MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(4), em2426
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/14380

Best practices for teaching the concept of the derivative: Lessons from experienced calculus instructors

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Received 29 August 2023 ▪ Accepted 09 March 2024

Abstract

Much research has reported on difficulties exhibited by students when working with the concept of the derivative in first-semester university calculus. In an effort to generate and share a resource on effective instructional practices related to the teaching of the concept of the derivative, a 12-item questionnaire was administered to experienced calculus instructors in the United States. Most of the 10 experts who participated in this study considered providing ample examples and practice problems, using graphing utilities such as Desmos and GeoGebra, and employing problem solving strategies in the teaching of the concept of the derivative to be effective in supporting students learn about the concept of the derivative, among other things. These experts further remarked on representations of the concept of the derivative and real-world contexts they typically use in their teaching of the concept, in addition to providing rationales for the choice of real-world contexts they typically use in their teaching of the concept. Recommendations for instruction are included.

Keywords: calculus, derivatives, technology, teaching strategies

INTRODUCTION

The concept of the derivative is not only the heart of differential calculus at the undergraduate level, but also a concept that has numerous applications in many fields of study, including mathematics, economics, physics, and engineering (cf. Berry & Nyman, 2003; Jones, 2017; Mkhatshwa, 2018, 2023a; Siyepu, 2013). Despite the importance of this concept in the aforementioned fields of study, a substantial number of studies have reported on students' difficulties related to the concept of the derivative when solving calculus-based problems. Among other things, several researchers have reported on students' lack of facility with calculus rules of calculating derivatives (cf. Clark et al., 1997; Engelke, 2004; Mkhatshwa, 2020; Piccolo & Code, 2013). Other studies have found that constructing or interpreting graphs of derivatives is problematic for students (cf. Habre & Abboud, 2006; García-García & Dolores-Flores, 2021; Haciomeroglu et al., 2010; Ubuz, 2007). Findings by Ibrahim and Rebello (2012) indicate that while students are often successful in making sense of derivatives in kinematics contexts, interpreting derivatives in non-kinematics contexts such as in economics is often challenging for students.

Given these widespread difficulties, one may wonder about the extent to which promising instructional approaches in mathematics education such as the flipped classroom, inquiry-based learning, project-based learning, team-based learning, and the Modified Moore method have been employed in the teaching of calculus topics such as the derivative (cf. Aryal, 2022; Cronhjort et al., 2018; McLoughlin, 2009; Mkhatshwa, 2021; Peter et al., 2020; Viirman & Pettersson, 2022; Wu & Li, 2017). McLoughlin (2009) presented a number of reasons that makes it difficult to implement an inquiry-based approach in the teaching of calculus. As an alternative, McLoughlin (2009) proposed the use of the Modified Moore method. We argue that, with the exception of the flipped classroom, most of the aforementioned teaching approaches have been underexplored in the teaching of calculus. Indeed, evidence from research suggests that the teaching of mathematics, including undergraduate-level calculus, is dominated by the use of the traditional lecture method (cf. Ashraf, 2020; Code et al., 2014; Speer et al., 2010).

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Contribution to the literature

- Research on students' difficulties with making sense of the concept of the derivative is well documented. There is, however, a paucity of research on effective instructional approaches related to this concept, which is the motivation for the current study.
- Specifically, the current study reports on effective teaching strategies in connection with the concept of the derivative based on the experiences of 10 veteran calculus instructors.
- Findings of this research have implications for different stakeholders, including calculus instructors, textbook authors, and textbook selection committees in mathematics departments, respectively.

Findings from another body of research indicate that the integration of different educational technologies such as graphing calculators, Desmos, GeoGebra MATLAB, Maple, and WolframAlpha in the teaching of calculus could help students develop robust understandings of various calculus ideas, including the concept of the derivative (cf. Leng, 2011; Wu & Li, 2017; Zbarsky et al., 2021). Motivated by the scarcity of research that specifically addresses effective instructional approaches related to the concept of the derivative, the current study reports on veteran calculus instructors' perspectives on effective instructional approaches and educational technologies related to the teaching of the concept of the derivative in calculus. On a related note, a recent study (Mkhatshwa, 2023b) investigated experts' views regarding effective teaching approaches in the context of related rates problems in calculus. This study found that using diagrams during classroom instruction could be helpful in supporting students' reasoning about rate quantities that could be represented by derivatives when working with this type of problems.

Drill et al. (2013) argued that teachers from a variety of disciplines, including mathematics, tend to look to research for help on how to best teach certain concepts/topics in their disciplines. Motivated by Drill et al.'s (2013) argument, and in an effort to generate a resource that calculus instructors could turn to for help on how to best teach the concept of the derivative, this study used a questionnaire to elicit experienced mathematics professors' views, based on their teaching experiences, of what they would consider to be effective teaching practices related to the concept of the derivative in calculus. Advances in computer technology have led to the development of dynamical mathematical software that are increasingly being used in the teaching of various topics in calculus, including the concept of the derivative (cf. Habre & Abboud, 2006; Thompson et al., 2013; Thurm & Barzel, 2022). As part of the investigation reported in this study, the aforementioned questionnaire included a question on what teaching technologies the study participants have used in their teaching of derivatives as well as how they used these technologies. Several studies have found that the presentation of mathematical content during classroom instruction tends to follow the presentation of the same content in mathematics textbooks (cf. Begle, 1973; Reys et al., 2004; Wijaya et al., 2015). Thus, as part of the investigation reported in this study, the study participants were asked about their perceptions regarding the presentation of the concept of the derivative in calculus textbooks. I remark that students are typically first introduced to the concept of the derivative in a first-semester calculus course, commonly known as calculus I, in the United States. The research questions guiding this study are:

1. What do experienced calculus instructors consider to be effective instructional strategies in their teaching of the concept of the derivative in calculus I?
2. What teaching technologies/tools do experienced calculus instructors use in their teaching of the concept of the derivative in calculus I?
3. What do calculus instructors identify as weaknesses and strengths in the presentation of the concept of the derivative in calculus I textbooks?

