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An Inquiry Approach to Construct Instructional Trajectories Based on The Use of Digital Technologies

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There are diverse ways to construct instructional activities that teachers can use to foster their students' development of mathematical thinking. It is argued that the use of computational tools offers teachers the possibility of designing and exploring mathematical tasks from distinct perspectives that might lead their students to the reconstruction of mathematical relations. In particular, a task that involves the construction of a simple dynamic configuration is used to introduce an inquisitive approach to identify mathematical conjectures or relations and ways to explore and support them. In this process, a hypothetical instructional route is sketched where visual, numeric, geometric, and algebraic approaches are utilized to validate those conjectures.

Keywords: Problem Solving, Computational Tools, Teachers' Knowledge, Instructional Trajectories.

INTRODUCTION

The significant development and availability of several digital tools have opened up diverse opportunities for teachers and students to approach and construct mathematical knowledge and to develop problem-solving strategies. How does the use of particular digital technologies help teachers promote their students' development of problem solving activities? What types of opportunities can the use of the tools offer the learners to engage in mathematical thinking? To what extent does the use of digital technologies become relevant for teachers to trace and explore potential instructional routes to guide their students learning experiences? I utilize the construct "instructional trajectories" to explore and discuss ways in which the systematic use of computational technologies can help teachers trace and examine potential instructional routes to frame and guide their instructional practices. It is argued that the use of the tools becomes important for teachers and students to be engaged in an inquiring or inquisitive approach to reconstruct or develop mathematical relations and enhance problem solving approaches. The hypothetical instructional trajectories that result from examining mathematical task with the use of computational tools are used to guide and promote the students' actual development of their own learning trajectories. In this context, an overarching principle that distinguishes the use of the tools is to conceptualize the tasks in terms of dilemmas or questions that need to be represented and explored through the use of mathematical resources and problem solving strategies. In this context, an inquisitive approach to work on the tasks becomes relevant to illustrate that the use of the computational tools can help teachers develop and employ a set of heuristics (Polya, 1945) that includes a dynamic representation of the task, finding loci, exploring partial goals, using the Cartesian system, quantifying relations, etc. In addition, it is shown that the construction of instructional trajectories can be a teachers' means to review their own mathematical knowledge and problem solving approaches and to openly discuss the paths or routes to approach and solve the tasks in their actual practice.

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Instructional Trajectories and Teachers' Mathematical Knowledge

What mathematical, technological, and pedagogical knowledge should the education of high school mathematics teachers include? Who should participate in the designs of educational programs to prepare and upgrade mathematics teachers? What should be the role of mathematics departments or the faculty of education in preparing prospective and practicing teachers? In what types of educational programs should practicing teachers participate in order to revise and extend their mathematical knowledge and to incorporate research results from mathematics education into their practices? Traditional ways to prepare high school teachers normally involve the participation of both mathematics departments and the faculty of education. Mathematics departments offer courses in mathematics while the faculty of education provides the didactical or pedagogical courses. This model of preparing teachers has not clearly provided them solid basis to exhibit the needed mathematical sophistication to interpret and efficiently guide their students in the construction or development of mathematical knowledge. As a consequence, teachers fail to organize and implement meaningful learning activities that foster their students' development of mathematical thinking. Indeed, it is common to read that university instructors complain that their first year university students lack not only fundamental mathematical knowledge; but also strategies or resources to solve problems that require more than the use of rules or formulae (Artigue, 1999, Selden & Selden, 2001).

Many practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. …Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are substantial, and often under-appreciated (Adler, et al., 2005, p. 361).

Davis and Simmt (2006) suggest that teachers' preparation programs should focus more on teachers' construction of mathematical ideas or relations to appreciate their connections, interpretations, and the use of various types of arguments to validate and support those relations, rather than the study of formal mathematics courses. Thus, the context to build up the teachers' mathematical knowledge should be related to the needs associated with their instructional practices. "…[mathematical knowledge] needed for teaching is not a watered version of formal mathematics, but a serious and demanding area of mathematical work" (Davis & Simmt, 2006, p. 295). In this perspective, we argue that teachers' mathematical knowledge can be revised and enhanced within an interacting intellectual community that fosters an inquisitive approach to develop mathematical ideas and to promote problem-solving activities. The core of this community should include the participation of mathematicians, mathematics educators, and practicing teachers. This community should promote collaborative work to construct potential instructional trajectories to guide or orient the teachers' instructional practices. Teachers need to be interacting within a community that supports and provides them with collegial input and the opportunity to share and discuss their ideas in order to enrich their mathematical knowledge and problem solving strategies. Regarding the use of computational tools, Bransford, Brown, and Cocking (Eds.) (1999) state that:

New tools of technology have the potential of enhancing learning in many ways. The tools of technology are creating new learning environments, which need to be assessed carefully, including how their use can facilitate learning, the types of assistance that teachers need in order to incorporate the tools into their classroom practices, the changes in classroom organization that are necessary for using technologies, and the cognitive, social, and learning consequences of using these new tools (p. 235).

In this context, we illustrate the importance of using computational tools to represent and explore various ways of approaching mathematical tasks. The task discussion leads us to show that the use of diverse computational tools offers teachers the possibility of working on mathematical tasks from perspectives that involve visual, numeric, geometric and formal approaches. And as a consequence, they can appreciate or value the advantages associated with the use of the tools and trace potential instructional routes that can guide and foster their students' development of mathematical thinking and problem solving approaches.

