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Changes in how prospective teachers anticipate secondary students' answers

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This study focuses on how prospective teachers learn about students' mathematical thinking when (i) anticipating secondary students' answers reflecting different characteristics of understanding and (ii) propose new activities in relation to the classification of quadrilaterals. The data were collected from forty-eight prospective secondary school teachers enrolled in an initial training programme. The results indicate three changes in how the prospective teachers anticipate secondary students' answers in relation to the role given to a perceptual or relational perspective of the classification of quadrilaterals. These changes are described considering how prospective teachers grasp the students' understanding of the inclusive relation among quadrilaterals as a conceptual advance. We argue that prospective teachers' learning was promoted after participating in a structured environment where they had the opportunity to discuss how to recognize the features of student's understanding.

Keywords: prospective teacher learning, student mathematical thinking, teacher knowledge of student thinking, teacher education

INTRODUCTION

Anticipating and interpreting students' mathematical thinking are teachers' teaching tasks in which teachers must generate hypotheses about how students' mathematical thinking can be developed. The ability to anticipate the possible responses of students with different characteristics of conceptual understanding is crucial in teaching practice. Stein and colleagues (2008) indicated that the ability to monitor group discussion in the mathematics class depends on the way in which the teacher anticipates the likely responses of students to highly cognitively demanding mathematical tasks. From another perspective, Ball and colleagues (Ball et al., 2008) have also stressed that the ability to anticipate student responses enables teachers to

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better select more suitable mathematical tasks. In this sense, Ball and colleagues (Ball, Thames, & Phelps, 2008) pointed out that:

> Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. […] Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking (p. 401)

This characterisation underlines the importance of teachers' anticipation of what students are likely to do when they have to solve certain mathematical problems in order to propose new activities to support students' learning. So, the anticipation and interpreting of students' mathematical thinking are key teaching tasks in which teachers must generate hypotheses about how students' mathematical thinking could be developed (Norton, McCloskey, & Hudson, 2011; Fernández, Llinares, & Valls, 2012).

Researchers have adopted several different approaches in order to determine how prospective teachers learn to identify evidence of students' mathematical understanding. The results of these studies provide insight into how prospective teachers learn to interpret students' mathematical thinking (Franke & Kazemi, 2001; Roth, Foote, Bolson et al., 2014; Sherin, Jacobs, & Philipp, 2010). A consequence of such research is that teacher trainers have begun to design resources that support prospective teachers' learning of how students learn mathematical concepts and how their understanding develops (Bartell, Webel, Bowen, & Dyson, 2013; Jacobs, Lamb, & Philipp, 2010; Sánchez-Matamoros, Fernández, & Llinares, 2015; Wilson, Mojica, & Confrey, 2013). One implication derived from these studies is that when prospective teachers are learning about students' mathematical thinking they should be able to recognize the role played by the understanding of specific mathematical elements for the students' conceptual progress (An & Wu, 2012).

In these contexts, how prospective teachers identify different features of students' understanding? and, how prospective teachers began to develop this competence? are relevant questions to better understand the prospective mathematics teachers' learning. Our study seeks to contribute to this emerging literature about prospective teachers' learning by describing changes in the way in which they learn to anticipate students' answers reflecting different characteristics of the conceptual understanding of a specific mathematics topic.

State of the literature

- Anticipating and interpreting students' mathematical thinking are teachers' teaching tasks in which teachers must generate hypotheses about how students' mathematical thinking could be developed
- In carrying out these activities teachers use different domains of Mathematical Knowledge for Teaching.
- When prospective teachers are learning about students' mathematical thinking they should be able to recognize the role played by the understanding of specific mathematical elements for the students' conceptual progress.

Contribution of this paper to the literature

- We identify three changes in how the prospective teachers anticipate secondary students' answers in relation to the role given to a perceptual or relational perspective of the classification of quadrilaterals when they are learning about the students' mathematical thinking.
- These changes are described taking into account how prospective teachers considered students' understanding of the inclusive relation among quadrilaterals as a conceptual advance (that is to say, as a Key Developmental Understanding).
- The recognition by prospective teachers of Key Developmental Understanding in mathematical topics could be considered as a benchmark in prospective teachers' learning about students' mathematical thinking.