LITERATURE REVIEW

Students' Thinking About Derivatives

In as much as the primary focus of the present study is to provide calculus instructors' perspectives on what they consider to be effective instructional strategies when teaching the concept of the derivative, it is paramount to review existing literature on students' thinking about derivatives for comparison. In other words, this will help determine the extent to which the teaching strategies proposed by calculus instructors could potentially address known students' difficulties when working with derivatives.

A reoccurring theme from a number of studies that have examined students' thinking about derivatives is that students tend to prefer algebraic methods, which are often more procedural and less conceptual, when working with derivatives (cf. Berry & Nyman, 2003; Dawkins & Epperson, 2014; Engelbrecht et al., 2009; Habre & Abboud, 2006; Haciomeroglu et al., 2010; Ibrahim & Rebello, 2012; Thompson, 1994; Weber & Thompson, 2014). In fact, one of these studies reported on students who "overused algebraic techniques even when they proved cumbersome or inappropriate"

(Engelbrecht et al., 2009, p. 839) when working with derivatives. Other studies have found that, among other rules of differentiation, using the chain rule in connection with implicit differentiation or the product rule of differentiation is problematic for calculus students (cf. Clark et al., 1997; Mkhatshwa & Jones, 2018; Picollo & Code, 2013). On the contrary, Maharaj and Ntuli (2018) reported on calculus students for whom using the chain rule to differentiate exponential and logarithmic functions was plain sailing. On a related note, findings by Orton (1983) and Mkhatshwa (2023a) indicate that differentiating polynomial functions is effortless for many calculus students.

Another reoccurring theme from research on students' reasoning about derivatives is that while derivatives are often well understood by students in kinematics contexts, the same cannot be said of other real-world contexts, including economics (cf. Cetin, 2009; Kertil et al., 2023; Mkhatshwa, 2018, 2023a; Stroup, 2002; Zandieh & Knapp, 2006). Specifically, these studies have found that interpreting derivatives in kinematics contexts is often easy for calculus students, while interpreting derivatives in non-kinematics contexts is typically challenging for the same students. Jones (2017) posited that "an overreliance on kinematics [during classroom instruction] may lead students to view derivatives only through that narrow lens, possibly limiting the applicability of derivatives in many other contexts" (p. 109). Jones (2017) went on to recommend that:

> … more non-kinematics derivatives be incorporated into first-year calculus instruction, as well as rich discussion about them, so that we may help students develop the conceptual and procedural tools to effectively apply, use, and interpret these types of derivatives in a range of contexts (p. 109).

Evidence from a related line of research shows that interpreting quantities represented by negative derivatives is problematic for students (cf. Beichner, 1994; Orton, 1983).

Students' propensity to confuse rate quantities with other rate quantities or with amount quantities is yet another reoccurring theme from research that has looked at students' understanding of derivatives in real-world contexts (cf. Flynn et al., 2018; Mkhatshwa, 2020; Prince et al., 2012; Rasmussen & Marrongelle, 2006). Rasmussen and Marrongelle (2006), for instance, provided evidence of a student who did not make a "distinction between rate of change in the amount of salt [a rate quantity represented by a derivative] and the amount of salt [an amount quantity]" (p. 408) in the context of solving a calculus-based application problem. While the source of students' tendency to confuse rate quantities with amount quantities may be unknown in other disciplines, research by Feudel and Biehler(2022) suggests that in economics, this confusion could originate from classroom instruction or economics textbooks. According to these authors, the derivative is "… commonly used in economics as the amount of change when increasing the production by one unit" (p. 437). A similar observation was made by Mkhatshwa (2016) who reported on opportunities to learn about marginal analysis (i.e., marginal cost, marginal revenue, and marginal profit) provided by a widely used business calculus textbook in the United States.

The Role of Technology in the Teaching of Derivatives

While the principal focus of the current study is on instructors' perceptions regarding effective teaching practices related to the concept of the derivative, it is important to review existing literature on the role of technology in the teaching of the derivative. In fact, one of the items in the questionnaire used in this study specifically asked instructors about the technologies they use, and how they use these technologies, in their teaching of the concept of the derivative. Several studies have reported promising results related to the integration of technology in the teaching of derivatives in calculus (cf. Berry & Nyman, 2003; Borji et al., 2018; Haciomeroglu & Andreasen, 2013; Hiyam et al., 2019; Illanes et al., 2022; Rosly et al., 2020; Sari et al., 2018; Tall et al., 2008). In their investigation of students' thinking about the relationship between a function and its derivatives in a geometric context, Berry and Nyman (2003) found that the use of two educational technologies (a TI-83plus graphing calculator and a calculator based ranger) was helpful in supporting students' understanding of the aforementioned relationship. This finding is consistent with that of Hiyam et al. (2019) who used Mathematica, a computerized algebra system, to develop students' ability to make connections between graphs of functions and graphs of the functions' derivatives, among other things.

Sari et al. (2018) reported on a teaching experiment that used GeoGebra, a dynamic mathematics software, to enhance students' ability to visualize the concept of the derivative. Sari et al. (2018) argued that using GeoGebra in the teaching of the derivative concept in calculus has the potential to support students' understanding of the concept in multiple function representations. Similar results were reported by Haciomeroglu and Andreasen (2013) who used GeoGebra to illustrate and explore the concept of the derivative with the goal of enhancing students' understanding of the concept. Rosly et al. (2020) reported on the use of WolframAlpha, a web-based computational knowledge engine, to support students' calculational knowledge of derivatives in differential calculus. Borji et al. (2018) found the use of Maple to be effective in supporting students visualize the concept of the derivative.

