, my own number … I can extract M20, Nka etsa (I can make) 20+20+10=M50.00 (writing on the chalkboard). We add up to M50.00. So, which other numbers can we decompose fifty Maloti into? Which numbers can we decompose fifty Maloti into? Tefo!
>
> Tefo: M10+M10+M10+M10+M10=M50.00

In this excerpt, Thandi encourages learners to 'decompose' M50 into various forms. As promised in her lesson plan, she demonstrates how to do so (20+20+10=50). This demonstration that Thandi made appeared helpful in aiding learners to comprehend the meaning of decomposing numbers. Tefo's response in the excerpt above may be considered evidence of this. However, this opening exercise of decomposing numbers might be coded as 'concentration on procedures' an element of the foundation dimension of the KQ (Rowland et al., 2005). It would be interesting to observe what Thandi would have done had some learners suggested 40+5+5, as an answer. This would be of interest because in real life there is no note that is worth M40 and in Thandi's fake money there were no coins such as M5. The fact that Thandi did not say anything about this possible answer (and other related ones), which learners could have provided, somehow shows that she did not think "deeply" about the concept which she was going to teach. She thus did not consider several "answers" that possibly could have been provided by the learners and also did not think of ways in which she would address such "possibilities". This somehow, shows lack of SCK on Thandi's part, in that, she seemed to be unaware of the need to examine the aspects of students' thinking that goes beyond the procedure and the pattern that she has shown them. However, the use of fake money in class shows that Thandi had a strong PCK which enabled her to identify even the strategies that were not highlighted in the textbooks, but also equally relevant to be used to achieve the goal of the lesson.

Connecting Ideas

If teachers are to make mathematics comprehensible to learners, they must always make efforts to present it as a series of connected concepts, procedures, ideas, and practices. Marshall et al. (2010) take a view that making connections between multiple representations helps students see mathematics as a web of connected ideas and not as a collection of arbitrary, disconnected rules and procedures. The connections can be made between concepts, operations, units, topics, and branches (e.g., geometry and arithmetic). For instance, the use of an iconic representation such a number square in class can help learners to recognize the connection that exists between two operations namely addition and subtraction of whole numbers. Rowland et al. (2009) argue that teachers must bear in mind the complexity and cognitive demands of mathematical concepts and procedures in their attempt to sequence and connect mathematical content. They further identify contributory codes for this dimension (connection) as: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; and recognition of conceptual appropriateness.

In Thandi's lesson, the main 'connection' that we observed is the way the use of money context managed to connect operations of addition and subtraction together. The other 'connection' was made between two topics namely whole numbers and decimal numbers, and this occurred as learners were buying fruit and had to determine their change. In another incidence during the lesson, Thandi made some connection between two representations namely money and ordinary numbers. The extracts below highlight the connections made in Thandi's lesson:

> Thandi: … He has M2.00 … and from the M2.00 he has ... the banana costs M2.00.
>
> Pupils (chorus): Yes madam.
>
> Thandi: From the M2.00 he has and the banana costs M2.00, is he going to get the change or not?
>
> Pupils (chorus): No.
>
> Thandi: No, because he has finished the money, ha ke re (isn't it so)? When you subtract M2.00 from M2.00 you get zero (0). Ha ke re (isn't it so)?
>
> Thandi: M2.00 take away M2.00 results in no change is the same as 2-2=0. In mathematics, when we perform subtraction using the number, the answer that we get after subtracting the numbers

we call it difference, but when we deal with money we call it a change.

Thandi: Take away here refers to subtract which we also refer to as minus and we use negative sign to represent it!

In the lesson above, Thandi taught the subtraction concept and to make it comprehensible to the learners she used fake money as a representation. The choice of the representation in this instance was motivated by the fact that the students were familiar with the concept of money and how it is used. Therefore, instead of dealing right away with the numbers (i.e., teaching the concept abstractly), Thandi used concrete object (i.e., the fake money which she had asked learners to create). During the lesson some connections were identified, which in that case was: "*Two Maloti take away two Maloti results in no change and 2-2=0*". The use of Maloti (fake money) served as a way of "customizing the display of information", which in turn made it easy for learners to grasp the concept (Moleko & Mosimege, 2021). What was also of interest in this lesson was to see the teacher also using money to address the mathematical vocabulary, hence the statement, "*in mathematics, when we perform subtraction using the number, the answer that we get after subtracting the numbers we call it difference, but when we deal with money, we call it a change*". Furthermore, the student teacher (Thandi) clarified the meaning of the operational sign (-) using an expression "take away" and also indicated that another mathematical term that can be used instead of subtracting is minus'. By so doing, the student teacher thus provided learners with options for language, mathematical expressions, and symbols (Center for Applied Science and Technology–CAST, 2011).