THEORETICAL BACKGROUND

In the Mathematical Knowledge for Teaching framework (MKT), Ball and colleagues (Ball et al., 2008) describe Specialized Content Knowledge as the ways of knowing mathematics that are particularly useful in understanding students' mathematics and the Knowledge of Content and Students as knowledge of the ways students make sense of a particular mathematical idea. When prospective teachers learn about students' mathematical thinking they should be able to recognize the role played by the understanding of specific mathematical elements in the students' learning. In this context, the construct Key Developmental Understanding (KDU) proposed by Simon (2006) could be used to examine how prospective teachers relate the Specialized Content Knowledge and the Knowledge of Content and Students when they anticipate hypotheses about student' mathematical thinking. The Key Developmental Understanding involves "a conceptual advance on the part of students", that is, "a change in students' ability to think about and/or perceive particular mathematical relationships" (p. 362). Therefore, "a KDU in mathematics is a conceptual advance that is important to the development of a concept. It identifies a qualitative shift in students' ability to think about and perceive particular mathematical relationships, in other words, a significant change in the assimilatory structures that students have available (p. 364)". From this perspective, knowing Key Development Understanding of a mathematical concept could help prospective teachers to understand the conceptual advance of students, that is, it could help prospective teachers to understand the ways students make sense of particular mathematical ideas. Knowing what could be considered a Key Developmental Understanding of specific mathematical topics is in the intersection of the Knowledge of Content and Student and Specialized Content Knowledge. We hypothesize that if prospective teachers focus their attention on a Key Development Understanding of a particular mathematics topic, they can learn to make hypotheses about how students' mathematical understanding is developed. In this study, we focus on how prospective teachers consider the students' understanding of inclusive relation as a conceptual advance in the understanding of the quadrilateral classifications.

Understanding the inclusive classification and how it relates to the process of defining geometric figures can be understood as a conceptual advance on the part of students, that is, a change in students' ability to think about and/or perceive mathematical relationships key in the process of classifying quadrilaterals. Understanding the inclusive relations between the quadrilaterals implies a qualitative shift in students' ability to think about and perceive particular relationships among geometrical figures. The understanding of inclusive relations derived from adopting a relational perspective (relations between geometrical properties of the quadrilaterals) in the quadrilateral set and is therefore a key element when prospective teachers are learning about how students' mathematical understanding of classification of quadrilaterals is developed.

Research on students' understanding of classification processes underlines the key role played by the inclusive relations among quadrilaterals in how students come to understand the inclusive (hierarchical) and exclusive (partition) classifications (De Villiers, 1994). Inclusive classifications result when the application of classifying criteria to a specific set creates subsets in which it is possible to establish an inclusion relation (hierarchical chain) among its elements. For example, in an inclusive classification of a set of parallelograms, the square can be considered a special type of rhombus; while in an exclusive classification (partition) the square and the rhombus belong to separate groups. In this sense, understanding the inclusion relations of quadrilaterals is an important aspect in the development of students' geometric thinking (Usiskin & Griffin, 2008).

The objective of this research is to characterize how prospective teachers learn about students' mathematical thinking related to the classification of quadrilaterals. In particular, we pose the research question:

To what extent does identifying Key Development Understanding of a specific mathematical topic help prospective teachers to develop hypotheses about students' mathematical thinking?

METHOD

Participants and context

The participants were forty-eight prospective secondary school teachers enrolled in an initial training programme. The participants comprised mathematics and engineering graduates pursuing training to become secondary school mathematics teachers. The programme included subjects such as school organisation, psychology of instruction, mathematics education and teaching practice in secondary schools. In relation to mathematics education, the prospective teachers were studying a subject focused on the characteristics of secondary school students' mathematical understanding. This subject was taught for four hours a week, for thirteen weeks, and focused particularly on secondary students' mathematical understanding and how to select activities that would promote conceptual understanding. One of the learning environments of this subject was about secondary students' understanding of the classification of quadrilaterals.

The learning environment about students' understanding of the classification of quadrilaterals

The learning environment about students' understanding of the classification of quadrilaterals consisted of six sessions each lasting two hours (with a total of twelve hours), and an online discussion in which prospective teachers participated for 10 days. The design incorporated a socio-cultural perspective (Wells, 2002) and considered four aspects: Experience, Information, Knowledge Building and Understanding. ''Experience'' is the prior knowledge that prospective teachers have constructed during their participation in learning and teaching situations. ''Information'' consists of our understanding (as a scientific community) of how students understand the processes of quadrilateral classification (theoretical information) that we provided to prospective teachers. ''Knowledge Building'' is related to how prospective teachers engage in meaning-making with others in an attempt to extend and transform their understanding of a student's mathematical thinking and their own understanding of mathematics. Finally, ''Understanding'' constitutes the interpretative framework in terms of which prospective teachers make sense of new situations, that is, what they mobilise to develop hypotheses about how students mathematical thinking could be developed and justify the problems that they had to propose.