Despite the overwhelming support by proponents of using technology in mathematics teaching at all levels, evidence from research suggests that there has been a limited integration of technology in mathematics education (cf. Agyei et al., 2022; Kamau, 2014; Marshall, 2012). Among other things, several studies have reported on critics' views regarding the use of technology in the teaching of mathematics, including in calculus (cf. Hegedus et al., 2015; Nabb, 2010; Sammartino, 2023; Serkan, 2013). More broadly, critics of the integration technology in mathematics education argue that the use of technology in mathematics classrooms is the reason why students have impoverished mathematical skills, especially algebraic skills, that are required for advanced mathematics (cf. Nabb, 2010; Serkan, 2013). In essence, one may argue that critics of using technology in mathematics education value procedural understanding of mathematical ideas, in addition to conceptual understanding of the same ideas. On a related note, one study (Sevimli, 2016) reported on students' attitudes pertaining to the use of technology in calculus. This researcher reported that "… students in the traditional group place more importance on procedural skills, while students in CAS group attach more importance to conceptual skills in terms of instructional objects" (p. 1).

The Role of Textbooks in the Teaching of Mathematics

In spite of the fact that the main focus of the present study is to examine experienced calculus instructors' views regarding effective teaching practices related to the concept of the derivative, it is important to highlight the role of textbooks in the teaching of mathematics in general. One of the items in the questionnaire used in this study specifically asked instructors about their perceptions regarding strengths and weaknesses in the presentation of the concept of the derivative in calculus textbooks. A common theme that emerges from research that has reported on student learning in mathematics classrooms and the opportunities to learn provided by mathematics textbooks is that the presentation of content, including calculus ideas such as the derivative, during course lectures tend to closely follow the presentation of similar content in mathematics textbooks (cf. Begle, 1973; Mkhatshwa, 2016; Reys et al., 2004; Robitaille & Travers, 1992; Wijaya et al., 2015). Begle (1973), for instance, asserted that "most student learning is directed by the text rather than the teacher" (p. 209). Begle's (1973) views regarding the role of textbooks in students' learning of mathematics were echoed by Reys et al. (2004) who posited that "… the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). On a slightly different note, Charalambous et al. (2010) argued that "textbooks afford probabilistic rather than deterministic opportunities to learn mathematics" (p. 118).

Thus, in light of the importance of textbooks in students' learning, the present study seeks to provide calculus instructors' perspectives regarding opportunities to learn about the concept of the derivative presented by calculus textbooks in the United States. I remark that to date, only one study (Mkhatshwa, 2022) has reported on the opportunity to learn about derivatives provided by calculus textbooks in the United States. Findings of the aforementioned study indicate that while calculus textbooks provide sufficient opportunities to engage in quantitative reasoning (Thompson, 2011), these textbooks provide limited opportunities for students to engage in covariational reasoning (Carlson et al., 2002) when working with derivatives.

METHODS

Questionnaire Design & Validation

This exploratory and qualitative study used a 12-item online Qualtrics questionnaire (**Appendix A**) to elicit experts' (calculus instructors) views on the teaching of the concept of the derivative in calculus I, among other things. The primary aim of the study was to examine, through the lens of experts, effective ideas and strategies related to the teaching of derivatives in calculus I. With the exception of question 1 and question 2 in the questionnaire that were included to elicit experts' basic information, including their experience teaching calculus I, the rest of the questions in the questionnaire were mostly informed by findings from existing literature on students' thinking about the concept of the derivative or its applications. The questionnaire was administered in the fall semester of 2022.

The goal for designing the questionnaire was seven-fold. First, to gain insight on what experts consider to be easy/challenging for students when working with derivatives (item 3 and item 4 in the questionnaire). Second, to gain insight on what experts consider to be effective instructional practices in the teaching of derivatives (item 5 in the questionnaire). Third, to gain insight on experts' perceptions regarding the role of technology, or other teaching tools, in the teaching of the concept of the derivative (item 6 and item 7 in the questionnaire). Fourth, to gain insight on the dominant representation, if any, of the concept of the derivative that is used often during classroom instruction (item 8 in the questionnaire). Fifth, to gain insight on the dominant real-world context, if any, used in the teaching of the concept of the derivative (item 9 in the questionnaire). Sixth, to gain insight on the range (wide or limited) of textbooks used by experts in the teaching of derivatives (item 10 in the questionnaire). Seventh, to gain insight on what calculus instructors consider to be strengths and weaknesses in the presentation of the concept of the derivative by calculus textbooks (item 11 and item 12 in the questionnaire).

Face and content validity (Martinez, 2017) of the questionnaire was assessed by three subject experts. It should be noted these experts are not among the experts who are participants in the current study. All the experts independently found the questionnaire to be appropriate for addressing the research questions addressed in the study, thus establishing the questionnaire's face validity. Additionally, the experts determined that the questionnaire addresses most, if not all, all aspects related to the teaching and learning of the concept of the derivative, thus establishing the questionnaire's content validity. The three subject experts have experience teaching various calculus topics, including the concept of derivative. Additionally, two of the experts have experience tutoring students (i.e., working with students on a one-to-one basis) taking introductory calculus courses, where the concept of the derivative is first introduced and covered in greater detail. Based on the subject expects' calculus teaching and/or tutoring experience, these experts are adequately qualified to assess the content and face validity of the questionnaire used in the study.

Sampling Method & Participants

Convenience sampling was used to recruit the 10 experts (herein identified as E1 through E10) who participated in this study. This was done in two stages. In the first stage, I sent out invitation emails to colleagues in the teaching profession who I know have experience teaching calculus I in the United States. At the time of the study, one of the experts was a colleague at the institution I am currently affiliated with, and another expert is from an institution I was previously affiliated with. Some experts are colleagues I met at professional conferences, and other experts are colleagues in the teaching profession that I have not met, but I happen to know of scholarly work they have done related to the teaching and learning of calculus. In the second stage, I searched the internet using key words such as "calculus director" or "calculus coordinator." I then sent out invitation emails to anyone who matched the aforenoted internet search criteria and was affiliated with an institution of higher education (i.e., college or university) in the United States.