Hypothetical Instructional Trajectories and Computational Tools

Problem solving activities that promote the use of digital tools represent an opportunity for practicing and prospective teachers to revise and extend their mathematical competences. What task representations are favored with the use of computational tools? To what extent does the use of computational tools become relevant in identifying and exploring conjectures or mathematical relations? To what extent does the use of particular tools shape a students' way of thinking about tasks and problems? These questions help explore ways of reasoning that can emerge or be developed in problem solving approaches that promote the use of computational tools.

It is argued that the development and availability of computational tools offers teachers and students the possibility of enhancing their repertoire of heuristic strategies to solve mathematical problems and to formulate or reconstruct some mathematical relations. "…guided reinvention [of mathematical knowledge] offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics" (Gravemeijer & Doorman, 1999). It is also important to recognize that different tools may offer distinct opportunities for students to represent and approach mathematical problems. Thus, it becomes relevant to show and discuss not only the potential associated with the use of diverse tools but also ways in which the distinct approaches to the tasks or problems can be related or complemented. For example, with the use of dynamic software, such as Cabri-Geometry or Sketchpad, some tasks can be represented dynamically as a means to identify and explore diverse mathematical relations or conjectures. Later, with the use of a hand-held graphing calculator those conjectures can also be analyzed graphically and algebraically. In this perspective, an underlying principle in any problem solving approach to learn mathematics is to look for distinct ways to represent and explore mathematical tasks and to contrast or discuss mathematical approaches that emerge from the use of diverse tools including the use of paper and pencil (Santos-Trigo, 2007). Thus, the problems or tasks are seen as opportunities to pose and pursue relevant questions that can lead to identify and explore mathematical relations (Schoenfeld, 1998). We identify and document the types of heuristic strategies that appear in problem solving approaches that promote the use of computational tools. In particular, the analysis and discussion of the strategies which emerge as result of constructing and exploring dynamic representations of problems.

Tasks are the key ingredients in promoting and tracing the students' development of problem solving strategies. Here, teachers first need to identify potential or theoretical instructional trajectories (Simon & Tzur, 2004) to frame and then discuss the distinct routes that their students can follow to approach the tasks.

> …[A]n overarching research goal in the field of learning trajectories is to generate knowledge of learning and teaching. Therefore, scientific processes (e.g., documenting decisions, rationales, and conditions; hypothesizing mechanisms; predicting events; and checking those predictions) must be carefully followed and recorded (Clements & Sarama, 2004, p.85).

The identification of potential instructional trajectories involves working on the tasks in detail and exploring various ways to represent and examine the tasks using computational tools. Working on these tasks requires that teachers recognize ways in which mathematics knowledge is connected, and a discussion of what constitutes a valid argument to support mathematical relations. Zbiek, Heid, & Blume, (2007, p. 1170) suggest that in experimental mathematics, computational tools can be used for:

> (a) gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results.

In this context, we illustrate the ways in which the use of Cabri-Geometry software and hand-held graphing calculators can help teachers represent and apply a set of heuristics to approach and solve the tasks. The solution process is presented around problem solving episodes where relevant questions guide the task solution process. The episodes are part of an inquiry framework that identifies instructional trajectories that teachers can use to structure and to guide the development of their lessons (Santos-Trigo & Camacho-Machín, in press). The task is representative of a set of problems that were used in a problem-solving seminar in which high school teachers used Cabri-Geometry software to identify and discuss potential learning trajectories.

The task involves the construction of a dynamic configuration that leads to relate a tangent circle to the study of two conic sections: The parabola and the hyperbola. Here, the use of two tools, the dynamic software and a hand-held calculator, becomes relevant to complement and relate ways of reasoning that involve visual, numeric, geometric, and algebraic approaches. The task is a variant of what Gravemeijer & Doorman (1999) call context problems since the problem solver has the opportunity to reconstruct a set of mathematical relations as a result of representing and examining mathematical objects dynamically.

An example: On the Construction of Possible Instructional Routes

An overall principle associated with the construction of potential instructional trajectories is that all problem representations should be constantly examined and interpreted in terms of responding questions that involve the use of mathematical resources or problem solving strategies. Thus, the formulation of questions and the search for diverse ways to respond to those questions are crucial activities that shape the development of potential routes of instruction. The next example illustrates ways in which the use of a tool (Cabri-Geometry software) can offer teachers the opportunity of reconstructing a set of mathematical relations that involves contents associated with the study of the conic sections. The problem solving episodes emerge within a community in which high school teachers together with mathematicians and mathematics educators worked on series of tasks to identify potential instructional routes and to discuss the strengths and limitations of using several computational tools. Thus, the goal is to characterize the community or group's problem solving approaches that emerged during the development of the sessions rather than analyzing in detail the individual contribution or performances of the participants.

The initial task. Given a line L and a point P not on the line (Figure 1) construct a dynamic configuration[1] that involves other mathematical objects and identify properties or mathematical relations that result from moving particular elements within the configuration.

This is an open activity where the construction of a geometric configuration might involve various initial routes. Thus, some departure attempts may include, for example: (i) Placing a point Q on line L and constructing an equilateral triangle with side PQ (Figure 2a) and add other objects and start moving some of them to identify invariants or changes produced as a result of that motion on other objects within configuration; or (ii) Situating also point Q on line L and drawing a circle that passes through point P and is tangent to line L at point Q (Figure 2b). In the latter, the initial goal can be to identify mathematical relations around the construction of a circle tangent to line L that passes through point P (Figure 2b). Thus, to draw a tangent circle to line L that passes through point P is the point of departure to identify and explore mathematical relations.

