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Analyzing Mathematics Textbooks through a Constructive-Empirical Perspective on Abstraction: The Case of Pythagoras' Theorem

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This study aims at analyzing how Pythagoras' theorem is handled in three versions of Taiwanese textbooks using a conceptual framework of a constructive-empirical perspective on abstraction, which comprises three key attributes: the generality of the object, the connectivity of the subject and the functionality of diagrams as the focused semiotic tool. The results show that Taiwanese textbooks intended to develop the object through the variation among the levels of generality as well as between more and less familiar connections. The procedural diagrams were less provided than the conceptual diagrams, and the diagrams required more processing were less provided than the diagrams required less processing. Nonetheless, there were differences among the three textbooks. On the basis of similarities and differences, we discuss issues related to the learning and teaching of mathematics.

Keywords: abstraction; constructive; empirical; Pythagoras; textbook

INTRODUCTION

In order to understand how textbook authors design mathematics textbooks, , there are some studies analyzing problem types, including routine problems versus non-routine, open-ended versus close-ended, traditional versus non-traditional, and application versus non-application problems (Zhu & Fan, 2006), problem-solving procedures including four problem-solving stages and heuristics such as 'acting it out', 'looking for a pattern', etc. (Fan & Zhu, 2007), procedural complexity denoting the number of steps in a common solution method and including three different levels of complexity (Vincent & Stacey, 2008), cognitive demand including memorization, procedures without connections, procedure with connections, and doing mathematics (Bayazit, 2013; Jones & Tarr, 2007).

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There are other studies focusing on introduction or description of mathematical concepts and reasoning in textbooks. That is, how some specific concept is introduced in textbooks or how opportunities to learn mathematical reasoning are provided. For instance, the concept of average is presented as a computational algorithm or as a representative of a data set (Cai, Lo, & Watanabe, 2002), the concept of fractions is treated as part-whole, ratio, operator, quotient, or measure (Charalambous, Delaney, Hsu & Mesa, 2010), and reasoning-and-proving is classified as identifying a pattern, making a conjecture, providing a proof and providing a non-proof argument (Stylianides, 2009). These analyses have gone beyond telling us the challenging levels of problems, the different strategies or purposes of solving problems in textbooks.

Nonetheless, these studies focused more on task analyses, and less on detailed accounts of the process of learning one mathematical topic from all information of the topic in textbooks. Many researchers indicate that mathematics textbooks not only constitute an essential part of the curriculum but play a significant role in teaching of mathematics (e.g. Ball & Cohen, 1996; Xenofontos & Papadopoulos, 2015). Drawing on the lack of detailed accounts of how the intended process of leaning mathematics is arranged by textbooks as well as the importance of textbooks for affecting students' learning and teachers' curriculum development, refining our understanding of mathematics textbooks should go on.

State of the literature

- Most of previous studies on textbook analysis do not focus on detailed accounts of the progression of learning to analyze mathematics textbooks.
- As abstraction has been recognized as a key adaptive mechanism of human cognition and an essential process in the learning of mathematics, to understand how mathematics textbooks manage abstraction is crucial.
- Skemp's and Piaget's view of abstraction can be recognized as a constructive-empirical perspective, which attends to the relationship between the specific experience from which something is to be abstracted and the abstracted generality and assumes that the latter is more abstract than the former.

Contribution of this paper to the literature

- This study is the first to analyze both description and work-examples in textbooks from the perspective of abstraction.
- The three attributes - generality of the object, the connectivity of the subject and the functionality of diagrams are substantiated by comparing three versions of mathematics textbooks.
- The comparison shows that different approaches are exploited to develop the subject's abstraction of Pythagoras' theorem.

A research forum held at the 26th international conference of PME discussed two different contexts for abstraction: in mathematics and in mathematics learning (Boero et al., 2002). At the forum, Gray and Tall, Hershkowitz, Schwarz, and Dreyfus, and Gravemeijer, presented three models of abstraction. Gray and Tall (2007) held a view of a natural process of mental compression as abstraction. Hershkowitz, Schwarz, and Dreyfus (2001) considered mathematical abstraction as an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure, and of elaborating a model for the genesis of abstraction. Gravemeier (2007) focused on how to encourage the use of abstraction in instruction and viewed abstraction as construction within which students construe mathematical knowledge grounded in their earlier informal experience. In addition to Gravemeijer's (2007) study, White and Mitchelmore (2010) proposed the model of teaching for abstraction, which has been applied to many topics of elementary school mathematics—decimals, angles, and ratios. Both cognitive and didactical research provide us insightful understanding of abstraction for learning and teaching. However, these ideas have not been used to shed light on the intended curriculum. As abstraction has been recognized as a key adaptive mechanism of human cognition and an essential process in the learning of mathematics, this study aims at understanding how mathematics textbooks manage abstraction.

In reporting this study, we begin with a conceptual framework used for the analysis of the intended abstraction behind mathematics textbooks. Then, with explanation of the data and methodology, we describe the results from the textbook analysis which shows the patterns of abstraction in three different versions of Taiwanese secondary textbooks in relation to Pythagoras' theorem. The topic is selected due to three reasons. First, Pythagoras' theorem can connect between abstract geometric ideas and concrete perceptual images, which are fused as figural concepts (Fischbein, 1993). This connection provides us with greater opportunity to observe intended abstraction, including generalization (ref. Radford, 2003), behind textbooks. Second, this theorem can be learnt through students' movements from contextual, pre-symbolic to symbolic generalizations (see Moutsios-Rentzos, Spyrou & Peteinara, 2014), which is part of abstraction. Third, the topic is important in the history of mathematics (Maor, 2007), and learning how to prove Pythagoras' theorem can be of good use in nurturing students' creativity in secondary schools (Tam and Wang, 2012).