In total, I sent out a total of 16 invitations. Of these invitations, 10 experts agreed to participate in the study. Of the 10 participants, three participants responded to at least 75% of the items included in the questionnaire-the rest of the participants responded to all the questions included in the questionnaire. Three participants stated that they had taught no more than five sections of calculus I. Two participants had taught at least six but no more than 10 sections of calculus I. One participant had taught at least 11 but no more than 15 sections of calculus I. One participant had taught at least 16 but no more than 20 sections of calculus I, and three participants had taught more than 20 sections of calculus I. Additionally, seven participants were affiliated with institutions classified as "very high research activity" institutions in the United States, also known as R1 institutions. Two participants were affiliated with institutions classified as "high research activity" in the United States, also known as R2 institutions, and one participant was affiliated with a medium-sized four liberal arts college.

Data Analysis

The data, consisting of experts' responses to Items 3 through 12 in the questionnaire, were analyzed using a thematic analysis approach. Braun and Clark (2006) define thematic analysis as "… a method for identifying, analyzing and reporting patterns (themes) within data" (p. 79). For the purpose of this study, a theme is a similar response to the questionnaire items given by at least two experts. For instance, if one expert noted velocity, another expert mentioned acceleration, and another expert remarked about speed or distance in response to item 9 in the questionnaire that elicited experts' perceptions regarding typical real-world contexts they use in their teaching of derivatives in Calculus I, I identified the kinematics context as a theme.

RESULTS

Students' Calculational Knowledge of Derivatives

When asked about their perspectives regarding what comes easy for students when working with derivatives (i.e., item 3 in the questionnaire), nine of the 10 experts commented on students' proficiency with using the power rule of differentiation (hereafter, power rule), among other rules of rules of differentiation, to calculate derivatives. That is, differentiating polynomial functions is generally viewed by experts to be straightforward for most students. The following is an exemplary remark made by one of the experts in response to item 3 in the questionnaire:

> Certain rules for finding derivatives (e.g., the power rule) seem to stick quite easily. Students love the power rule so much that they apply it even in contexts, where it is not appropriate (E4).

Although using the power rule is generally easy for many students, E4 remarked that sometimes students have a propensity to apply it inappropriately. Drawing on my experience teaching calculus I, for example, I have observed students inappropriately using the power rule to calculate derivatives of exponential functions. Although E4 specifically noted the power rule, this expert's remark also suggests that the are other rules of derivatives students find easy to use when working with derivatives as indicated by the phrase "certain rules for finding derivatives …" in the proceeding remark. Indeed, three experts identified the product rule of differentiation (hereafter, product rule) and quotient rule of differentiation (hereafter, quotient rule) in their response to item 3 in the questionnaire. The following is a reproduction of an exemplary response to the aforementioned item given by one of the three experts:

> Students tend to find using the differentiation rules more straightforward. They are usually quick to identify and apply the product, quotient, and chain rules without much difficulty. They also tend to do well with more straightforward applications of derivatives such as position-velocity-acceleration and L'Hospital's rule (E6).

It should be noted that although E6 noted two other rules of differentiation (i.e., the chain rule and L'Hospital's rule), in addition to the product rule and quotient rule, none of the other nine experts remarked on these rules of differentiation as being generally straightforward for students. Furthermore, I interpreted E6's remark that students "… tend to do well with more straightforward applications of derivatives such as position-velocity-acceleration …" to mean that working with derivatives in kinematics contexts is generally easy for many students. In fact, two other experts remarked (in response to item 4 in the questionnaire) on students' difficulties with interpreting derivatives in non-kinematics contexts such as in economics. The following is a reproduction of one of the two experts' responses to item 4 in the questionnaire:

> More advanced methods of finding derivatives: logarithmic differentiation, any implicit differentiation question that requires the chain rule, and anything that requires interpretation of a derivative in some context other than displacement/velocity/acceleration. On this last point, what I mean is something like this: let $f(x)$ be the average number of new COVID-19 infections per day, x days after the pandemic began; what units is $f(x)$ measured in? Explain what quantity $f'(x)$ represents, and what units it should be measured in? Or, let $C(x)$ be the cost per unit when a company produces x units. What is $C'(x)$? Obviously in both examples there is a red herring: f or C already represents a rate, so I'm asking about a rate of change of a rate, but I feel this concept is very important and is quite difficult for students to grasp when they first encounter the idea of "rate of change" as a mathematical quantity (E3).

Consistent with E3's remark regarding the chain rule of differentiation (hereafter, chain rule), five other experts identified the chain rule (or its application thereof) to be problematic for many students when working with derivatives.

Effective Ways to Support Students Develop a Solid Understanding of the Concept of the Derivative

When asked about their perspectives regarding effective teaching practices they have used to support students develop a solid understanding of the concept of the derivative (i.e., item 5 in the questionnaire), six of the 10 experts remarked on creating ample opportunities (both during classroom instruction and outside the classroom) for students to work with average rates of change or derivatives (also known as instantaneous rates of change). Following are exemplary responses to item 5 in the questionnaire given by three of the six experts:

> I tend to focus quite a bit on the meaning of a derivative and often come back to the definition of a derivative. I emphasize that a derivative is a quotient. We always have change in y (or another quantity) divided by change in x (or another variable). This tells us "how fast" y is changing as x changes. I provide a lot of class time for students to work on problems and explore ideas. I also assign quite a bit of HW [homework], where students can practice and/or extend their understanding (E1).

> We do a lot of in-class worksheets, where they have to calculate average rates of change, and then discuss how to get better and better approximations. We have a nice worksheet that ties the rates of change to the slope of a line, and we have several exercises, where they work through that and eventually develop the idea that taking the limit would give us the "best" answer (E7).

> I provide students with a lot of practice. Lots of in-class and outside-of-class worksheets (E8).

In addition to spending time unpacking the meaning and definition of the derivative, E1 creates opportunities in the form of homework for students to practice working with derivatives outside the classroom as indicated by his remark: "I also assign quite a bit of HW [homework], where students can practice and/or extend their understanding." E7 provides ample opportunities to help students understand the relationship between the concepts of average rate of change and the derivative via in-class worksheets. E8 uses worksheets during classroom instruction as well as outside the classroom to help students practice working with the concept of the derivative. Two other experts mentioned using real-life examples to support students develop a robust understanding of the concept of the derivative. The following is a reproduction of one of these experts' responses to item 5 in the questionnaire:

> I tend to rely mostly on real-life examples and problems that they can relate to. These real-life contexts provide opportunities to talk about units and how to interpret derivatives in meaningful ways. I try to use technology to help visualize some abstract ideas in relation to derivatives (e.g., continuity of functions) (E4).