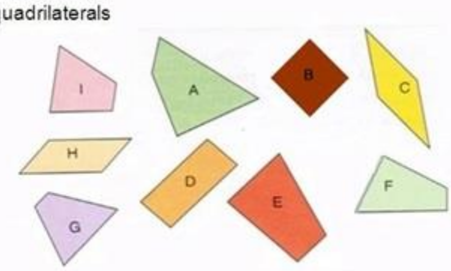
To begin with, the prospective teachers completed a professional task (task 1) individually. The aim of this task was to analyse in what extent and in which way the prior experience of prospective teachers helps them to anticipate students' answers. In the second session, prospective teachers were divided into groups of four to share and discuss their answers to task 1. In the third and fourth sessions, the teacher trainer presented and discussed information about secondary school students' understanding of the classification of quadrilaterals. The provided information was a synthesis and elaboration from knowledge of students' thinking on inclusive and exclusive classifications gleaned from research findings (Battista, 2007; De Villiers, 1994; Fujita, 2012; Usiskin & Griffin, 2008). In the fifth session, prospective teachers worked in groups to revise their initial responses to task 1 and identified changes based on the information about students' understanding of quadrilaterals classification (Knowledge Building). In the sixth session, prospective teachers completed a new task (task 2) individually. Lastly, the prospective teachers took part in an online debate lasting ten days which was aimed at reaching a consensus on the answer to task 2 and producing a summary report. We used the summary report from this last task to analyse how prospective teachers identify and use the students' understanding of inclusive inclusion as a Key Development Understanding of the classification of quadrilaterals to anticipate students' answers and provide activities to improve students' understanding. The structure of this learning environment allows prospective teachers to begin to use the language that enabled them to share with peers. In other words, one of the aims of the learning environment was that prospective teachers appropriated a professional discourse about students' understanding of the classification of quadrilaterals and about how to make teaching decisions to promote students' conceptual development.

Task 1

Task 1 consisted of two quadrilateral classification textbook problems (ages 14-15) from secondary school textbooks (Figure 1), and six professional questions aimed at prompting prospective teachers to (i) anticipate the response of students reflecting

Problem 1

Which of these quadrilaterals



have the properties described below:

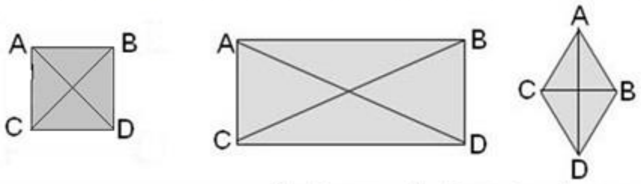
a. Only have two parallel sides

b. Have two parallel sides and two right angles

c. Their sides are equal, but their angles are not

Problem 2

2.1 Look at the figures and complete the table below, putting YES or NO in the empty boxes:



	Square	Rectangle	Rhombus
The four sides are equal			
The four angles are equal			
The opposite sides are equal			
The diagonals are equal			
The diagonals intersect at the middle point			
The diagonals form right angles			

2.2. Indicate the similarities and differences of the diagonals when comparing:

a) rectangle and square

b) rhombus and rectangle

c) rhombus and square

2.3. How are the parallelograms classified when using the diagonals as criteria? Justify the answer.

Figure 1. The two quadrilateral classification problems from secondary school textbooks used in task 1

different characteristics of conceptual understanding and (ii) provide activities/problems to improve their understanding. The six professional questions were:

A1. Anticipate what Maria, a 3rd year secondary school student (aged 14-15), would have to do and say in each problem in order to demonstrate that she has achieved the learning objective assigned for the problem (classify quadrilaterals using different criteria).

A2. Explain which aspects of Maria's answer to each problem make you think that she understands the process of classification of quadrilaterals. Explain your answer.

B1. Anticipate what Pedro, another 3rd year secondary school student (aged 14-15), would have to do and say in each problem in order to demonstrate an understanding of certain elements of the classification of quadrilaterals but he does not yet understand the classification of quadrilaterals (learning objective). Explain your answer.

B2. Explain which aspects of Pedro's answer to each problem make you think that he does not understand the process of classification of quadrilaterals. Explain your answer.

C. If you were the teacher of these students,

How would you modify/extend these problems in order to confirm that Maria has achieved the intended learning objective? Explain your answer.

How would you modify/extend these problems so improve the Pedro's understanding about the classification of quadrilaterals? Explain your answer.

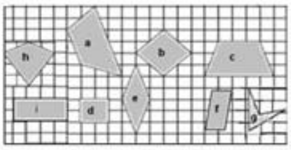
The first four questions professional questions referred to the teacher's ability to anticipate what students might be thinking when they solved the problems (anticipating the answers of students with different characteristics of understanding of the classification of quadrilaterals). These questions asked the prospective teachers to develop hypotheses about the students' mathematical understanding. The last two professional questions were related to teaching decisions that the teachers should make in order to promote student conceptual development.