Taiwanese students' performance in mathematics literacy is internationally ranked at the top level (OECD, 2009; 2012). Besides, around 36% of ninth graders could construct a correct proof which required combining several geometric arguments (Heinze, Cheng & Yang, 2004). Textbooks are one important source for Taiwanese students to learn mathematics when 92 % of Taiwanese secondary teachers reported that they used textbooks as basis for teaching mathematics (Mullis, Martin, Foy & Arora, 2012). Mathematics textbooks are developed by private publishers who invite university professors and school teachers to collaboratively write textbooks. Then, all textbooks must pass through a reviewing process by university professors appointed by National Academy for Educational Research in Taiwan. Teachers can decide which textbook series is going to be used, but the same textbook is used at the same school. In recognition of students' good performance, the influence of textbooks on classroom practice as well as both developing and reviewing process of textbooks, this study pays attention to Taiwanese mathematics textbooks.

CONCEPTUAL FRAMEWORK

Before conceptualizing the constructive-empirical perspective on abstraction, we emphasize the function of textbooks in transposing knowledge with the assumption that "Bodies of knowledge are, with a few exceptions, not designed to be taught, but to be used." (Chevallard, 1988, p. 6). No mathematics presented in textbooks is the original invention of some mathematicians, and textbooks need to transpose knowledge from being used in development and practice to being taught and learnt in class. Consequently, the textbooks are a didactic transposition and reflect how knowledge is learned (Kang & Kilpatrick, 1992). In this study, we treats textbooks as playing an active role in taking the object, the subject, and the semiotic tool into account, and then explored how mathematics textbooks transpose mathematical knowledge.

According to Skemp (1986), abstraction is a mental activity through which humans become aware of similarities in their experiences, which are classified to create classes of experiences against which new experiences are compared and assimilated. Skemp describes these classes of experience as concepts, which fall into two forms, primary and secondary. The former forms of concepts are derived directly from experience, whereas the latter are abstracted from the former. Mitchelmore and White (2007) call this view of abstraction an *empirical abstraction* which is literally one of three forms of abstraction described by Piaget but more profound than Piaget's notion of empirical abstraction. Empirical abstraction is referred to the identification of the superficial properties of physical objects by Piaget, whereas empirical abstraction is referred to the identification of underlying structures of general experience by Skemp.

In addition to empirical abstraction, much of the work on abstraction in the development of mathematical understanding also draws on Piaget's (1985) description of two other forms of abstraction: pseudo-empirical and reflective abstraction. Empirical abstraction focuses on objects and their properties, whereas both pseudo-empirical and reflective abstraction focuses on the actions and their relations. However, there is qualitative difference in what is acted. Pseudo-empirical abstraction teases out actions on external objects, and reflective abstraction teases out actions on internal objects. In particular, reflective abstraction is "drawn from the general coordination of actions or of operations" (Piaget, 1980, pp. 89-97). In addition to coordination, Piaget distinguishes three other processes: interiorization, encapsulation and generalization. Furthermore, Dubinsky (1991) identifies reversing as the fifth process of reflective abstraction and extends Piaget's theory to analyze the development of concepts in advanced mathematics. Those processes involve both inductive (from the specific to the general) and deductive reasoning (from a hypothesis to generate logically necessary inference). It implies that abstraction is an important construct for analyzing the learning of any mathematical content at any level. For instance, Simon, Tzur, Heinz, and Kinzel (2004) elaborated Piaget's reflective abstraction in order to describe a basic mechanism for pedagogical theory. Silverman and Thompson (2008) applied it not only to students' learning but also teachers' learning. Herein, we investigate through what abstraction processes are arranged in textbooks.

Yang (2013) recognizes Skemp's and Piaget's view of abstraction as a constructive-empirical perspective, which attends to the relationship between the abstracted generality and the specific experience from which something is to be abstracted, and assumes that the former is more abstract than the latter. From this perspective, the relationship between what is to abstract and what is abstracted can be discriminated by the relative degree of abstractness of the object, and it can be perceived or constructed by the subject with the help of the semiotic tool (in the sense of Peirce's representamen). That is, the subject (the learner) can develop the understanding of the object (the mathematical content) from physically or mentally interacting with semiotic tools. Moreover, three key components of abstraction are identified: the object, the subject and the semiotic tool, which are derived from not only the constructive-empirical but also the dialectic perspectives on abstraction (see Yang, 2013). However, we first analyze how Taiwanese mathematics textbooks arrange abstraction based on the constructive-empirical perspective due to the limit of space, and refer to the essential attribute of each component – the generality of the object, the connectivity of the subject and the multiplicity of the semiotic tool, to formulate research questions for textbook analyses.

As the generality of the object is related to the connotation and extent of the object and assumed that the relative degree of abstractness of the object can be distinguished, we can investigate *how the* sequence *of less and more generality is arranged by textbooks*. As the connectivity of the subject is concerned with the connection made to the subject's experience and assumed that the subject can make unfamiliar ideas familiar based on the connection to previous experience (Hazzan, 1999), we can investigate *how the sequence of less and more connectivity is arranged by textbooks*. As for the semiotic tool, which includes words, numbers, symbols, figures, graphs, pictures, tables, real scripts, manipulative instruments, and so on (Goldin & Kaput, 1996; Lesh, Post, & Behr, 1987), we mainly focus on figures, graphs, pictures and tables, which are all viewed as diagrams, due to that the geometric topic is much related to diagrams (Jones & Fujita, 2013). Hence, we shift from the multiplicity of the semiotic tool to the functionality of diagrams, and investigate *how textbooks arrange diagrams to represent the object*.