It is worth noting is that E4 not only uses real-life examples to help students make sense of various aspects of the derivative such as their units of measure or their interpretations in real-world contexts, but this expert is also selective when it comes to the examples he uses as indicated by his remark: "I tend to rely mostly on real-life examples and problems that they can relate to." Additionally, E4 uses technology to help students "visualize some abstract ideas in relation to derivatives…" Two other experts mentioned using group problem solving strategies to help students master the concept of the derivative. The following is a reproduction of one of these experts' responses to item 5 in the questionnaire:

> We have one day a week ("workshop") devoted to group problem solving, and when we are near the beginning of the derivative topic, I have students spend some of the time solving verbal problems about interpreting derivatives. Not word problems, per se, but *explanatory* problems, e.g., here's a graph of $y = f(t)$. Suppose the graph represents a bear's weight at time t. Draw a line L_1 through $(1, f(1))$ and $(3, f(3))$, and a tangent line L_2 at $t = 4$. What aspect of the bear's weight does L_1 represent? L_2? Make a statement about line L_2 in terms of a derivative of f (E3).

It is evident from E3's response that group problem solving plays a key role in her teaching of not just the concept of the derivative, but rather calculus I in general. Specifically, this is indicated by her opening remark: "We have one day a week ("workshop") devoted to group problem solving…" In light of the fact that experts' responses to item 5 in the questionnaire are the central focus of the present study, I have reproduced all the responses to the aforenoted item provided by the 10 experts in **Appendix B**.

Technologies Used to Support Students' Understanding of the Derivative

When asked about their perspectives regarding technologies they have used to support students' learning about the concept of the derivative in their calculus I classes (i.e., item 6 in the questionnaire), four experts mentioned using various types of graphing utilities. The following are exemplary responses to item 6 in the questionnaire given by three of the four experts:

> … When learning what a derivative is (graphically), I always show them a video or an app that lets them drag points (my own drawing is limited, and I think it's important for them to see something done well at least once). If I do not have technology in my classroom, I will just email them a link to the videos, but I typically have the ability to project in whatever classroom I'm in. For derivative rules, I have a "gateway" assessment that students can take multiple times. This is also done via WebAssign, but in a proctored environment (E1).

> I use graphing tools like Desmos and GeoGebra to help visualize some of the concepts about derivatives. I also teach my students how to use these tools so they can use them on their own (E3).

> The primary technology I use in my calculus classes is Desmos, particularly when it comes to graphical derivatives or finding the equation of the tangent line. The process for finding the equation of the tangent line is often technical for students, so it is satisfying when they can plot the curve in question and the line they found to verify their algebraic work using a graphing utility (E5).

Although E1 did not specify the graphing utility he uses i.e., "an app that lets them [students] drag points", E3 and E5 specifically noted using Desmos to support students' learning about the concept of the derivative. In addition to using the Desmos, E3 noted using another graphing utility (i.e., GeoGebra). This expert further highlighted a multi-purpose for using the Desmos and GeoGebra technologies, namely "… to help [students] visualize some of the concepts about derivatives" and to teach students "… how to use these tools [Desmos and GeoGebra] so they can use them on their own." Four other experts provided responses that suggested that they either do not use technology all or that their use of technology when teaching the concept of the derivative is minimal. The following is an exemplary response given by one of the four experts: "I do not use much technology to support students' learning about derivatives, but I would be happy to learn how to do it" (E7). I interpreted the latter part of E7's remark i.e., "but I would be happy to learn how to do it" to mean that some calculus I instructors' minimal usage of technology in their teaching of derivatives could be attributed to lack of formal training in the usage of available technologies such as Desmos and GeoGebra.

Although very interesting and hopefully helpful for both experienced and novice calculus I instructors, there were no obvious themes (besides using real-world examples students could relate to and assigning practice problems that gradually build in difficulty) in the experts' responses to item 7 in the questionnaire. To reiterate, item 7 asked about experts' perspectives regarding other tools, besides technology, they have used to support students' learning about the concept of the derivative in their calculus I classes. Nearly all the responses given by the 10 experts appear to be additional ways these experts have found effective in supporting students' leaning about the concept of the derivative. Consequently, I have reproduced these responses in **Appendix C** and labelled them as additional effective ways for teaching the concept of the derivative.

Representations & Contexts Used in the Teaching of the Concept of the Derivative

When asked about a representation they use often in their teaching of the concept of the derivative (i.e., item 8 in the questionnaire), seven experts noted using most of the representations of a derivative (i.e., algebraic-using explicit formulas, geometric/graphical-using graphs, numerical-using numeric tables of values, or verbal-using verbal descriptions). Furthermore, responses given by five experts suggests that these experts rarely or never use the numerical table of values representation of the derivative at all. The following is a reproduction of three exemplary responses:

> I motivate derivatives at first verbally by comparing the derivative to a rate of change like driving so students can conceptualize the reasoning why we study such things. Afterwards there is usually a lesson or two on graphical representations, where students examine the slopes of a graph qualitatively (positive/negative/zero slopes, how slopes change, etc.) Most of the time spent will be on the algebraic formulas: given a function, use appropriate differentiation rules to find the derivative. Once all differentiation rules have been covered, I tie it back into the verbal descriptions with applications such as the free-fall model (E6).

> I use algebraic and geometric representations most often. I use numeric tables the least. I use verbal descriptions when doing related-rates problems (E8).

> I am using most of them except numeric tables of values (E9).