It seems that the connection of "zero change" and the "number zero (0); the connection of "change" and difference" and the connection between the terms "take away, subtract and minus" respectively here are critical. The first connection here is between the two sentences: Two Maloti take away two Maloti results in no change and *2-2=0*. The second connection is between the two expressions: "change" and "difference". The third connection is between the terms; 'take away', 'subtraction' and 'minus'. The fake money used thus made it possible for all these connections to be discovered, thus making learning meaningful.

Even though the teacher managed to procedurally teach the subtraction concept and showed how to arrive at "zero" as an answer, the explanation of zero was not ideal. The teacher explained zero as *"haho letho" ("nothing")*. This is the mistake that most teachers do because it gives learners the impression that zero is not a number but "nothing". This later creates confusion especially when learners progress to higher grades where they have to deal with a "zero concept" in other contexts. For instance, it becomes difficult for learners to understand why any number raised to zero as an exponent is equal to one (1), especially when they were told that zero is nothing. Thandi's lack of SCK was observable in this regard wherein she could not be able to identify this as an error in her explanation, which could potentially create difficulties for understanding the zero concept when learners move to higher grades. However, the teacher's choice of representation (money) (PCK) made it easy for learners to understand the addition and subtraction concepts, which she taught them. It also made it possible for learners to make connections between these concepts.

Transformation as a Dimension

Of the four dimensions of KQ, the key dimension remains to be 'transformation' in this study that interrogates the choice and use of the mathematical representations in teaching. When unpacking the notion of SCK in mathematics education Ball et al. (2008) emphasize the key role representations play in making mathematics concepts understandable to learners. These scholars further argue that because some representations are more powerful than others in affording learners' access into mathematics concepts, teachers who have rich SCK choose and use "appropriate representations" that make content comprehensible to learners.

Every mathematics teacher must make choices in planning and delivering a lesson. The choices involve among others selecting a key representation for the concept intended to be taught. For instance, a number line can be used for teaching addition, subtraction, multiplication, and division. But a number line is a key representation for addition and subtraction while a multiplication square or a multiplication array might prove to be key representations for performing multiplication at any level of the primary school mathematics. During lesson preparation, a teacher must therefore think carefully about the examples, illustrations, and analogies that he/she can use in class to make concepts, procedures, or even core vocabulary comprehensible to learners. Rowland et al. (2009) identify contributory codes in this dimension of the KQ as, choice of representations, teacher demonstrations, and teacher's choice of examples.

In her lesson, Thandi chose to use fake money as a key representation to scaffold learners' skills of addition and subtraction of whole numbers and decimal numbers. During her extensive lesson on addition and subtraction, Thandi taught learners about decomposing the numbers and demonstrated what she meant by decomposing numbers.

Thandi: Yes, when we decompose a number, we break it into pieces, ha ke re (is that so)?

Pupils (chorus): E-ea 'm'e (yes madam).

Thandi: Re ya e tjhwatla (we break it)!

Thandi: If you break it up into pieces, we just take out any numbers that can add up to fifty, ha ke re (is that so)? So, my own number … I can extract M20, Nka etsa (I can make) 20+20+10=M50.00. We add up to M50.00. So, which other numbers can we decompose fifty Maloti into? Which numbers can we decompose fifty Maloti into? Tefo!

Tefo: M10.00+M10.00+M10.00+M1.000+M10.00= M50.00

Sarah: M5.00+M10.00+M15.00+M20.00=M50.00

Thandi: Yes Majara!

Majara: Oh, I was going to say M20.00+M15.00+ M10.00+M5.00

Thandi: That is good Majara! You see class…the answers which Sarah and Majara have provided are the same. The only difference is that the order of the numbers is not the same.

The demonstration that Thandi made appeared to be helpful in terms of helping learners to comprehend the meaning of "decomposing numbers". The evidence is seen by Tefo's response as shown in the excerpt above. What was interesting to note was to hear her also code-switching (using Sesotho language) which was the home language of all the learners in her class to reinforce understanding of the term decompose. According to Moleko and Mosimege (2021), promoting understanding across languages is important to facilitate understanding of the mathematical concepts. The question which she posed to the whole class namely, "*Which numbers can we decompose fifty Maloti into"* shows that she afforded the learners the opportunity to expand on the example that she made earlier when demonstrating how to decompose fifty Maloti. Tefo's response and other learners' correct responses (representations) enabled the learners to recognize the multiple ways in which they can express or represent 50 Maloti (M50). Thandi further addressed the issue of "commutative law" in her teaching by indicating that *M5.00+M10.00+M15.00+ M20.00 is the same as M20.00+M15.00+M10.00+M5.00* drawing from the answers provided by the two students, Sarah and Majara.