METHOD

Content analysis

This study used a content analysis methodology to analyze the selected topic, Pythagoras' theorem. In order to define the analysis units, we first classified texts by categories of text goals which include: (1) to introduce something related to one object [I], (2) to explore one object [Exp], (3) to describe one object [D], (4) to explain or elaborate one object [EL], (5) to show worked examples related to one object [W], (6) to provide questions for practice [P], and (7) to talk about one object [T]. Most of the previous studies on mathematics textbook analysis focused on the fifth and sixth text goals which are realized through problems. However, we focused on the whole text as to the topic in relation to Pythagoras' theorem. In general, one paragraph of the same text goal or one problem was identified as one analysis unit. It should be noted that several sub-questions of the same problem context is treated as an analysis unit. Then, each analysis unit was coded regarding each attribute. The last step was to enter the data into an Excel file for statistical analysis. The classification scheme of each attribute would be elaborated after Samples section.

Samples

This study compares three different mathematics textbooks: Kang-Hsuan (Hung, 2013), Han-Lin (Chang, 2011) and Nani (Tso, 2011). The three versions of mathematics textbooks for the junior high school follow the same national curriculum and are published by three main textbook publishers in Taiwan. In Kang-Hsuan and Han-Lin, Pythagoras' theorem is used as a title of Section 2-3. In Nani, Gougu (Leg-leg) theorem[1] is used as a title of Section 2-3. This topic covers 18 pages in each of the three textbooks. One research assistant identified the analysis units, and the author checks them again. There are 53, 60 and 50 analysis units respectively in Kang-Hsan, Han-Lin and Nani. The topic is assumed to teach in 5 to 6 sessions (45 minutes per session).

Development of a coding scheme

In order to develop a coding scheme that allow for a practical analysis of abstraction, we not only refer to Yang's (2013) framework but also review some literature for formulating the operational definitions of the three attributes of abstraction – the generality of the object, the connectivity of the subject and the functionality of diagrams in the section.

How can the levels of generality be distinguished? According to Gray and Tall (2007), "perceptions of and actions on objects are reflected upon, producing an increasingly sophisticated mental framework (p. 29)", which leads to the formulation of three mathematical worlds. Each world entails its own particular way of developing greater and greater sophistication from a lower to a higher abstract level of warrants for mathematical truth (Tall, 2004). The substantial content of the generality can be derived from comparing sub-objects presented in textbooks regarding one topic. For instance, the generality of Pythagoras' theorem can increase from one right-angled triangle as an example without the focus on the logic of the theorem (if p then q), one right-angled triangle as a generic example embodying the logic of the theorem, to a conditional statement for any right-angled triangle and its proof. Accordingly, each analysis unit can be classified into specific, generic or formal level. For the formal level, we further distinguish whether or not proof is presented due to proof is a special genre of mathematics and difficult for students to learn.

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[1] Pythagoras' theorem was known as the "Gougu Theorem" (勾股定理) in China during the Han Dynasty (202 BC to 220 AD) (see http://en.wikipedia.org/wiki/Pythagorean_theorem)

White and Mitchelmore (2010) explicitly express that "without a strong link between fundamental mathematical concepts and students' experience, any abstraction approach is likely to falter (p. 209)." However, the textbooks rather than students' learning or thinking are analyzed in this study. How can the degree of connectivity in textbooks be evaluated? We focus on to what extent the assumed subject's prior knowledge is connected, and the connections are classified as either less or more familiar. The more familiar the subject is and the more connections the subject has formed to it, the more concrete the object becomes. Hence, the connectivity is classified as either less or more familiarity. The former form focuses on the modification of the subject's experience to reconstruct meaning while the later implies the addition of familiar information to the subject's experience. For instance, using previous knowledge to calculate the areas of the squares generated from each side of one specific right-angled triangle is more familiar than to generalize or reason the relationship among the three sides of one specific right-angled triangle. Applying the Pythagoras' theorem to one typical right-angled triangle is more familiar than to one non-typical right-angled triangle.

Lastly, the diagrammatic representation can be distinguished as two categories: procedural diagrams that represent some relations in a sequence of diagrams and conceptual diagrams that represent one concept or some relations in one diagram. For instance, textbooks may use several sequential figures or one complicated figure to prove Pythagoras' theorem. In order to distinguish the requirement of diagrammatical processing, we further classify conceptual and procedural diagrams into either less or more requirement to deduce new information from diagrams for further reasoning. When analyzing the analysis units, one more category of functionality of diagrams is added: diagrams for constructing the object. All of the categories are noted as functionality of diagrams.