Although not explicit, a close examination of E6's response to item 8 in the questionnaire might suggest that this expert does not use the numerical table representation of the derivative at all as this expert did not make any comment to this effect in her response. On the other hand, E8 specifically noted rarely using this representation as indicated by his remark: "I use numeric tables the least." Like E8, two other experts mentioned using the algebraic and geometric representations of the derivative more often in their teaching of calculus I. E9's remark, "I am using most of them except numeric tables of values", suggests that this expert does not use the numeric table representation of the derivative at all in her teaching of the concept of the derivative in calculus I. In response to the same item in the questionnaire, one expert only remarked "algebraic-using explicit formulas" (E2), while another only remarked "verbal-using verbal descriptions" (E10). Arguably, this may suggest that E2 and E10 do not emphasize the importance of multiple representations of the concept of the derivative in their teaching of calculus I.

When asked about real-world contexts they often use in their teaching of the concept of the derivative (i.e., item 9 in the questionnaire), seven experts provided responses that suggest that they frequently use the kinematics context in their teaching of the concept of the derivative or in applications of this concept. The following is a reproduction of exemplary responses to item 9 in the questionnaire given by six of the 10 experts:

> kinematics is most common (by far), but I do assign HW problems that use other contexts (cost per item, population per year, etc.) (E1).

> Kinematics is one of the contexts that many students easily relate with. I also use other contexts such as growing rectangles, plot measuring, among others. I use these contexts by having students explore a problem rooted in these contexts and then share their thinking with the whole class (E4).

> Driving vehicle to motivate the idea. After students are comfortable with the differentiation rules, I discuss kinematics and the free-fall model and using derivatives to answer related questions (E6).

> We use a lot of position, velocity, acceleration problems (E7).

> Physics contexts (e.g., velocity and acceleration) seem to work the best (E8).

> Linear motion (position-velocity-acceleration) (E9).

In the preceding responses, E1 noted that while the kinematics context is most common he also uses other contexts such as the economic context. This is indicated by the following claim in his response "… but I do assign HW problems that use other contexts (cost per item, population per year, etc.)". Similarly, E4's response suggests that this expert incorporates other contexts, in addition to the kinematics context. Furthermore, E4 provided a rationale for using the kinematics contexts in his teaching of derivatives i.e., it "… is one of the contexts that many students easily relate with." E6, E7, E8, and

E9's responses are clear on the fact that the kinematics context is mostly used in these experts' teaching of the concept of the derivative in their calculus I classes.

Calculus Textbooks' Strengths & Weaknesses in Their Presentation of the Concept of the Derivative

When asked about potential strengths in the way calculus textbooks present the concept of the derivative (i.e., item 12 in the questionnaire), five experts expressed appreciation for the incorporation of illustrative diagrams/pictures and the abundance of examples and practice problems provided in calculus textbooks they have used. Before I present exemplary responses to buttress this claim, I note that there was not much variation in the textbooks used by the 10 experts as eight of the textbooks have the same author (James Stewart), another textbook has a different author (Matthew Boelkins), and another textbook has different authors (Joel Hass, Christopher Heil, and Maurice Weir). The textbook information is based on the 10 experts' responses to item 10 in the questionnaire that asked about details of textbooks (i.e., title, author(s), and edition) the experts have used in their teaching of the concept of the derivative in calculus I. The following is a reproduction of exemplary responses to item 12 in the questionnaire given by three of the 10 experts:

> Stewart's figures are nice. It shows tangent lines clearly. The examples and practice problems are also good (there are lots from which to choose) (E1).
>
> Stewart has plenty of problems that build in difficulty that I incorporate into my activities and assignments (E5).
>
> Lots of good graphs and graphical representations of things (E6).
>
> Enough number of examples and problems (basic and application) (E7).

In addition to commenting on the adequacy of practice problems provided by calculus textbooks they have used in their teaching of the concept of the derivative, E5 and E7 were careful to note that there is a variation (in terms of the level of difficulty) in the examples or practice problem provided by these textbooks. Furthermore, while E6 only commented about the abundance of helpful illustrative graphs and graphical representations, E1 commented on the nature of diagrams (which he refers to as figures) and examples and practice examples provided in calculus textbooks. Other than four experts who noted that they have not observed weaknesses in the presentation of the concept of the derivative in calculus textbooks (i.e., the experts' response to item 11 in the questionnaire), there is no obvious pattern in what the experts identified as weaknesses in calculus textbooks' presentations of the concept of the derivative. The following is an exemplary response from the four experts who have not noticed weaknesses in calculus textbooks' presentations of the concept of the derivative: "I do not think Stewart has any weaknesses in presenting the concept of the derivative. I do not like how Stewart handles limits or the introduction to integrals, however" (E6). As a remark, I note in passing that James Stewart is the lead author of most of the commonly used calculus textbooks in the teaching of calculus I in the United States.

DISCUSSION & CONCLUSIONS

This qualitative case study used a Qualtrics questionnaire to examine calculus instructors' perspectives regarding effective instructional practices in the teaching of the concept of the derivative in the United States, among other things. The following is a discussion of the key findings of the study. First, several of the experts in this study reported that students tend to do well when working with derivatives in kinematics contexts, and that interpreting derivatives in non-kinematics contexts is problematic for students. This is consistent with findings of several studies that have reported on students' thinking about the concept of the derivative (cf. Cetin, 2009; Kertil et al., 2023; Mkhatshwa, 2018, 2023a; Stroup, 2002; Zandieh & Knapp, 2006). As previously reported in the results section, seven of the 10 experts in this study noted that they use the kinematics context more often compared to other real-world contexts such as economics when teaching the concept of the derivative. I thus recommend that calculus instructors consider using a variety of other real-world contexts such as economics, in addition to the kinematics context, in their teaching of calculus I. Such a move could be effective in helping students have a broader understanding of the concept of the derivative and its applications in other real-world contexts. Furthermore, a majority of the 10 experts who participated in this study reported that using the chain rule is generally challenging for students when working with derivatives. Again, this is consistent with findings of previous research that indicate that using the chain rule in connection with implicit differentiation or the product rule of differentiation is problematic for calculus students (cf. Clark et al., 1997; Mkhatshwa & Jones, 2018; Picollo & Code, 2013).