First episode: Dynamic representation and partial goals. An important strategy that is used often in problems or tasks that can be represented dynamically is to identify and analyze loci that result when some components (points, segments, lines, etc.) of the problem representation are moved along well defined paths. Thus, the construction of a dynamic representation of problems, whenever possible, is a heuristic that need to be considered in problem solving approaches. The use of the software for the construction of a dynamic representation is based on conceptualizing the problem in terms of relevant mathematical properties. What does it mean to draw a circle that passes through a point and is tangent to a given line? In this task, a heuristic, that involves focusing on a partial goal of drawing a circle with center point C situated on a perpendicular to line L and radius

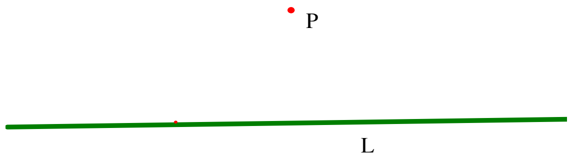

Figure 1. Construct a dynamic configuration that includes a given line L and a point P out of the line

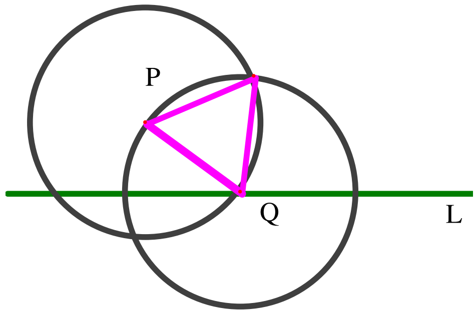

Figure 2a. Drawing an equilateral triangle with side PQ

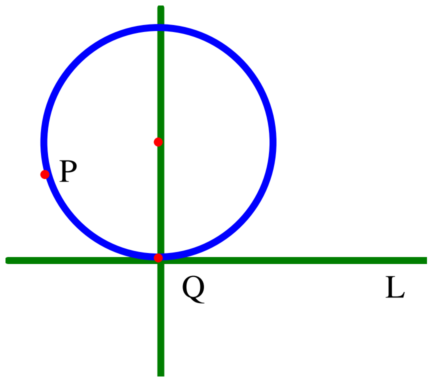

Figure 2b. Drawing a circle that passes through point P and is tangent to line L

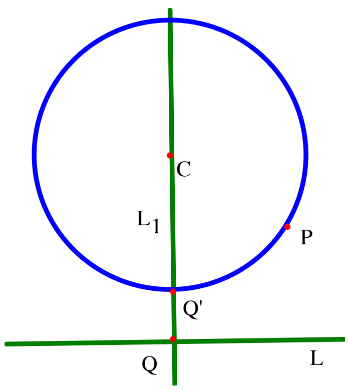

Figure 3. The center of the circle must lie on the perpendicular line to L

[1] A dynamic configuration consists of simple mathematical objects (points, segments, lines, triangles, squares, circles, etc.) arranged in such a way that one can move a particular element within the configuration and observe what happens to others elements as a result of that movement.

segment CP (Figure 3), is pursued to identify ways to construct a tangent circle.

A visual approach. The circle (Figure 3) satisfies the condition that passes through point P but it is clear that it is not tangent to line L. However, when either point C or point Q are moved along lines L_1 or L respectively, there will be visually a position for the circle in which it is tangent to line L (Figure 4a and 4b). This visual solution is useful to make explicit a set of properties associated with the construction of the tangent circle.

Second episode: Identification of geometric properties, a bisector approach. What geometric properties does the tangent circle satisfy? Is there any particular relation between the center of the tangent circle and the tangency point and point P? The visual approach becomes important to identify relevant properties embedded in the representation. It is observed that when the circle is tangent to line L (Figures 4a and 4b), then d(C,Q) must be equal to d(C,P). Based on this fact, the center of the tangent circle must be the intersection of the perpendicular bisector of segment QP and L_1 (perpendicular line to L that passes through Q) (Figure 5).

The above solution involves an Euclidean construction since it can be drawn with straightedge and a compass. With the use of the software it is possible to identify and examine the path left by particular points when other points are moved within the representation. What is the locus of point C' (center of the tangent circle) when point Q is moved along line L? (Figure 6). The locus of point C' when point Q is moved along line L seems to be a parabola; however, it is important to prove that the locus satisfies the definition of this conic section.

Third Episode: The use of empirical and formal arguments. To verify empirically that the locus is a parabola, we choose a point R on the locus and assume that point P is the focus and L is the directrix of the parabola. We calculate the distance from R to P and from R to line L and notice that for distinct positions of point R both distances are equal. Figure 7 shows two positions of point R. In this example, another heuristic method appears: To measure attributes (lengths, distances, areas, perimeters, angles, slopes, etc) associated with particular objects in order to identify invariants. In this case, the use of the software helped us to measure and compare distances from a point on the locus to line L and from the point to the center of the tangent circle.

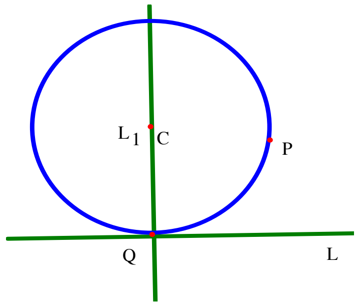

Figure 4a. Moving point C along line L_1 to visually identify the tangent circle to line L

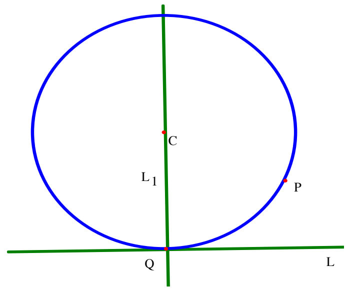

Figure 4b. Moving point Q along line L to visually identify the circle tangent to line L