In sum, table 1 shows the three attributes and their classifications. As for generality, the specific level refers to specific examples without generalization. The generic level refers to a generalization from several specific examples and a verbal description of the old object which had been taught in the textbook. The formal level refers to a verbal description of a new object which had not been taught in the textbook as well as an algebraic or symbolic description of an object. As for connectivity, the content which looks typical or not difficult based on the assumed subject's previous knowledge is classified as more familiar, whereas the content which looks new or complicated, e.g. generating a new object and introducing meta-

Table 1. Three attributes of abstraction and their classifications

Attribute	Description	Operational Classifications
Generality	the connotation and the extent of objects	G0: specific level G1: generic level G2: formal level G2-1: without proof, G2-2: with proof
Connectivity	connections of subjects' experience with objects	C0: no connection C1: more familiar C2: less familiar
Functionality	diagrams used for representing concepts or procedures	F0: no diagram F1: conceptual diagrams F1-1: less processing, F1-2: more processing F2: procedural diagrams F2-1: less processing, F2-2: more processing F3: diagrams for presenting the object

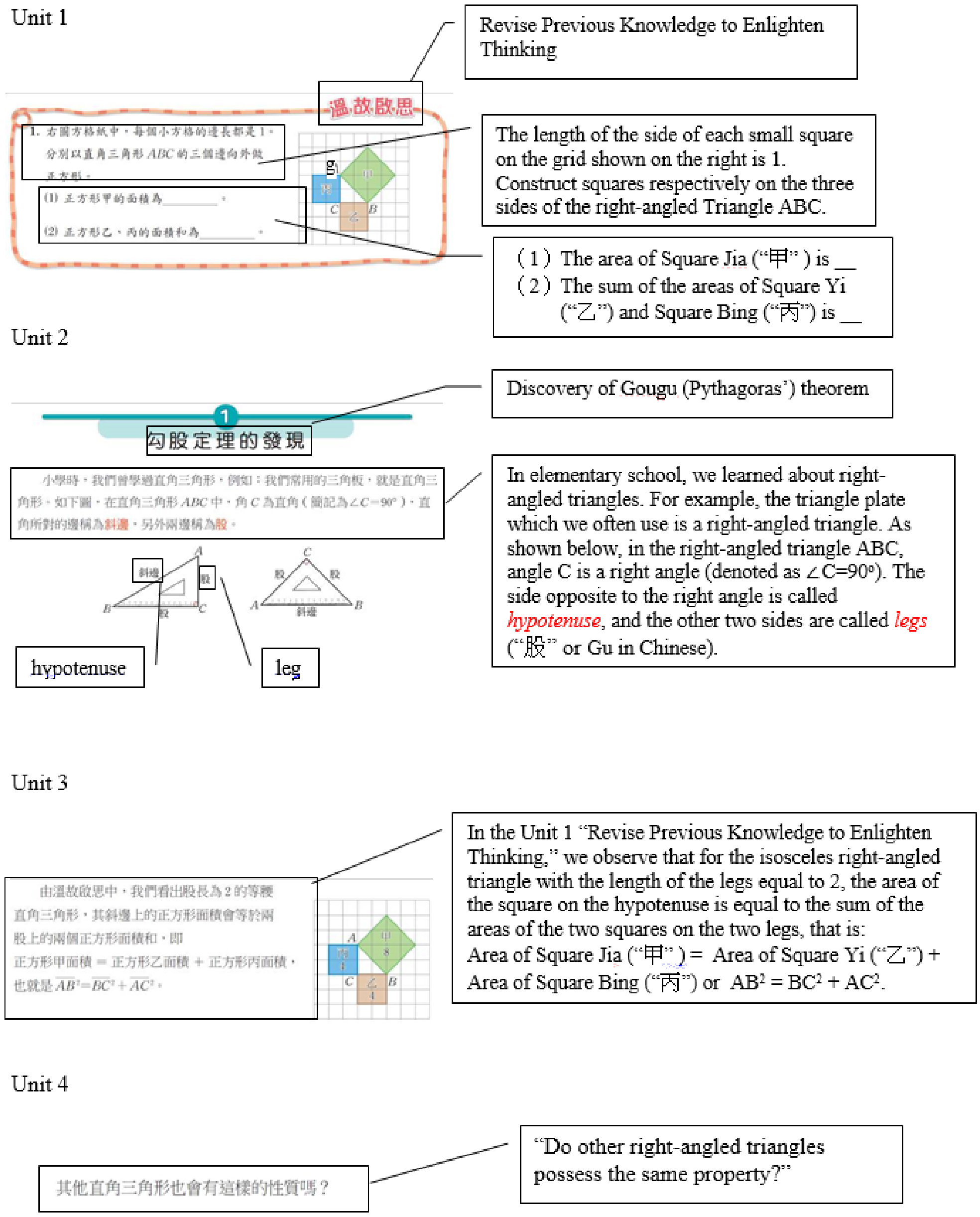

Figure 1. First four analysis units in Nani (Adapted from Yang, 2013)

knowledge of the object, is classified as less familiar. As for functionality, one analysis unit which has one diagram or several independent diagrams is classified as conceptual diagrams required either less or more processing, except that it is used for the subject to present the object. One analysis unit which has a sequence of diagrams is classified as procedural diagrams required either less or more processing.

Exemplary coding procedure

Taking the first page of this topic in the Taiwanese textbook, for example, we identified four analysis units, Units 1 to 4 (U1 to U4). As shown in Figure 1, first of all, U1 provides practice for students to revise their familiar and related concept of square areas. Then, U2 introduces mathematical terms related to sides of right-angled triangles. Next, U3 describes the relationship among the three square areas, which is presented with one specific example. At the end of this page, U4 explores the generality of this relationship by asking one question. Accordingly, the first four analysis units were identified as [P], [I], [D] and [Exp] respectively.