Second, the experts in this study remarked on several instructional strategies they have found helpful and effective in their teaching of calculus I. These include, among others, providing ample opportunities in the form of work-sheets for students to explore the relationship between the concept of average rate of change and the derivative, creating opportunities to work with the concept of the derivative both in class and outside the class, using group problem solving strategies to help students master the concept of the derivative, and using examples that have real-world contexts that students could relate to when teaching the concept of the derivative. A complete list of effective strategies related to the teaching of the concept of the derivative, based on the experiences of the 10 experts who participated in this study, have been reproduced in **Appendix B** and **Appendix C**, respectively. I believe that these teaching strategies could serve as a useful resource for all calculus I instructors, especially for novice calculus I instructors such as graduate students who typically serve as teaching assistants in mathematics departments. Taken together, the reported effective strategies seem to align well with established pedagogical approaches in mathematics education. For instance, using group problem solving strategies to help students learn about the concept of the derivative is one effective way to implement active learning in calculus teaching, an effective instructional approach that has received much attention in recent years (cf. Duran et al., 2022; Fuchs & Sahmbi, 2024; Stanberry, 2018). Furthermore, the use of real-world contexts by the experts who participated in the current study is much appreciated. This is because evidence from research suggests that students often exhibit challenges when tasked with applying mathematics in real-world contexts (cf. Buddo et al., 2019; Kertil et al., 2023; Klymchuk et al., 2010).

Third, several experts in this study noted using graphing utilities such as Desmos and GeoGebra to support students in their learning about the concept of the derivative. A number of studies have reported promising results related to the integration of technology in the teaching of derivatives in calculus (cf. Berry & Nyman, 2003; Haciomeroglu & Andreasen, 2013; Hiyam et al., 2019; Rosly et al., 2020; Sari et al., 2018; Tall et al., 2008). Other educational technologies that have been used successfully in other studies to enhance students' learning about various calculus ideas including the derivative include MATLAB, Maple, and WolframAlpha, graphing calculators (cf. Leng, 2011; Wu & Li, 2017; Zbarsky et al., 2021). Given the multitude of benefits associated with using technology in the teaching of mathematics such as helping students visualize mathematical ideas, I recommend that calculus instructors take advantage of available opportunities and resources in their institutions or departments to integrate technological tools such as Desmos, WolframAlpha, and graphing calculators in their teaching of the concept of the derivative in calculus I.

Fourth, with the exception of the numerical representation of the concept of the derivative, a majority of the experts in this study reported using multiple representations of the concept of the derivative, namely algebraic-using explicit formulas, geometric/graphical-using graphs, and verbal-using verbal descriptions in their calculus I classes. On the contrary, evidence from research that has examined students' thinking about the concept of the derivative in multiple representations shows that the general inclination for students when working with derivatives or applications of derivatives is to use the algebraic representation of the concept of the derivative (cf. Berry & Nyman, 2003; Dawkins & Epperson, 2014; Engelbrecht et al., 2009; Habre & Abboud, 2006; Haciomeroglu et al., 2010; Ibrahim & Rebello, 2012; Thompson, 1994; Weber & Thompson, 2014). To some extent, and arguably, this may suggest that students' propensity to work with derivatives algebraically over other representations stems from other sources such as calculus textbooks, and not from classroom instruction. I thus recommend that calculus instructors not only use multiple representations of the concept of the derivative in their teaching of calculus I, but also create opportunities (e.g., in-class worksheets, homework assignments, projects, quizzes, and exams) for students to practice working with multiple representations of the concept of the derivative both during course lectures and outside the classroom.

Fifth, five of the 10 experts who participated in this study expressed appreciation for the abundance of examples, practice problems, and illustrative diagrams/graphs provided in textbooks they have used in their teaching of derivatives in calculus I. Furthermore, four other experts remarked that they have not observed any weakness in the presentation of the concept of the derivative in calculus textbooks they have used in their teaching of calculus I. As previously noted, mathematics textbooks, including calculus textbooks, play a crucial role in mathematics instruction, often determining what instructors teach, how they teach it, and ultimately what students learn during classroom instruction (cf. Robitaille & Travers, 1992; Wijaya et al., 2015). Consequently, I recommend that calculus instructors or textbook selection committees in mathematics departments consider, among other things, adopting textbooks that provide ample opportunities for students to work with the concept of the derivative in multiple representations and real-world contexts, respectively.

Study Limitations & Directions for Future Research

Finally, I discuss two limitations of the study. First, none of the experts were interviewed, something that would have enabled the experts to elaborate on some of their responses such as how they use graphing utilities such as Desmos and GeoGebra in their teaching of the concept of the derivative in calculus I. To alleviate this limitation, the interested reader is encouraged to see research by Liang (2016) who provide a detailed account regarding how calculus instructors could use Desmos to support students' learning about the concept of the limit, a foundational concept related to the concept of the derivative. The interested reader is further encouraged to see the work of Sari et al. (2018) regarding the use of GeoGebra to support students' learning about the derivative. Several other studies were cited before that provide detailed accounts of how to use other learning technologies such as Mathematica to support students' learning about the concept of the derivative (cf. Hiyam et al., 2019). Second, inter-rater reliability was not assessed, something that could have further strengthened the validity of the questionnaire used in the study. I recommend that future research that uses the questionnaire designed for this study (or its adaptation thereof) assesses both the validity (i.e., face validity and content validity) and the reliability (i.e., interrater reliability) of the questionnaire.

In light of the findings of the current study, future research might examine the effectiveness (or lack thereof) of the instructional strategies reported in this study in diverse instructional settings (e.g., large enrollment classes, recitation sections, discussion sections, and online classes) or with different student populations. Additionally, future research might examine the effectiveness (or lack thereof) of some of the promising instructional approaches related to the teaching of calculus that have been underexplored in the teaching calculus. These instructional approaches include the Modified Moore method, team-based learning, and project-based learning (cf. McLoughlin, 2009; Peters et al., 2020; Wu & Li, 2017).

Funding: No funding source is reported for this study.

Acknowledgements: The author would like to thank the experts who volunteered their time and expertise responding to the items included in the questionnaire. The author would also like to thank the three subject experts who assessed the face and content validity of the questionnaire used in the current study.