Later in the lesson when learners were struggling to subtract decimal numbers from whole numbers, Thandi encouraged learners to use other forms of representations:

Motsamai: The banana is 1 Loti and 50 cents. We subtract M1.50.

Thandi: We subtract 1.50 Loti. It's 150 lisente (cents), from M5.00 he has, we subtract 1.50 lisente ha ke re (isn't it so)?

Pupils (chorus): Yes madam.

Thandi: So, what is the change? What is Makoro's change? So how much is he going to get as the change for Makoro? How much is he going to get? Use your fingers, use our money … just think, think, use your fingers, your head, whatever! What do you think is going to be Makoro's change? What do you think is going to be Makoro's change, when we subtract 1.50 from the M5.00 he has? Nkele!

Nkele: Makoro's change is going to be M3.50.

Here Thandi asked learners to use fingers and their heads to think about the correct answer for the change. The excerpt suggests that the use of any of these representations (fingers or/and heads) afforded the learner (Nkele) with strategies that made it possible for her to obtain the correct answer (M3.50). It is possible that Nkele uses these representations in her daily buying to determine her change. Based on this therefore we would like to argue that Thandi's choice of the "selling and buying situation" (which learners are familiar with) assisted learners to manage subtracting decimal numbers from whole numbers, which could have been more cognitively challenging if it was only presented/represented symbolically as 5-1.5=? Another interesting point to note is that to enable the learners to determine the answer, Thandi converted loti into cents *("… 1.50 Loti. It's 150 lisente [cents]")*. This conversion made it easy for learners to perform subtraction since many learners often find working with decimals difficult as opposed to working with whole numbers. Converting Loti into cents was another way of customizing the representation of M1.50 loti into a simple one, namely, (150 cents) to enable learners to then use their heads and fingers (as the teachers encouraged them) to determine the answer. The money conversion, encouraging learners to use their heads and fingers thus served as a good way of guiding information processing, visualization, and manipulation of the given problem (CAST, 2011). Thandi's realization that some learners might find the wrong answer due to being unable to work with decimals hence using a conversion idea made it easy for learners to determine the answer. This to a certain extent, shows that she had some SCK even though it was not impeccable.

Contingent Moments

The work of teaching is complex in that things do not always go according to plan. There are classroom incidents that might "pop up" unplanned. Such incidents are usually as a result of learners' contributions either in a question or comment. Rowland et al. (2005) have argued that experienced teachers are more likely to handle such unexpected moments than novice teachers especially student teachers. The following excerpt illustrates one of the contingent moments in Thandi's lesson:

> Motsamai: The banana is 1 Loti and 50 cents. We subtract M1.50.
>
> Thandi: We subtract 1.50 Loti. It's 150 lisente (cents), from M5.00 he has, we subtract 1.50 lisente ha ke re (isn't it so)?
>
> Pupils (chorus): Yes madam.
>
> Thandi: So, what is the change? What is Makoro's change? So how much is he going to get as the change for Makoro? How much is he going to get? Use your fingers, use our money … just think, think, use your fingers, your head, whatever! What do you think is going to be Makoro's change? What do you think is going to be Makoro's change, when we subtract 1.50 from the M5.00 he has? Nkele!
>
> Nkele: Makoro's change is going to be M3.50.

Up to this far, the lesson was going well as the class was dealing with whole numbers. The challenge began when learners were asked to subtract decimal fractions from whole numbers such as *5-1.5*. So, when learners could no longer give answers right away Thandi became frustrated. That pushed her to deviate from her agenda which was to use fake money. She then resorted to the use of other forms of representations such as *"…use your fingers; your head; use whatever…"*. Here, Thandi asked learners to use their fingers and their heads to think about the correct answer (the change). The excerpt suggests that she believed that the use of any of these representations would afford the learners strategies that would make it possible for them to obtain the correct answer (M3.50). The fact that Thandi became frustrated when learners could not spontaneously give answers when dealing with decimals as they did with the whole numbers, shows that she was not aware of the difficulties the learners encounter when dealing with decimals. This lack of awareness is noticeable in her instruction; instructing learners to bring the notes of fake money leaving out the coins which could have been useful when dealing with decimals.

Contribution of the Study to the Body of Knowledge

Many studies have focused on the use of manipulatives, pictorial use, counters, and artefacts as some forms of representations that can be used in the classroom and highlight the benefits thereof (Bartolini Bussi, 2011; Darnis & Dodd, 2021; Salingay & Tan, 2018; Strom, 2009). However, in this study we unpack and deepen the understanding of the choice and use of representations, by practically demonstrating the operationalization of the four dimensions of KQ namely, foundations, connections, transformation, and contingency. Through these dimensions, the study provides a practical guide on how teachers can help learners make connections of the mathematical concepts, thus seeing mathematics as a web of related ideas and not as a collection of arbitrary, disconnected rules and procedures. The study further provides a guide on how to choose the appropriate mathematical examples, select the appropriate representations to teach the identified examples as well as indicate how the selected representations can be used to teach the identified examples.