The textbook problems used in the task 1 asked secondary students to examine the properties of parallelograms and classify them using several criteria. The figures that appeared in the problems from textbooks were "prototypical figures" providing information as, the four sides and angles are equal in a square, the four sides are equal and the opposite angles are equal in a rhombus, and the four angles are equal and the opposite sides are equal in a rectangle, but without using quantifiers. The problem 1 asks students to classify a set of nine quadrilaterals using three different criteria. The different items could be resolved by identifying the figures that met a criterion and grouping them together (Usiskin & Griffin, 2008). The items in this problem could be solved by considering the use of quantifiers (only two parallel sides, the number of equal sides, the angles) and the existence of parallel sides and the measure of the angles.

The problem 2 asks students to identify various characteristics of figures and to classify parallelograms using the diagonals. The goal of this problem was to explore the properties of parallelograms. To solve item 2.1, students had to consider the existence of quantifiers (number of sides, and angles), the measure of the sides and angles (for example, the existence of right angles), the relation between the sides (in a parallelogram, opposite sides are equal), if the diagonals intersect at the middle point, and use the words "square", "rectangle" and "rhombus" to designate the figures. In order to solve item 2.2, it was necessary to identify similarities and differences between the parallelograms. The aim of this item was to demonstrate how students establish relationships between the elements and properties of the different figures. Furthermore, classification using the diagonals (item 2.3) illustrated how students use the relationship between the different elements to make classifications.

Problem 1

1.1. Classify the following quadrilaterals according to the congruence of all their sides and angles

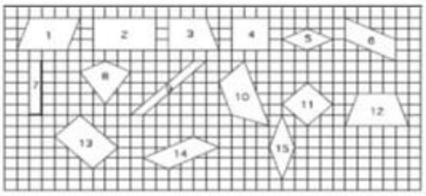


		All the angles are equal	
		YES	NO
All the sides are equal	YES		
	NO		

1.2. Answer the following questions:
A. Are there any quadrilaterals in which all the sides and angles are equal? Which ones?
B. Are there any quadrilaterals in which all the sides are equal but not all the angles are equal? Which ones?
C. Are there any quadrilaterals in which all the angles are equal but not all the sides are equal? Which ones?
D. Are there any quadrilaterals in which all the sides and angles are unequal? Which ones?

Problem 2

Which of the following quadrilaterals:



A) Are parallelograms? Explain your answer.
B) Are rhombuses? Explain your answer.
C) Are rectangles? Explain your answer.
D) Are squares? Explain your answer.

Figure 2. The two quadrilateral classification problems of task 2 (adapted from the questionnaire developed by Fujita, 2012; p. 63).

Task 2

Task 2 also consisted of 2 quadrilateral classification problems (Figure 2) and the same six professional questions.

The two problems in task 2 were adapted from the items used by Fujita (2012). These items were designed to determine students' cognitive development in relation to understanding the inclusion relations of quadrilaterals. The two problems in task 2 intended to activate secondary students' knowledge about parallelograms in terms of their images and their definitions, enabling them to consider parallelograms in terms of their properties. The means by which students could solve these problems makes it possible to determine how they learn to identify quadrilaterals through prototypical examples and how they use their properties to explain relationships between quadrilaterals. In this context, the relationship between prototypical examples and definitions is crucial to identifying characteristics of conceptual understanding of the inclusion relations of quadrilaterals. The aim of problem 1 was to classify nine quadrilaterals in terms of the congruence of their sides and angles. The aim of problem 2 was to activate knowledge related to the understanding of inclusive classifications of quadrilaterals. The different items attempted to demonstrate that two figures corresponding to different prototypical examples could share several characteristics. In this case, secondary school students who identified all parallelograms correctly would be able to identify rhombuses, rectangles and squares as parallelograms and establish an inclusion relation (sets in items B, C and D are subsets of the set obtained in item A). For these features, we conjectured that these problems are suitable as a context for prospective teachers to think about students' understanding of quadrilateral classification.

Analysis

The data analysed in this study were the prospective teachers' written answers to tasks 1 and 2, both individually and in groups (face-to-face and virtual). Data analysis was carried out in two phases by four researchers, using a constant comparison method (Strauss & Corbin, 1994). In the first phase, we identified what prospective teachers considered evidence of different characteristics of conceptual development and the activities proposed to support the students' progress in the understanding of the classification of quadrilaterals (in tasks 1 and 2).

In the second phase, we focused on how prospective teachers changed through their participation in the learning environment in the way they considered the students' understanding of inclusive relation as a conceptual advance to think about and/or perceive particular mathematical relationships among the quadrilaterals. The objective of this second phase was to identify changes in how prospective teachers anticipated students' answers and understood how students' conceptual understanding of the classification of quadrilaterals could be developed. We identify three changes that reflected the way in which prospective teachers employed inclusive relations to make hypotheses about secondary school students' conceptual understanding and how selected activities that supported such development. These changes were characterized through the way in which prospective teachers considered the students' understanding of inclusive classification as a "Key Developmental Understanding," and are described in the results section.