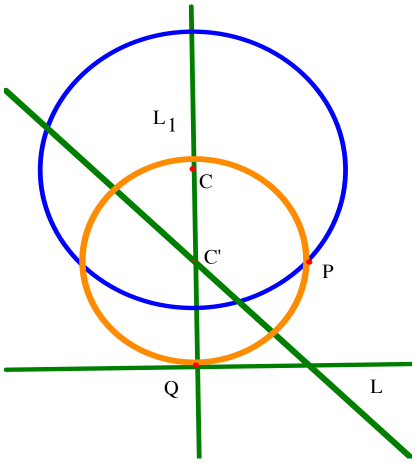

Figure 5. The center of the tangent circle is the intersection of the perpendicular bisector of PQ and the perpendicular line to L that passes through point Q

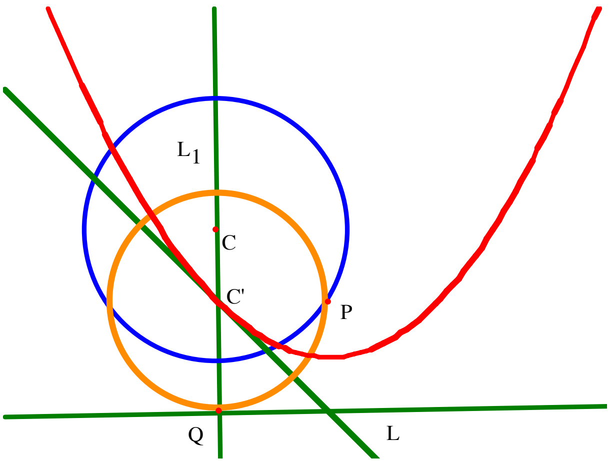

Figure 6. The locus of point C' when point Q is moved along line L is a parabola.

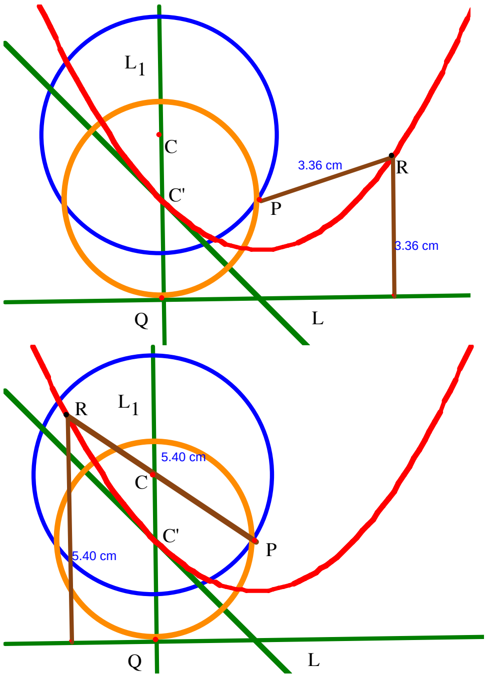

Figure 7: Verifying the definition of parabola

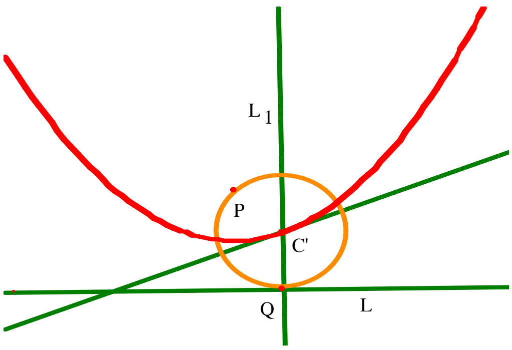

Figure 8: Using the definition of perpendicular bisector to show that the locus satisfies the definition of the parabola

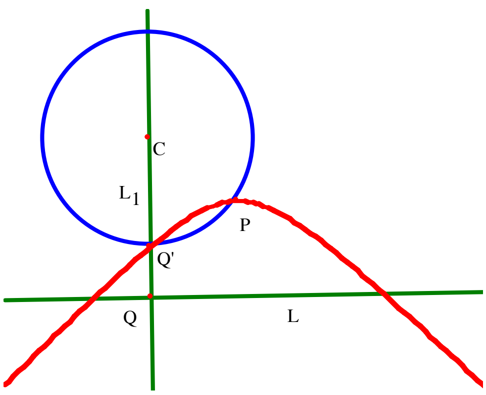

Figure 9. the locus of point Q' when point Q is moved along line L.

A geometric argument to show that the locus is a parabola is based on observing that point C', which generates the locus, is on the perpendicular bisector of segment QP (Figure 8). Therefore, the distance from point C' to point P is equal to the distance from point C' to line L (the definition of perpendicular bisector). Therefore, the locus of point C' when point Q is moved along line L is a parabola.

Fourth episode: Connections. On figure 3 it is observed that the circle intersects the perpendicular line to L at Q' and when point Q is moved along line L, then point Q' describes a unique path. What is the locus of point Q' when point Q is moved along line L? Again the software helps us identify this locus (Figure 9).

When point Q moves along line L there are two positions, the intersection points of the locus and line L, in which the circle is tangent to line L. Thus, the center of each tangent circle will be the intersection points (C' and C") of the perpendicular lines to line L drawn from the intersection points of the locus and line L and the perpendicular line to L_1 that passes by point C respectively (Figure 10).