The study highlights the three important actions to be taken by the student teachers (and the in-service teachers) to promote meaningful learning in a mathematics classroom. These actions are i) to "identify/determine" the sound mathematical example(s) that will enable the teacher to teach the lesson or the mathematical concept in a comprehensible manner; ii) to "choose" the appropriate type(s) of representation(s) to teach the mathematical concepts and iii) to "use" the chosen representation(s) to teach the mathematical concept. The study provides knowledge of the various use of representations when teaching mathematical concepts. Knowledge of the various use of representations include using representations to *make abstract mathematical concepts and processes accessible to learners*; using representations as *resources for teaching and learning*; using representations for *the purpose of promoting learner engagement*; using representations to *make the concepts simple, intuitive, and perceptible*; and using representations to *provide flexible teaching*. Although the study was intended to operationalize the four dimensions of KQ as already alluded, other important findings were discovered regarding the use of representations. These findings contribute to the enhancement of the existing KQ typology, which has four dimensions namely, foundation, connection, transformation, and contingency. In line with this, the use of "fake money" as a representation in the current study afforded learners opportunities to engage with the addition and subtraction concepts and to practically solve the related problems. The representation therefore promoted learner engagement and increased participation in class. The use of representation also afforded learners opportunities to use the money context (which they are familiar with) to learn the mathematical concepts (addition and subtraction). The fake money used as a representation thus optimized the relevance of the mathematical concepts in real life, value of learning and authenticity. The use of representations further enabled the student teacher to clarify the mathematical vocabulary and to address the mathematical notations and symbols. Furthermore, the representation made it possible for the student teacher to represent information in various ways, thus customizing the display of the mathematical concepts. The use of "fake money" afforded the learners the opportunities to engage in visual and discovery learning. The study thus proposes the addition of 'learner engagement' and 'linguistic' dimensions to the existing KQ typology. The learner engagement dimension, therefore, requires the teacher to know how to select the appropriate representations that will subsequently enable the learners to effectively engage in meaningful learning. The linguistic dimension on the other hand, requires the teacher to have knowledge of the mathematical language, language of learning and teaching as well as the home language(s) to effectively use the representations.

CONCLUSION

Analysis of the data reveals that Thandi chose to use fake money as a representation to support the teaching and learning of mathematical operations namely addition and subtraction. Thandi had various resources at her disposal to use in this lesson; however, she carefully chose to use fake money made of paper to facilitate the teaching of whole numbers and decimal numbers. The homework that was given to learners namely, to construct fake money prior to this lesson, had the potential of helping learners to represent money in an inactive way (hands on activity) while at the same time helping them to think more seriously of real money used in their community. When learners encountered some subtraction difficulties, Thandi encouraged them to use the 'fingers' of their hands as representations to assist them to get to the correct answer. It is also noted in the analysis that the effective use of fake money in this lesson afforded learners opportunities to make crucial connections between subtraction of whole numbers and decimal numbers. Again, the use of fake money in the lesson encouraged discussion. Based on this therefore we conclude that the fake money helped Thandi to effectively achieve her lesson objectives.

Although Thandi seemed to be in control of her class, some of the weaknesses were revealed in her teaching. She seemed to lack SCK and that was evidenced by being unaware of the need to examine the aspects of students' thinking that goes beyond the procedure that she taught her learners. Her lack of SCK was also observable when she could not be able to identify and realize that her explanation of zero as an answer was inaccurate and that such an answer could potentially create difficulties in terms of understanding the "zero concept" especially when learners move to higher grades. The fact that she became frustrated when learners could not spontaneously provide answers when dealing with decimals as they did with the whole numbers, shows that she was not aware of the difficulties that the learners encounter when dealing with decimals. This lack of awareness was noticeable when she instructed learners to bring the notes of fake monies leaving out the coins which could have been useful when they were dealing with decimals. Nonetheless, her lesson also revealed some of her strengths which were admirable as a student teacher. She demonstrated knowledge of choosing the appropriate representations to be used to teach the subtraction concept. This is one of the significant strengths that she portrayed. She also demonstrated knowledge of context by applying the money concept in the teaching of mathematics to teach subtraction. Furthermore, her PCK enabled her to be innovative and thus ask learners to make money and bring it to the class to illustrate the subtraction concept. This enabled learners to understand the concept and develop its mastery. She further seemed to know the other types of representations other than the money concept which is why she was able to provide some suggestions on what the students can use to calculate the change. While the findings of this study cannot be generalized, we believe that this work might give some useful insights with regard to student teachers' mathematical understanding and use of representations in teaching.

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