RESULTS

The results are organized into three sections, each of which describes a change in how prospective teachers identified students' understanding of inclusive relationships as a Key Developmental Understanding of the classification of quadrilaterals. In task 1, prospective teachers reflected three different standpoints. For one group of prospective teachers, understanding the classification of quadrilaterals was linked to identifying and defining prototypical figures considering all the properties that distinguished them from one another. These generated singleton subsets without relations between them. This group of prospective teachers supported their arguments from a perceptual perspective. The second group of prospective teachers held the understanding of the classification of quadrilaterals was linked to being able to form non-singleton sets but without specifying any relationship between figures within a set. Lastly, a third group of prospective teachers linked understanding of the classification of quadrilaterals to students' ability to establish relationships between some properties of quadrilaterals. This third group of prospective teachers used a relational perspective to characterize the students' understanding of classification processes of quadrilaterals. This made it possible to link students' understanding of the classification of quadrilaterals to the ability to recognize, for example, that squares can be considered a particular type of rhombus.

By the end of the learning environment, some prospective teachers became aware that a student's ability to establish inclusion relations between the figures within a set constituted evidence of a better understanding of the classification of quadrilaterals (changes 2 and 3, Figure 3). However, this change did not occur identically for all the prospective teachers. During the working sessions, discussions arose about the significance of forming sets of several figures that shared a given property but were perceptually different (e.g. placing rhombuses and squares together because they were quadrilaterals with four equal sides). However, even after completion of the tasks and discussions in small and large groups, one group of prospective teachers still did not recognize the role played by the students' understanding of inclusive relations as a Key Developmental Understanding of the classification of quadrilaterals (change 1). The three changes are described below, together with an explanation of why some prospective teachers did not recognize students' understanding of inclusive relations as a Key Developmental Understanding.

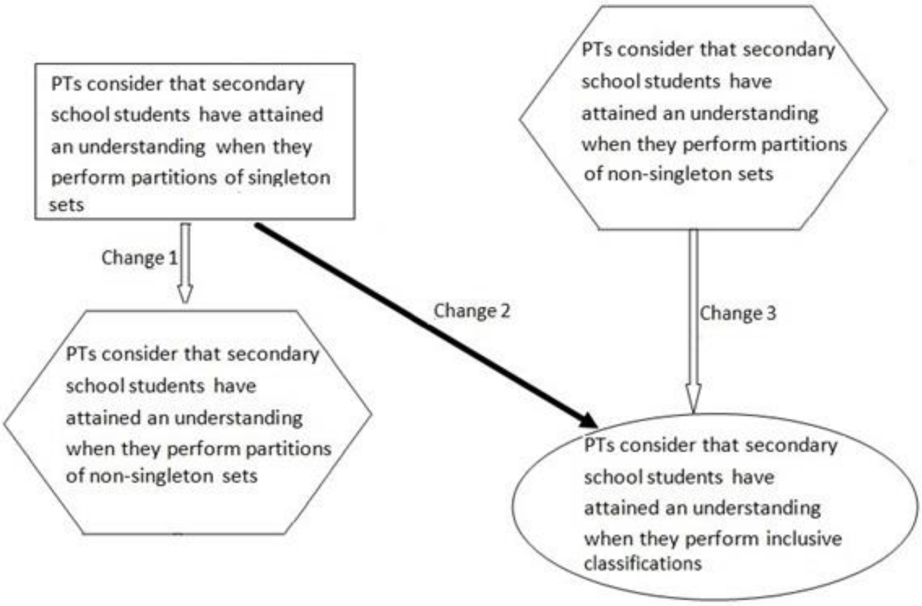


Figure 3. The changes identified that characterized prospective teachers' learning

Change 1

From considering singleton sets to considering partitions formed by non-singleton sets

Here, prospective teachers initially considered that secondary school students' understanding of the process of classification was evidenced using prototypical figures. Thus, each prototypical figure was defined by a set of properties that distinguished it from the others. With this approach, prospective teachers did not consider it relevant to establish relationships between the properties of quadrilaterals. Thus, squares and rhombuses were unrelated figures.

Subsequently, some of these prospective teachers came to consider that secondary school students' understanding of the classification of quadrilaterals was evidenced when they could identify sets of figures that included different figures which nevertheless shared several properties. In this situation, the prospective teachers did not consider it relevant to establish an inclusion relation within these sets of figures, and thus they did not recognize a square as a particular type of rhombus.

For example, when initially anticipating student responses (task 1), prospective teacher 17 (PT17) considered that the understanding of the classification process was evidenced by the use of two characteristics (length of the diagonals and the angle they formed) applied to the set of parallelograms to perform classifications that enabled the generation of a classification of singleton subsets without establishing relationships between the figures and generating unrelated sets of figures. Thus in the task 1, when anticipating a possible response, he considered that the use of diagonals as a criterion (item 2.3 of problem 2) would reflect a whole understanding. For example, PT17 stated:

> [the student] should put that the criterion for classifying the parallelograms according to their diagonals is [that the diagonals have] the same length, and that the angle that they form [is equal]. The diagonal in the rectangle and in the square has the same length but they differ in that the diagonals of the square form a right angle but those of the rectangle do not.