With the use of the conic command from the software, we select five points on the locus and draw the corresponding conic section (Figure 10). In this case the conic section is a hyperbola. To show that the locus satisfies the definition of hyperbola, we draw a perpendicular line to L that passes through point P. This line intersects the locus at point P' and point M is the midpoint of segment PP'. We draw the perpendicular line to line PP' that passes through point M and a circle with center at point M and radius MP. This circle intersects that perpendicular at point K. We draw a perpendicular to line MK that passes through point K and a perpendicular to line PP' that passes through point P, these lines get intersected at point K'. We draw a circle with center point M and radius MK'. This circle intersects line PP' at points $F_1$and F_2. F_1 and F_2 are the foci of the hyperbola (Figure 11). This geometric construction can be validated through an algebraic approach (Santos-Trigo, et, al., 2006).

Again to show empirically that the definition of hyperbola is satisfied, we take a point S on the locus and calculate the absolute value of the difference between the distances from that point to each focus. It is observed that for different positions of point S the difference is a constant (Figure 12).

It is also observed from figure 10 that the loci of points C' and C" (centers of the tangent circles), when point C is moved along line L_1, is a parabola (Figure 13).

The argument used to show that the locus is a parabola is based on the fact that points P and R are on the circle with centre C', therefore, d(C',P) = d(C',R). That is, the focus of the parabola is point P and its directrix is line L.

A triangle approach. Another way to draw the tangent circle to line L that passes through point P involves drawing Q on line L, a circle with center point Q and radius QP, and a parallel line L' to line L that passes

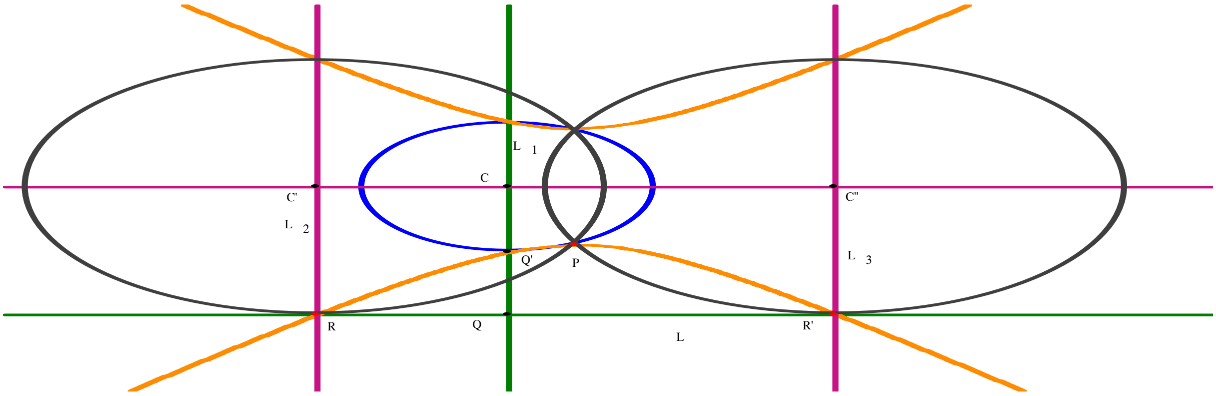

Figure 10. Drawing the tangent circles to line L

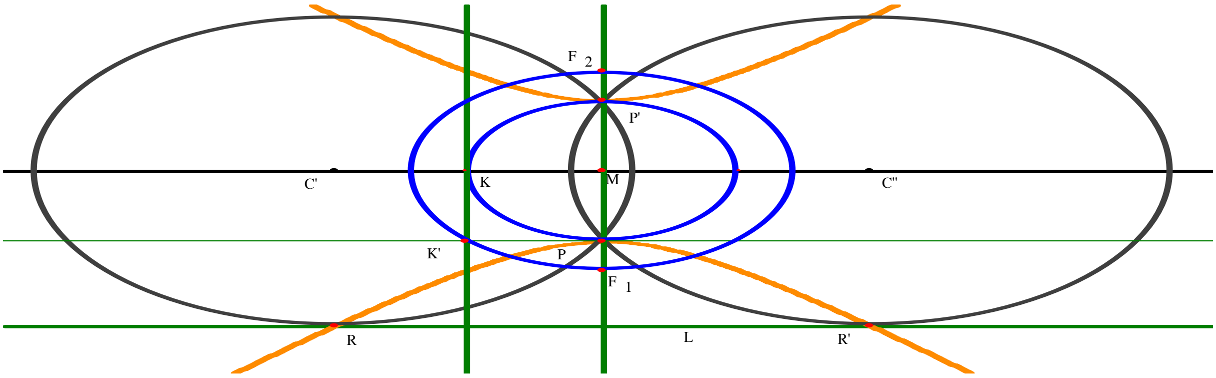

Figure 11. The locus satisfies the definition of hyperbola

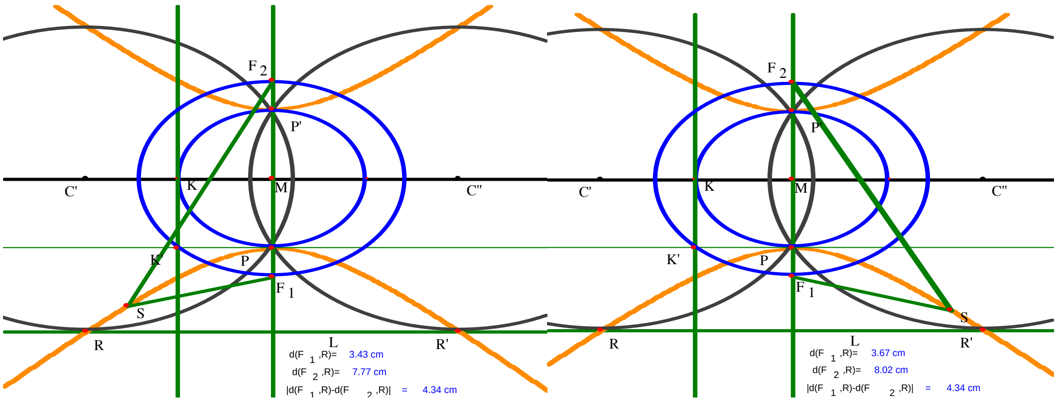

Figure 12. For distinct positions of point S on the locus the definition of hyperbola is satisfied