When analyzing these analysis units from the generality of the object, we pay attention to the levels of generality. The common and main object in the analysis units is right-angled triangles. U1 and U3 are coded as G0 (specific level) due to their reference to a specific right-angled triangle without generalization. Although algebraic symbols appear in U3, they are just used to represent the side length of the specific triangle. U2 and U4 are both coded as G1 (generic level) due to their reference to two right-angled triangles for introducing the terms "hypotenuse" as well as "leg" and other right-angled triangles for generalizing the relationship respectively.

When analyzing these analysis units from the connectivity of the subject, we pay attention to the degree of familiarity. We consider the subject from the point of view of a 'typical' student who has previous knowledge in mathematics textbooks. U1 is coded as C1 (more familiar) because it asks the subject to practice using previous knowledge. U2, U3, and U4 are all coded as C2 (less familiar) because they were about the development of the new sub-objects. U2 is used to know the new terms, U3 is set for observing the relationship among the three squares and the length of three sides, and U4 askes the subject to generalize the relationship to other right-angled triangles.

When analyzing these analysis units from the functionality of diagrams, we pay attention to both the types of diagrams and the demand of processing. The diagrams in U1 and U3 are static figures on the grid and thus belong to the conceptual type. Both diagrams are used to count or figure out the areas of the squares and thus require less processing of diagrams based on the assumed subject's previous knowledge, so coded as F1-1. The diagrams in U2 are two triangle plates and thus belong to the conceptual type. They are used to introduce new mathematical terms and thus require less processing of diagrams, so coded as F1-1. U4 was coded as F0 due to no diagram.

We give the diagrams in figure 2 and 3 to show the difference between F1-1 and F1-2. The two diagrams are similar; however, they were coded as F1-1 and F1-2 respectively due to that the diagram in figure 2 provided all of the required side length, but, in figure 3 required the subject to reason the side length of one right-angled triangle. The majority of analysis units with a sequence of diagrams were classified as F2-1, except that the sequential diagrams, as shown in figure 4 coded as F2-2, due to the requirement of processing figures to prove a theorem.

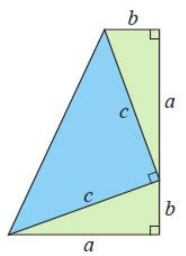

Figure 2. The diagram in the 52nd analysis unit in Han-Lin

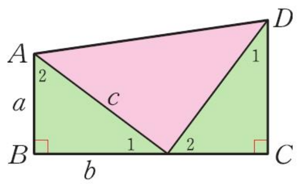

Figure 3. The diagram in the last analysis unit in Nani

As one part of the work in the Third International Mathematics and Science Study, Valverde, Bianchi, Wolfe, Schmidt & Houang (2002) proposed three aspects: content (number, measurement, geometry,…), performance expect (knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating) and perspective (attitudes, careers, participation, increasing interest, and habits of mind), to form a framework for comparing textbooks of many countries. This framework has examined the textbooks as a whole, focusing on the organization of the content across textbooks, and pursued both horizontal and vertical approaches to textbook analysis (Charalambous, Delaney, Hsu, 2010). Nonetheless, how textbooks treat the generality of the object, the connectivity of the subject, the functionality of diagrams and the detail of their sequence within each analysis unit or the change of the details from unit to unit is still under question. While textbook analysis is deepened into abstraction, our analytical framework serves a function of analysing abstraction in textbooks without student data. We agree that textbooks can not really represent the implemented curriculum which relies on interactions between teachers' and students' use textbooks. Nonetheless, this framework puts textbooks in an active role to investigate interactions of teachers' and students' use of textbooks.

Reliability of coding

For testing the reliability of the coding, two research assistants were trained to apply the coding scheme in content analysis of the two textbooks. The first six analysis units were used to help the two coders understand the codes. The other analysis units were double-coded. As a measure of reliability of the coding, the percentage agreement between the coders was calculated by dividing the number of agreements by the number of agreements plus disagreements. As to the generality, the connectivity, and the functionality, the percentage agreements of coders were 88%, 80% and 93%. The lower percentage agreement of coders for the connectivity was due primarily to that one coder limited previous knowledge to the pervious topics which has been taught whereas the other coder limited it to both previous topics and the content of this topic prior to the analysis unit. All disagreements were resolved by discussing among the coders and the author by discussion to achieve consensus.

RESULTS

Before reporting the results of the analysis on each attribute of abstraction in turn, we first show the frequencies of text goals in the three textbooks.

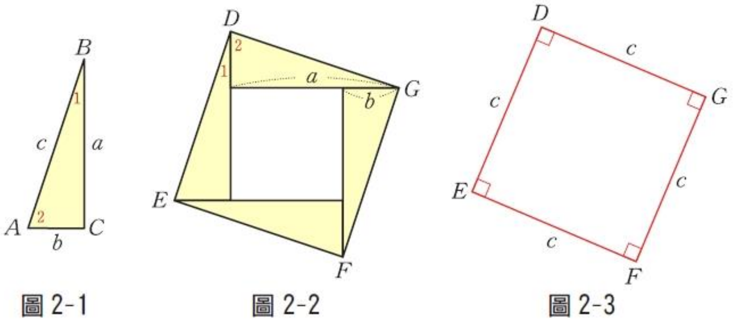

Figure 4. The diagram in the seventh analysis unit in Nani

Text goals

Table 2 showed that the three textbooks did not have similar percentages of the analysis units regarding the text goals of exploring one object [EXP] and showing worked examples related to one object [W]. The text goals of 13%, 8% and 6% of the analysis units were identified as [EXP] in Kang-Hsuan, Han-Lin and Nani. The text goals of 64%, 70%, 68% of the analysis units were identified as [W] and practice [P] in Kang-Hsuan, Han-Lin and Nani. Accordingly, Kang-Hsuan more likely adopted an exploratory approach to developing a new object, e.g. applying this theorem to construct a segment of irrational length and the distance formula. Han-Lin and Nani more likely adopt a worked-example approach.