Furthermore, when giving an example of a student's response that might reflect incomplete understanding of the process of classifying parallelograms, PT17 indicated that this would be evidenced when students had difficulties in visualising some elements of the figures (equal sides, angles and diagonals, point where the diagonals intersect). For example, in the item 2.1. of problem 1 (Figure 1), in which it was necessary to identify the properties of three parallelograms and complete a table, PT17 indicated:

> When completing the table for problem 2, students may encounter difficulties such as the following:
> -They may not see the 4 sides of the rhombus as equal
> -They may see the 4 angles of the rhombus as equal
> -They may not see the opposite sides of the rhombus as equal
> In short, they may experience problems in classifying the rhombus...

These difficulties will render them practically incapable of answering the questions of item 2.2 (indicate similarities and differences between the parallelograms and classify them using the properties of the diagonals).

The prospective teachers in this group felt that difficulties in identifying the elements of the figures (equal opposite sides, diagonals that intersect or not at the middle point, the angle formed by the diagonals) might reflect an incomplete understanding. As a result, they argued that if students identified these properties in the figures, they would therefore be able to classify quadrilaterals correctly.

In response to task 2, some of these prospective teachers began to use the words "inclusive classifications" but without clearly specifying in the anticipated responses what this might be. For example, in the problem of identifying rhombuses and rectangles in a set of 15 figures (problem 2, items B and C; Figure 2), PT17 argued that students should be able to see the figures forming the sets, and thus a conceptual advance in the students' understanding of the classification of quadrilaterals was the ability to identify subsets of quadrilaterals. However, by not explicitly considering relations among properties of quadrilaterals, these prospective teachers were actually referring to partitions in the set of quadrilaterals from a perceptual perspective. Thus, PT17 stated:

> For classification, we will consider an inclusive classification, in other words, we will not make a distinction between squares and rhombuses, and we will not make a distinction between rectangles and squares; therefore, we will consider that the student will perform inclusive classifications...

In these cases, the prospective teachers seemed to continue relying on the "appearance"; as PT17 said, "They would not make a distinction between rectangles and squares", generating partitions of the set of quadrilaterals. At this point, these prospective teachers considered, for example, that the figures 4 and 5 in problem 2 could belong to the same set, indicating that they shared a property (e.g., equal sides).

Emphasis was placed on the fact that they might share a property, but no relationship was established between the properties in the explanations given.

The emphasis on the perceptual and not about the relational perspective was reflected in the type of activities that they proposed to promote student conceptual development. For example, PT17 proposed an activity based on visual recognition of properties without mentioning the relation between the figures' properties that would help students advance in their conceptual understanding of the classification of quadrilaterals. The proposed activity (Figure 4) focused on the identification of two properties (equal sides and equal opposite angles) in a set of figures. The logical conjunction "and" employed in this activity required the consideration of two conditions at the same time, and therefore generated a partition. Thus, the learning objective was to create partitions in the set of parallelograms by identifying sets containing more than one figure, but without establishing a relation between the properties of the figures. This is evidenced by the fact that the prospective teachers did not use the word "rhombus" to refer to the set formed by the two prototypical figures of the square (4 and 11) and the rhombus (5 and 15). In this situation, prospective teachers recognised that the ability to perceive that some figures shared the same property and could thus be grouped was an important element in conceptual understanding of the classification process. One example would be having equal sides, in the proposal made by PT17 (Figure 4). These prospective teachers avoided defining the square as a particular kind of rhombus, which would determine the understanding of inclusive relations as a Key Developmental Understanding in learning classification processes. Here, the emphasis was placed on the identification of properties and not on recognition of relationships between the figures. For example, PT17 proposed a recognition activity as a means to support the progress in conceptual understanding of classification (we assume that the figures provided by this prospective teacher are the prototypical ones given in textbooks: a square, a rhombus and a parallelogram):

> "If we define the rhombus as a quadrilateral with 4 equal sides and equal opposite angles, which ones are rhombuses?"