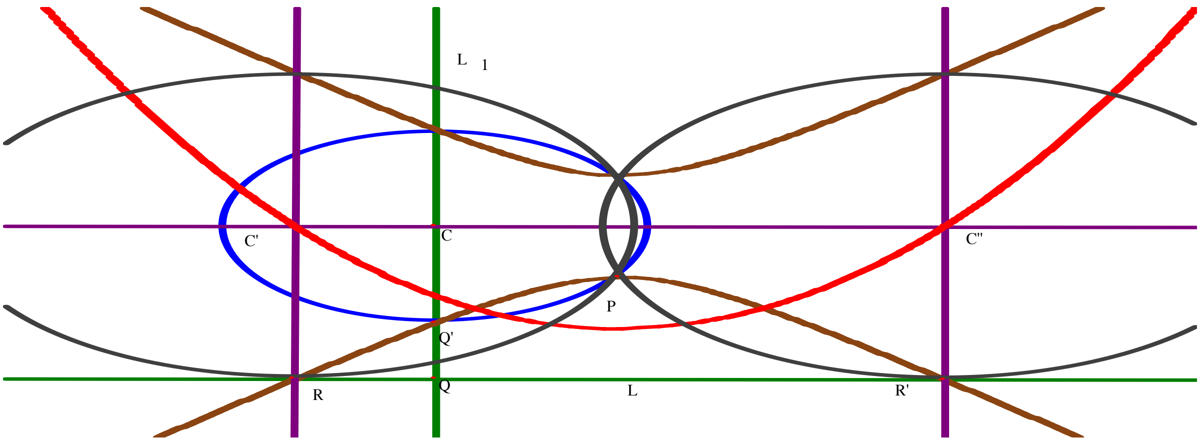

Figure 13. The path left by points C' and C" when point C is moved along line L1 is a parabola

through point P (Figure 14). Thus, the tangent circle to line L that passes though point P is the circle that inscribes triangle PQR. The center of the tangent circle is then the intersection of the perpendicular bisectors of the sides of triangle PQR (Figure 14a).

It is observed that when point Q is moved along line L, a family of triangles and circles tangents to L appeared. At what position of Q does triangle PQR become equilateral? To respond to this question, we identify the intersection of the heights of sides PQ and QR (orthocenter) and observe that the loci of point C and O when point Q is moved along line L are two parabolas (Figure 15). Thus, at the intersection point of those parabolas is the position where points C and O (the circumcenter and orthocenter) coincided. There the triangle PQR is equilateral (Figure, 15a).

A pattern approach. Yet, another approach to draw tangent circles to line L that pass through point P involves a construction pattern. The pattern is based on constructing initially a perpendicular line to L that passes through point P. This perpendicular line intersects line L at point Q. Thus, the midpoint of segment PQ is the center of the tangent circle to line L that passes through point P (Figure 16).

The grid on Figure 16a was constructed by drawing a perpendicular line to line PQ that passes through point C. This perpendicular intersects the circle with center C at point R. From point R a perpendicular to line L is drawn. By using the command *Reflection*, all the other lines are constructed. It is also observed that if line L and line PQ are the axis of a coordinates system, then the centers of the tangent circles to line L are given as C(0, 1); D(2, 2); E(4, 5), etc. This sequence leads us to observe that sequence of the first entries (0, 2, 4, 6, etc.) has constant difference of 2; while the second difference of the second entries (1, 2, 5, 10, 17, 26, etc.) was also of 2. Here, if segment QC is taken as one unit, then the equation of the curve that passes through the centers of the tangent circle to L is $y = \frac{x^2}{4} + 1$ which represents a parabola equation.

It is observed that a simple task that involves drawing a tangent circle brings into the discussion not only the use of diverse mathematical concepts but also the application of distinct mathematical processes and problem solving strategies to formulate and pursue relevant questions.

An algebraic approach. The initial task can also be represented algebraically. A heuristic here will be to set the Cartesian system in such a way that the algebraic calculations can be made easy. Thus, we choose the x-axis as the line L and the y-axis to be the perpendicular line to L on which the centre of the tangent circle is located. On Figure 17 line L is the x-axis and the perpendicular line to x-axis that passes through point Q is the y-axis, point P has coordinates (x_1, y_1) and M is

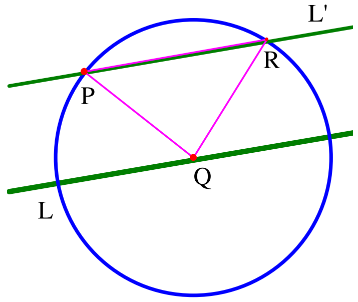

Figure 14. Drawing a circle with centre at Q and radius QP

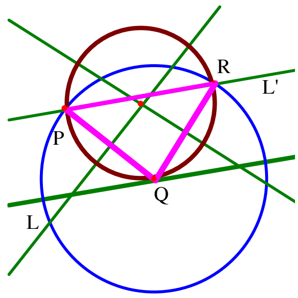

Figure 14a. The intersection point of the perpendicular bisectors of segment PQ and QR is the center of the circle tangent to L

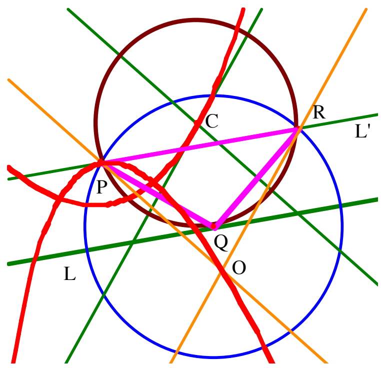

Figure 15. When triangle PQR does become equilateral?