The three textbooks arranged the [EXP] text goal differently. Kang-Hsuan arranged four units of [EXP] in the beginning and three units of [EXP] interweaving in the middle, and Han-Lin arranged three units of [EXP] in the beginning, one in the middle and one in the end. In contrast, Nani just arranged three units of [EXP] in the beginning. Kang-Hsuan adopted an exploratory approach for finding and proving the Pythagoras' theorem, whereas Han-Lin and Nani adopted this approach just for finding the theorem. Nonetheless, the three textbooks commonly arranged [EXP] in the beginning, and this pattern has been also found in the Japanese textbook (Jones & Fujita, 2013). Moreover, another common patterns among the three Taiwanese textbooks were that one worked example was followed by one question for practice in order to develop the subject's understanding of the object, and a sequence of questions were provided for practice in the end. The high percentage of [P] (all above 40%) reflects one underpinning pedagogy of using mathematics textbooks as a source of questions for practice.

Generality

Table 3 showed that the three textbooks had different distributions of the generality. Regarding the specific level, Nani provided the most percentage of the analysis units. Regarding the generic level respectively, Han-Lin provided the most percentages of the analysis units. Regarding the formal level, Kang-Hsuan and Nani

Table 2. Frequencies of text goals

Textbook	Text Goal						
	I	EXP	D	EL	W	P	T
Kang-Hsuan	2/53 (0.04)	7/53 (0.13)	6/53 (0.11)	2/53 (0.04)	11/53 (0.21)	23/53 (0.43)	2/53 (0.04)
Han-Lin	4/60 (0.07)	5/60 (0.08)	6/60 (0.10)	2/60 (0.03)	13/60 (0.22)	29/60 (0.48)	1/60 (0.02)
Nani	2/50 (0.04)	3/50 (0.06)	7/50 (0.14)	2/50 (0.04)	12/50 (0.24)	22/50 (0.44)	2/50 (0.04)

Table 3. Frequencies of generality

Textbook	Generality			
	G0	G1	G2-1	G2-2
Kang-Hsuan	38/53 (0.72)	5/53 (0.09)	9/53 (0.17)	1/53 (0.02)
Han-Lin	42/60 (0.70)	12/60 (0.20)	4/60 (0.07)	2/60 (0.03)
Nani	38/50 (0.76)	3/50 (0.06)	6/50 (0.12)	3/50 (0.06)

provided the most percentages of the analysis units. It implied that Nani was inclined to arrange the generality with the two extreme levels. Contrarily, Han-Lin was inclined to arrange more analysis units of the generic level to bridge between specific and formal levels.

The Pythagoras' theorem was abstracted with specialization and generalization in the three textbooks. Nonetheless, the arrangement of specialization and generalization was different. In Kang-Hsuan, content analysis showed that introducing mathematical terms related to sides of right-angled triangles (G1) was followed by asking one question about the possible relationship among the three sides of any right-angled triangle (G2-1). Next, several examples were shown at the same analysis unit for finding the relationship (G1) and followed by two more examples for finding the relationship (G1). Then, the Pythagoras' theorem was described in a formal way (G2-1). After that, one question about the generalization of the relationship (G1) was followed by exploring the relationship in a formal way (G2-1). Lastly, the theorem was stated formally (G2-1).

In Han-Lin, the first analysis unit was similar to the one in Kang-Hsuan, but followed by a historical work about observing the pattern of tiles (G1), rather than by asking a general and abstract question. Next, two different equal-sided right-angled triangles were explicitly presented in tiles (G1) and followed by one question about the generalization of the relationship among the three squares to any right-angled triangle (G1). Then, one specific right-angled triangle was provided for exploring the relationship among the three squares of the right-angled triangle (G0). The relationship extended from equal-sided right-angled triangle to any right-angled triangle was described verbally (G1). Lastly, an algebraic proof (G2-2) was presented, and the theorem was stated formally (G2-1).

In Nani, the Pythagoras' theorem was developed from one specific example (G0), followed by introducing mathematical terms related to sides of right-angled triangles (G1). Next, the relationship among the three square areas, which was presented with one specific example, was described (G0) and followed by one question about the extension of the relationship to other right-angled triangles (G1). Then, two specific right-angled triangles were provided for exploring the relationship among the three sides of the right-angled triangle. Because the two exploratory tasks provided a sequence of partial questions specific to one right-angled triangle, the two analysis units were identified as the specific level (G0). Lastly, an algebraic proof (G2-2) was presented, and the theorem was stated formally (G2-1).

When counting the number of changing levels between two adjacent units, it was found that Kang-Hsuan Han-Lin and Nani provided 37%, 29% and 35% of two adjacent units to change levels of generality. That is, Kang-Hsuan and Nani provided more opportunities for the subject to transit among the levels than Han-Lin. Moreover, we found that Kang-Hsuan and Han-Lin developed the subject's understanding of Pythagoras' theorem with a generalization-directed approach, starting from generic and general examples. On the contrary, Nani developed it with a specialization-directed approach, starting from specific examples one by one. Although all the three textbooks provided examples for the subject to explore the relationship, further analysis indicated that the examples were designed differently. To show visual proof of the theorem in one analysis unit, Kang-Hsuan and Han-Lin provided, respectively, three and one examples where one right-angled triangle was integrated with visual proof, whereas Nani provided one example where one right-angled triangle and its visual proof were separate. In addition, the three textbooks showed different methods to prove Pythagoras' theorem (see Tam & Wang, 2012).