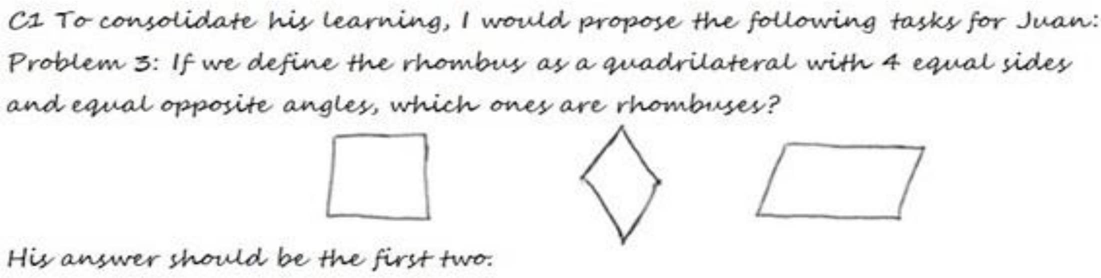


Figure 4. PT17's proposal to consolidate learning of the classification process [translated from the original].

From a mathematical point of view, the second part of the definition "and equal opposite angles" is redundant. However, the definition provided by this prospective teacher allows considering "the rhombuses and squares" in the same set. So, he considered that a partition could be formed by non-singleton sets but without considering the inclusion relation between the square and the rhombus.

Although the structure of the learning environment allows prospective teachers to begin to use the language that enabled them to share with peers in large group discussions (professional discourse), they did not go beyond the rhetorical use of theoretical terms. In the last example, PT17 used the term "inclusive classification" for referring to the squares and rhombuses set but without identifying the inclusive relation between squares and rhombuses.

Change 2

From singleton sets to considering "inclusive relations"

The prospective teachers experiencing this change initially occupied the position described earlier (from singleton sets) and after participating in the learning environment came to recognize that understanding the inclusive relationships was a Key Developmental Understanding in the progress of conceptual understanding of quadrilaterals classification. Thus, these prospective teachers became aware that understanding was evidenced when students were capable of setting relationships between properties of figures.

The prospective teachers experiencing this change became aware (in task 2) that students' understanding was evidenced by the use of criteria that generated non-singleton subsets, thus establishing inclusion relations within a set. In other words, they came to recognize that understanding of classification was evidenced when students considered that according to certain criteria, a square can be a particular kind of rectangle. Thus, the example they gave of a student's response to item c of problem 2 (are they rectangles? Explain your answer") that could show the student's understanding of the classification process was:

> Figures 2, 4, 11, 7 and 13 are rectangles because they all have right angles.

This response shows that students have understood the classification of quadrilaterals because they have used inclusive definitions, for example in item c of problem 2, inncluding the squares as rectangles (PT8, professional task 2).

In this response, these prospective teachers explicitly established that the consideration of inclusive relations ("...including the squares as rectangles") was an indicator of the students' understanding of the classification. This perception was based considering that the use of a sole criterion could be considered an indicator of understanding the classification of quadrilaterals, e.g. "all the angles are right angles", which leads to a classification in which rectangles and squares are grouped in the same set and which furthermore specifies the inclusion relation "including squares and rectangles".

Similarly, when PT8 anticipated a response to the problems in task 2 reflecting an incomplete understanding of the classification of parallelograms, considered that this was evidenced by the student's inability to generate inclusive classifications that permitted definitions whereby a square could be considered a rhombus. For example, when anticipating a student's response to item c of problem 2, PT8 stated that "2, 7 and 13 are rectangles because "Uall their angles are right angles and two pairs of sides are equal (the opposite sidesU)", explaining that "this makes me think that Isabel [the secondary student] has not reached the objective because in problem 2 she has not performed an inclusive classification since she has not considered squares as rhombuses or rectangles". In other words, PT8 considered that not generating inclusive classifications demonstrated an incomplete understanding of the classification process.

In this change, prospective teachers became aware that understanding the inclusive relations was an indicator of the students' progress in the conceptual understanding of the classification of quadrilaterals.

Change 3

From recognising partitions with non-singleton sets to considering inclusive relations.

At the beginning of the learning environment, some prospective teachers considered that students' understanding of the classification of quadrilaterals was evidenced when students grouped different figures that shared some properties but without establishing inclusion relations between them. Initially, these prospective teachers did not mention the relations between properties as a means for identifying different levels of conceptual understanding of the classification of quadrilaterals. For example, these prospective teachers indicated that a student response reflecting an incomplete understanding was when the students would only be able to identify singleton subset classifications:

> ...an incorrect response to item b (of problem 1) would indicate to me that although the students recognise parallel sides and equal angles (since they have responded correctly to a and c), they are unable to understand what happens in item b... based on this criterion, defined in b, squares and rectangles may correspond to the same classification.

Subsequently, in task 2, these same prospective teachers became aware that understanding the classification of quadrilaterals may be evidenced when students identified inclusion relations within a set. In other words, these prospective teachers focused on the understanding of the relations between properties as evidence of understanding classifications. Thus, a hypothetical response to items B and C of problem 2 was:

> (a possible correct answer would be) 4, 5, 11 and 15 are rhombuses. Note that 4 and 11 can also be sub-classified as squares, considered as special cases of rhombuses in which all of their angles are equal.
> 2, 4, 7, 11 and 13 are rectangles since their opposite sides are parallel and all their angles are equal. NNote that this includes 4 and 11, which are squares, particular kinds of rectangles in which all sides are equal (PT23, task 2).