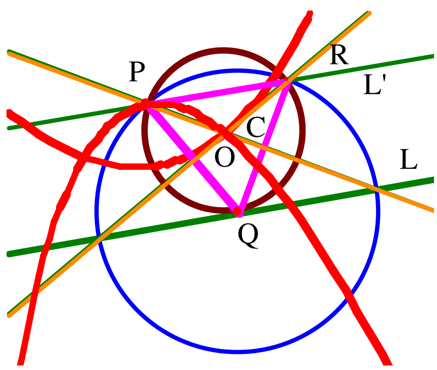

Figure 15a. Triangle PQR is equilateral when the circumcenter and orthocentre get intersected

the midpoint of segment QP and has coordinates $\left(\frac{x_1}{2},\frac{y_1}{2}\right)$.

Based on this information, the slope of line QP is $m=\frac{y_1}{x_1}$ and the slope of the perpendicular bisector of segment PQ is $m_1=-\frac{x_1}{y_1}$. Therefore, the equation of the perpendicular bisector of PQ can be expressed as: $y-\frac{y_1}{2}=-\frac{x_1}{y_1}(x-\frac{x_1}{2})$, and we take $x = 0$ then $y=\frac{x_1^2}{2y_1}+\frac{y_1}{2}$. Thus, the centre of the tangent circle will be $\left(0,\frac{x_1^2}{2y_1}+\frac{y_1}{2}\right)$. A simpler approach can also be applied by recognizing that the point P needs to satisfy that d(Q, C) = d(C, P). That is, if C(y, 0) and P (x_1 y_1) then we have that $y=\sqrt{x_1^2+(y_1-y)^2}$ which implies that $y=\frac{x_1^2}{2y_1}+\frac{y_1}{2}$

Commentary: The dynamic representation of mathematical objects or problems is a heuristic that can guide the problem solver in the search of mathematical relations. The partial goal of drawing a circle with its center on a perpendicular to line L and radius the distance from the center to the given point (P) becomes relevant to visualize and examine properties of the solution. Based on those properties the tangent circle was constructed. In addition, the dynamic configuration is used to relate the problem to other mathematical objects (parabola and hyperbola). The problem solver must show and justify that the objects that are visualized through the loci satisfy the corresponding definitions. To accomplish this task, an important heuristic that gives an empirical verification is to measure distances between objects in order to observe invariants when particular objects are moved along specific paths. In this case, the process of measuring and comparing distances was a relevant strategy to verify empirically the definition of both conic sections. In addition, the dynamic representation of the tasks becomes a departure point to identify and examine a set of relations that emerge as a result of moving mathematical objects within the same configuration. The use of the software not only can help teachers and students identify important mathematical relations; but also to provide a route to support or prove them. In this task, the route involves ways to first visualize a relation, later to verify it empirically and finally to use geometric and algebraic arguments to prove it.

Schoenfeld (1985; 1992) reports that in general students tend to copy or redraw figures that appear in the statement of the problems and use them to make conjectures or to identify relations. With the use of paper and pencil the sketches or representations drawn not necessarily capture the objects' precision and

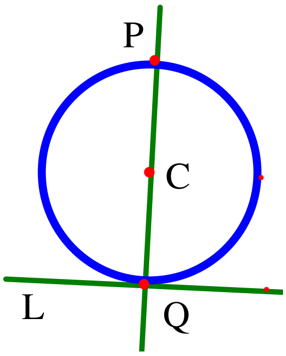

Figure 16. Drawing a tangent circle to line L that passes through point P

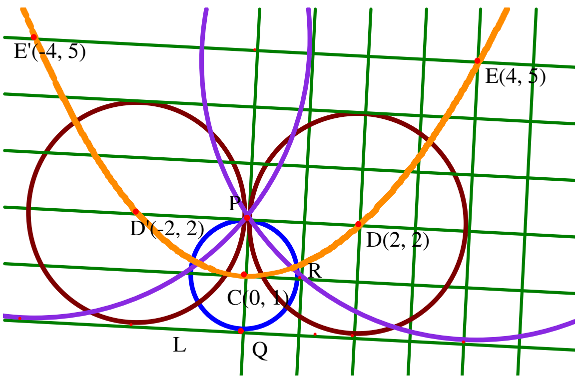

Figure 16a. Drawing other tangent circles based on a symmetry pattern of the initial construction

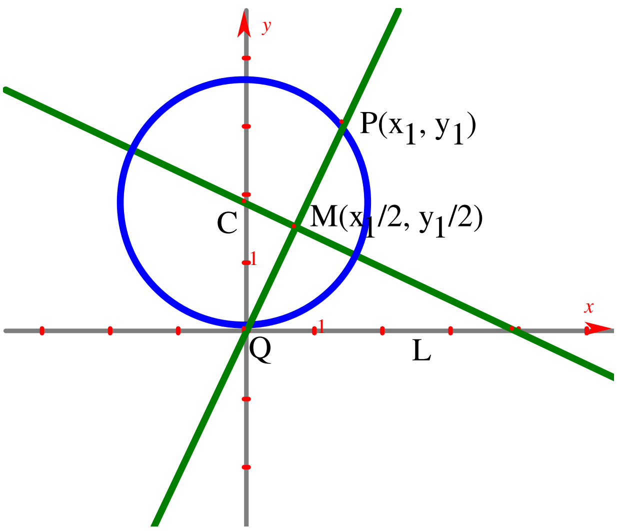

Figure 17. Approaching the task algebraically

students often assume or perceive false conjectures or statements. However, the use of the software allows us to accurately represent and draw mathematical objects. Also these representations can facilitate the process of looking for mathematical relations and the visual exploration of their plausibility. In addition, with the use of the software it is easy to change size or positions of the original objects to explore whether invariants or conjectures are maintained for a family of those objects. For example, in the task, the position of point P can change and the way to construct and generate the conic sections is preserved. As a consequence, with the use of computational tools, the problem solver or students might develop a method of thinking of how to represent and approach a family of isomorphic problems.