Connectivity

Kang-Hsuan and Han-Lin started and ended with the less familiar units, whereas Nani started with the more familiar unit and ended with the less familiar unit. This difference was due to that Nani always started with an exercise for practicing pre-knowledge related to the new topic in each sub-section. Regarding the text goal of [EXP] for developing the relationships among the three sides of an right-angled triangle, students were asked to recognize the area of each square generated from each side of the right-angled triangle and then identify the relationship among the three squares in Kang-Hsuan and Han-Lin. On the contrary, students were asked to justify a quadrangle is square, to recognize the square side length and its area, to recognize the side length of one right-angled triangle, and then identify the relationship among the square of the three sides of the right-angled triangle in Nani. The exploratory tasks in Kang-Hsuan and Han-Lin were easier for students to make connections with prior knowledge of areas of squares and forming a relationship among the three squares of one specific right-angled triangle than the task in Nani. In Nani, students were required to find the hypotenuse length of one right-angled triangle by reasoning from the area of one square composing of four right-angled triangles and one smaller square. Thus, the analysis units of [EXP] for developing the relationships among the three sides of a right-angled triangle in Kang-Hsuan and Han-Lin were coded as more familiar (C1) and the unit in Nani was coded as less familiar (C2).

Table 4 showed that the ratio of the connectivity with more familiar to less familiar was around 2 : 1. The chi-square test confirms no significant ($p = 0.746$) difference among the three textbooks. In each textbook, around 40% of the analysis units classified as less familiar appeared in the first 10 analysis units, and either 70% or 80% of the first 10 analysis units belonged to less familiar. This implied that the textbooks tried to arrange the object from less familiar to more familiar in the beginning. Most of the analysis units in the midst of the sequential analysis units were classified as more familiar, and some were classified as less familiar due to the requirement of strategies to solve problems and the development of the algebraic formula of the distance between two points on a Cartesian plane. The three textbooks arranged the last two or three analysis units as less familiar connection. When counting the number of changing the strength of connectivity between two adjacent units, it was found that Kang-Hsuan, Han-Lin and Nani provided 31%, 27% and 35% of two adjacent units to change the strength of connectivity. Nani provided the most opportunities for the subject to transit between more and less connection.

Functionality

To develop the subject's understanding of the distance between two points on a Cartesian plane, an exploratory activity was provided in Kang-Hsuan, worked-examples were provided in Han-Lin and Nani. In addition to the difference in text

Table 4. Frequencies of connectivity

Textbook	**Connectivity**	
	C1	**C2**
Kang-Hsuan	36/53 (0.68)	17/53 (0.32)
Han-Lin	41/60 (0.68)	19/60 (0.32)
Nani	31/50 (0.62)	19/50 (0.38)

goals, the diagrams in Kang-Hsuan was distinct from it in Han-Lin and Nani. Two points were located on a Cartesian plane with the origin and one unit length marked on each axis in Kang-Hsuan. On the contrary, Han-Lin and Nani provided a Cartesian plane with integer units or grids as well as the two supplementary lines parallel to the two axes respectively. Thus, the diagram in the analysis unit for developing the understanding of the distance between two points in Kang-Hsuan was coded as the conceptual diagram required more processing (F1-2) and the unit in Han-Lin and Nani were coded as the conceptual diagram required less processing (F1-1).

Table 5 showed that the three textbooks had different distributions of the functionality. Han-Lin provided the least percentage of the analysis units without diagrams, and the most percentages of the analysis units with conceptual diagrams required less processing, 27% and 55% respectively. There were more percentages of the analysis units with conceptual diagrams required more processing in Kang-Hsuan and Nani than in Han-Lin. Two similar diagrams for constructing two points on a Cartesian plane, shown in figure 5, were both provided by Han-Lin. It implied that Han-Lin more intended to utilize conceptual diagrams to support the subject's understanding of the object. On the contrary, Kang-Hsuan and Nani provided more opportunities for the subject to process conceptual diagrams.

CONCLUSION AND DISCUSSION

The results show that the three textbooks showed the pattern that one worked example was followed by one question for practice in order to develop the subject's understanding of the object. In the end, the three textbooks provided a sequence of questions for practice. As for the generality of the object, Kang-Hsuan and Han-Lin developed the subject's understanding of Pythagoras' theorem with a generalization-directed approach, and Nani developed it with a specialization-directed approach in the beginning. The generalization-directed approach may be more abstract in the beginning, but provide the need for coordination (one construction in reflective abstraction), whereas the specialization-directed may be less abstract, but require more generalization (one construction in reflective abstraction) from specific

Table 5. Frequencies of functionality

Textbook	Functionality					
	F0	**F1-1**	**F1-2**	**F2-1**	**F2-2**	**F3**
Kang-Hsuan	20/53 (0.38)	19/53 (0.36)	10/53 (0.19)	2/53 (0.04)	2/53 (0.04)	0/53 (0.00)
Han-Lin	16/60 (0.27)	33/60 (0.55)	5/60 (0.08)	3/60 (0.05)	1/60 (0.02)	2/60 (0.03)
Nani	16/50 (0.32)	20/50 (0.40)	10/50 (0.20)	3/50 (0.06)	1/50 (0.02)	0/50 (0.00)

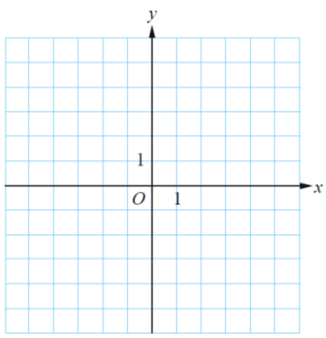

Figure 5. The diagram for constructing two points and calculating their distance in Han-Lin.

examples. It is difficult to determine which approaches is better without teaching experiment, and different approaches in textbooks may need different teaching skills.