Ethical statement: The author stated that permission (Project Reference Number: 04364e) to conduct this study was granted by Miami University's Research Ethics & Integrity Program. Informed consent was sought and received via a consent form that was signed by each of the participants in the study.

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

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APPENDIX A

Questionnaire

1. According to Carnegie classification system, what is the rank (e.g., R1) of the institution you are affiliated with?
2. How many sections of calculus I have you taught?

   1 to 5, 6 to 10, 11 to 15, 16 to 20, or over 20
3. Drawing on your experience teaching calculus I, what have you found to be easy/straightforward for students when working with derivatives? Explain.
4. Drawing on your experience teaching calculus I, what have you found to be particularly difficult/challenging for students when working with derivatives? Explain.
5. What are some of the effective ways you have used to support students develop a solid understanding of the concept of the derivative in your calculus I classes? Explain.
6. What teaching technologies, if any, do you use to support students' learning about derivatives in your calculus I classes? How do you use these technologies? Explain.
7. What other tool(s), besides technology, do you use to support students' learning about derivatives in your calculus I classes? How do you use this tool(s)? Explain.
8. Which representation (algebraic-using explicit formulas, geometric/graphical-using graphs, numerical-using numeric tables of values, or verbal-using verbal descriptions) of the derivative do you use often in your calculus I classes? Explain.
9. What real-world contexts (e.g., kinematics), if any, do you typically use in your teaching of derivatives in calculus I? Explain.
10. Please state the title, author(s), and edition (if applicable) of the calculus textbook you have used in your recent teaching of calculus I.
11. What do you consider to be a weakness(es), if any, in how calculus I textbooks you have used present the concept of the derivative (e.g., in expository sections, examples, or exercises, respectively)? Explain.
12. What do you consider to be a strength(s), if any, in how calculus I textbooks you have used present the concept of the derivative (e.g., in expository sections, examples, or exercises, respectively)? Explain.

APPENDIX B

Effective Strategies for Teaching Concept of Derivative in Calculus

Table B1. Experts' responses to item 5 in questionnaire

Expert	Response to item 5-What are some of effective ways you have used to support students develop a solid understanding of concept of derivative in your calculus I classes? Explain.
E1	I tend to focus quite a bit on the meaning of a derivative and often come back to the definition of a derivative. I emphasize that a derivative is a quotient. We always have change in y (or another quantity) divided by change in x (or another variable). This tells us "how fast" y is changing as x changes. I provide a lot of class time for students to work on problems and explore ideas. I also assign quite a bit of HW [homework] where students can practice and/or extend their understanding.
E2	Practice more problems with different solution process.
E3	We have one day a week ("workshop") devoted to group problem solving, and when we are near beginning of derivative topic, I have students spend some of time solving verbal problems about interpreting derivatives. Not word problems, per se, but *explanatory* problems, e.g., here's a graph of $y = f(t)$. Suppose the graph represents a bear's weight at time t. Draw a line L_1 through $(1, f(1))$ and $(3, f(3))$, and a tangent line L_2 at $t = 4$. What aspect of the bear's weight does L_1 represent? L_2? Make a statement about line L_2 in terms of a derivative of f.
E4	I tend to rely mostly on real-life examples and problems that they can relate to. These real-life contexts provide opportunities to talk about units and how to interpret derivatives in meaningful ways. I try to use technology to help visualize some abstract ideas in relation to derivatives (e.g., continuity of functions).
E5	I've tried to always have them try to be more creative with how they approach topics in calculus. I've also stressed the importance on infinitesimals and infinity.
E6	From outset of covering differentiation, I compare average and instantaneous rates of change with driving a car or running a race. This real-life example is close to their prior experience, and they tend to see a reasoning behind why studying calculus is practical and interesting. This makes the more computational parts of the class more meaningful to them.
E7	We do a lot of in-class worksheets where they have to calculate average rates of change, and then discuss how to get better and better approximations. We have a nice worksheet that ties the rates of change to the slope of a line, and we have several exercises where they work through that and eventually develop the idea that taking the limit would give us the "best" answer.
E8	I provide students with a lot of practice. Lots of in-class and outside-of-class worksheets.
E9	**Emphasizing the importance of using proper notations (as in other topics) **Provide many repeated practice problems **Provide many repeated assignments that encourage them to memorize rules and use them.
E10	Group quizzes.

APPENDIX C

Additional Effective Strategies for Teaching Concept of Derivative in Calculus

Table C1. Experts' responses to item 7 in questionnaire

Expert	Response to item 7-What other tool(s), besides technology, do you use to support students' learning about derivatives in your calculus I classes? How do you use this tool(s)? Explain.
E1	A large portion of my class time is spent with students working on problems during class. I have developed several activities over the years that range from basic practice problems (derivative rules) to much more exploratory problems (inquiry oriented or active learning).
E2	1. Providing classwork for each topic taught & 2. Providing solutions online for each classwork.
E3	I do not use what you'd call tools. To me, issue with derivatives (in addition to making sure that students always know it is defined as a limit) is connecting the concept of rate of change to real world examples. I mainly take examples from the news. The last few years have been a rich trove of examples in terms of COVID-19 graphs.
E4	Real-life objects such as a description of driving between two towns sometimes come in handy. Sometimes I ask my students to come up with contexts in their own lives and they bring a lot of them. We explore these one by one and discuss how the knowledge of derivatives may be applied.
E5	I do not have many tools that I use that are not technology-related.
E6	Most of my lessons are guided notes/activities I've written to scaffold the day's lesson. It often begins with me introducing the topic, working out a few examples, then I let them work through several practice problems that gradually build in difficulty. After several minutes of them working collaboratively on the problems, I bring the class back together to discuss a subset of the problems they just did. The in-class practice lets them develop their skills while an expert(s) in the field is present to guide their learning.
E7	Lots of pictures. Drawing lots of secant lines. Both of us doing it and also having the students do it.
E8	In-class and out-of-class worksheets. Online homework. Quizzes. Exams.
E9	Nothing special tools other than practice problems
E10	I video record my lecture and post links on homepage.