CONCLUDING REMARKS

I use the construct "hypothetical instructional trajectory" to identify and examine potential routes that teachers can initially trace with the use of computational technology. How can an instructional route be constructed? Who can participate in such construction? And what is the role of the use of computational tools in constructing them? The initial task is used as a departure point to construct a dynamic configuration that leads us to constantly formulate and explore questions from diverse angles or perspectives. In this process, there is an attempt to identify crucial themes and ideas that teachers and researchers could relate and consider in their practice and research agendas:

Inquiry Process. There is ample evidence that the use of the tool offers the problem solver the opportunity of becoming engaged into an inquiry process that guides him/her to look for mathematical relations and means to support them (Santos-Trigo, et, al., 2007). Thus, learning mathematics and problem solving are processes in which students constantly pose or formulate questions to identify, examine, and support conjectures or mathematical relations. In the task discussed, there is no initial given question or problem to solve, instead the problem solver begins by assembling or putting together a geometric configuration which becomes the source to be engaged into an inquiry process in order to develop or reconstruct a set of mathematical relations. The use of the tools provides, in general, instantaneous response to the problem solver' s queries and as a consequence it can foster the discussion of results within the learning community. Thus, such community should not only value or pay attention to the emerging relations or results; but also to the search for arguments to support them.

Heuristic Strategies. An important heuristic associated with the use of the tools is to think of the tasks or problems in terms of mathematical properties. If the problem solver is to represent the problem dynamically it is necessary to identify relevant mathematical properties to guide the construction of that representation. What does it mean to draw a circle that is tangent to a given line? Is there a relation between the tangency point and the center of that circle? These are examples of questions that helped problem solvers to represent the task with the use of the tools. In addition, other heuristics such as identifying and exploring partial goals, assuming the task solved, or finding loci of particular objects are easy to implement with the use of the tools and are useful to explore and generate mathematical relations.

The Use of Various Computational Tools. The efficient use of a tool to represent and explore mathematical problems is a process in which the problem solver identifies and recognizes the power and advantages to think of a given problem in terms of the software commands. The use of the tool also shapes the way students or problem solvers think of the problem (Kaput, Lesh, Hegedus, 2007). Since each tool offers particular advantages to deal with each problem, then it is relevant to utilize more than one tool to enhance the teachers or students' ways to approach and solve problems (Santos-Trigo, et, al., 2006). For example, the use of dynamic software facilitates the construction of dynamic representations of objects while the use of hand-calculator offers certain advantages to represent and deal with the problem algebraically. Thus, it is important for the problem solver to utilize various computational tools to search for and complement different approaches to the problem.

Curriculum Fundamentals. The task presented in this paper was discussed during two problem-solving sessions of three hours each. Some of the approaches emerged during the development of the session; but other ideas and task extensions emerged out of the sessions' work where the participants continued commenting, exchanging, and testing other task ideas. Here, the participants pointed out that to promote their students work along the lines that appeared while approaching the task, it is necessary to reduce the curriculum contents that teachers are asked to cover in their regular courses. In this perspective, the participants suggested that the contents to be studied need to be structured and organized around fundamental mathematical ideas and problem solving processes that are relevant for students to construct and develop in depth (NCTM, 2000). It is also recognized that the use of the tools can help students to foster strategies and ways to formulate and pursue questions and eventually identify a set of mathematical relations.

Teachers' Use of the Tools and Mathematical Knowledge. How should in-service teachers incorporate the use of computational tools in their instructional practices? There is evidence that the construction of potential instructional trajectories is a problem solving activity in which the teachers have the opportunity of recognizing the potentials and limitations associated with the use of the tools to represent and explore mathematical relations (Santos-Trigo, 2006). In addition, the use of the tools seems to promote the discussion of mathematical contents in terms of identifying potential routes for students to comprehend and apply the acquired knowledge. For example, in the initial task, the appearance of the conic sections while drawing a tangent circle not only promoted the discussion of the properties of those figures; but also the consideration of instructional paths in which the study of the conic sections could be structured or organized for students. Thus, a clear hypothetical route that emerges while approaching the task might focus on guiding the students to initially construct a dynamic representation of the problem to comprehend and make sense of relevant information associated with the problem situation. Later, the configuration becomes a source or instance to identify visually a set of relations or conjectures whose plausibility and validity can be validated empirically (quantification of those relations). Further, the use of tools not only facilitates the visualization and exploration of mathematical relations, but also provides important information to represent and analyze the relations in terms of geometric properties or algebraically.

The use of computational tools offers teachers the possibility of guiding their students to develop an inquiry approach to interact with mathematical ideas or problems. In this process, problem solving and constructing mathematical ideas require more than responding particular questions, they demand that the students become engaged into a reflection activity to search for multiples ways to solve problems or to explain mathematical ideas, and to look for possible connections and means to communicate results.

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