For thinking mathematically, both specialization and generalization are necessary (Mason, Burton & Stacey, 1984). Mathematics textbooks can provide specialization and generalization by changing the levels of generality, e.g. from the formal to the specific level, or from the generic to the formal level. Changing the levels of generality could provide more challenge than support during abstraction. For instance, students generalize the specific case to the formal property, and specialize the formal property to the specific application. We found that Kang-Hsuan and Nani more frequently changed the levels of generality to challenge the subject's abstraction than Han-Lin, while Han-Lin and Nani provided more opportunities for the subject to experience the formal level.

As for the connectivity of the subject, the three textbooks showed a similar pattern. They started with a concise development of Pythagoras' theorem and its proof by arranging less familiar connection, proceeded to one worked example and one corresponding practice by arranging more familiar connection, next followed by the application of the theorem to developing the concept of the distance between two points on a Cartesian plane, and ended with review exercises and the appreciation of another proof for the theorem. However, Nani arranged more familiar units in the beginning although provided more percentage of analysis units with less connectivity and more opportunities for the subject to transit between less and more connections than the other two textbooks. It could indicate that the underpinning pedagogical strategies in Nani included to utilize both more familiar content as the initial stage for the subject to learn mathematics meaningfully (White & Mitchelmore, 2010), and less familiar content as opportunities for the subject to learn how to make unfamiliar mathematical knowledge more familiar (Hazzan & Zazkis, 2005). Nonetheless, it is still an issue how to arrange and balance between familiar and unfamiliar content in mathematics textbooks.

As for the functionality of diagrams, more than 60% of the analysis units provided diagrams, and there were less procedural diagrams in the three textbooks. It could imply that the textbooks less portrayed mathematics as a doing subject in virtue of that the procedural diagrams are more likely interpreted as suggesting that mathematics is constructed by doing than the conceptual diagrams (Morgan, 1996). Moreover, textbooks utilizing conceptual diagrams required more processing may be more difficult for the subject to understand in virtue of that more operative apprehension is required to mentally operate on the conceptual diagrams and then to meaningfully look at the object (Duval, 1995).

Mathematical objects become known through semiotic means (Radford, 2002). On one hand, multiple semiotic tools are used to support the subject's understanding. On the other hand, the multi-semiotic nature of mathematics may result in difficulties inherent in the learning of mathematics (O'Halloran, 2000). We conjecture that textbooks could make the object visible by conceptual diagrams required less processing, as well as operable by procedural diagrams required more processing. As we found in this study, the Han-Lin more intended to support the subject by making the object visible, and the three textbooks commonly less intended to support the subject by making the object operable.

In terms of the three attributes: the generality of the object, the connectivity of the subject and the functionality of diagrams, we found that Nani was the most challenging textbook in comparison to the other two textbooks, and there were two common features of Taiwanese mathematics textbooks. One was that the variation among the levels of the generality as well as between less and more familiar connections was arranged to learn Pythagoras' theorem. We interpret the finding by referring to the discernment of variations as one key to learning (Marton & Booth, 1997). Through experiencing the variation arranged in textbooks, students are assumed to "shift from seeing relationships as specific to the situation to seeing them as potential properties of similar situations (Wason & Mason, 2006, p. 94)".

The second common feature is related to the functionality of diagrams. That is, the procedural diagrams were less provided than the conceptual diagrams, and the diagrams required more processing were less provided than the diagrams required less processing. According to the suggestion that "provide sequences of micro-modeling opportunities, …, that nurture shifts between focusing on changes, relationships, properties and relationships between properties (Wason & Mason, 2006, p. 109)", it is suggested to provide more procedural diagrams required more processing in textbooks. Nonetheless, empirical studies are necessary to justify the implication.

According to Pingel's (2010) work, the main difference in textbook analysis is between didactic analysis and content analysis. While the former focuses on the pedagogy behind the text, the latter examines the content of the text itself. One contribution of this study is that attention has been given, as far as possible, to didactic analysis, i.e. the process of abstraction in this study. A constructive-empirical perspective on abstraction provides a way of conceptualizing the intended process of abstraction behind textbooks. We must admit that we cannot make any claims about an ideal distribution for each attribute of abstraction and about a perfect sequence of classifications for each attribute. Nonetheless, we elaborate the three attributes for evaluating the features of textbooks based on the above conclusion.

Although this study is descriptive, it is the first to analyze both description and work-examples in textbooks from the perspective of abstraction. At fine-grained analysis of abstraction behind textbooks, the attributes of generality, connectivity and functionality can be viewed as alternative dimensions of variation to design a sequence of didactic texts. In addition, the three attributes of abstraction elaborate how mathematics textbooks approach to an overarching dilemma of the balance between supporting and challenging the subject's learning of the object represented by the semiotic tools.

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