Similarly, a students' hypothetical answer reflecting an incomplete understanding of the classification of quadrilaterals was based on the non-recognition of inclusive relations in one of the sets of parallelograms. Thus, in relation to item D of problem 2, a hypothetical answer which would demonstrate this lack of understanding was as follows:

> Item a) The quadrilaterals 1, 6, 9, 13 and 14 are parallelograms, since they have two pairs of parallel sides and their opposite angles are equal.
> Item b) The quadrilaterals 5, 11 and 15 are rhombuses because their sides are equal.
> Item c) The quadrilateral 2 is a rectangle because it is the only one with two pairs of equal sides and right angles.
> Item d) The quadrilateral 4 is a square because it is the only one with four equal sides and equal angles which are right angles.

In problem 2, [the student] has demonstrated that he is not capable of performing inclusive classifications or recognising when Ua property is the consequence of another. For example, in item d): The quadrilateral 4 is a square because it is the only one with four equal sides and equal angles which are right angles.

In this case, the learning objective established by the prospective teachers was to understand the relationships between the figures' properties, thus focusing attention on hierarchical classifications and "how properties can be the consequence of others". This way of proceeding displayed how the prospective teachers became aware that understanding inclusive relations was an indicator of the progress of conceptual understanding of the quadrilateral classification.

DISCUSSION

This study provides information on how prospective teachers learn about students' mathematical thinking. The results indicate that prospective teachers' learning was not uniform since they modified what was considered evidence of students' understanding, and as a result, changed their teaching decisions about how they could promote the conceptual advance on the part of students of understanding of the classification of quadrilaterals. We identified three changes related to the role given to a perceptual or relational perspective in the understanding of quadrilaterals classification. Although most prospective teachers used the students' understanding of inclusive relations as an indicator of the progress of students' conceptual understanding, there remained a group of prospective teachers who made rhetorical use of the relational perspective. For the latter prospective teachers, the development of secondary school students' conceptual understanding was linked to visual recognition of the figures' properties without underscoring the relationship between them. In these cases, the prospective teachers relied on visual aspects and prototypical examples linked to the definitions in order to generate indicators of understanding, and were therefore unable to consider the understanding of relational perspective as a learning goal. These prospective teachers did not recognize the students' understanding of the inclusive relations as a Key Development Understanding. One possible explanation for this fact could be that prospective teachers had a weak understanding of the classification of quadrilaterals, which is what underlines the relation between the specialized mathematical knowledge and the content and students' knowledge in prospective teachers' learning, as other researches have also pointed out for different mathematical topics (Fernández, Llinares, & Valls, 2013; Magiera, van der Kieboom, & Moyer, 2013; Wilson et al., 2014). From these results, the recognition of Key Developmental Understanding (Simon, 2006) in specific concepts could be considered as a benchmark in prospective teachers' learning about students' mathematical thinking.

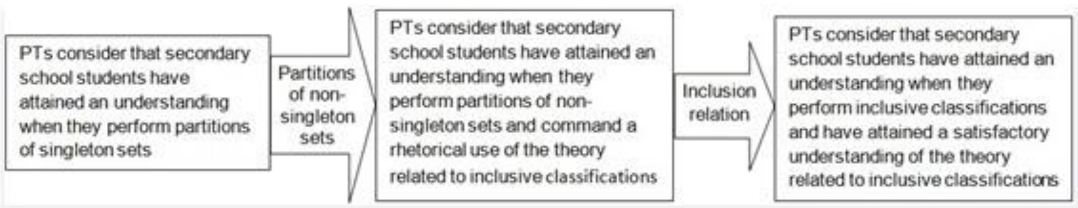


Figure 5. Benchmarks in prospective secondary school mathematics teachers' hypothetical learning trajectory of students' understanding of the classification of quadrilaterals.

In this sense, the results suggest that recognition of students' understanding of inclusive classifications as a Key Developmental Understanding enabled prospective teachers to construct a point of reference for anticipating the possible responses of students with different characteristics of conceptual understanding of the classification of quadrilaterals.

Finally, we think that prospective teachers' learning was promoted by participation in a structured environment where they had the opportunity to discuss how to recognize the different characteristics of understanding from the students' responses. The learning environment provided prospective teachers with the language necessary for talking about students' mathematical thinking. The use of this specific language enabled them to discuss particular aspects of secondary school students' behaviour when talking about the conceptual understanding. In this case, the explicit identification of students' understanding of inclusion relations as a point of reference when talking about understanding the classification of quadrilaterals (in contrast to adopt a perceptual perspective) enabled prospective teachers to provide more detailed description of students' hypothetical answers.

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