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Cognitive obstacles in the learning of complex number concepts: A case study of in-service undergraduate physics student-teachers in Zimbabwe

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Abstract

Physics and mathematics are interrelated as part of the science, technology, engineering, and mathematics (STEM) disciplines. The learning of science is supported by mathematical skills and knowledge. The aim of this paper is to determine the cognitive obstacles of in-service undergraduate physics student-teachers' understanding of the concept of complex numbers which is part of linear algebra. A case study is presented involving 10 undergraduate student-teachers at a university in Zimbabwe studying for a Bachelor of Science Education Honours Degree in physics. Data were generated from the 10 participants' answers to structured activity sheets and interviews. Action, process, object, schema (APOS) theory was used to explore the possible ways that students may follow to understand the concepts of complex numbers and how they concur with the preliminary genetic decomposition. It was observed that most of the participants were operating at the action level, with a few operating at the process and object levels of understanding. Recommendations are made in this study that instructors should pay more attention to the prerequisite concepts and the "met afters" so that students can encapsulate processes into object understanding of division of complex numbers and polar form.

Keywords: APOS theory, genetic decomposition, complex numbers, case study

INTRODUCTION

From our experience as mathematics lecturers, students' learning of linear algebra at university level is a complex process. It is considered the first course of advanced mathematics at university level meant to transfer students in their way of thinking from school mathematics towards advanced mathematical thinking (Klapsinou & Gray, 1999). This is also supported by Salgado and Trigueros (2015), who said that linear algebra has become a mandatory course in many undergraduate degrees since it is widely recognized to have many important applications in different disciplines. Trigueros (2019) also commented that linear algebra is a mandatory course for STEM students. Hannah et al. (2016) referred to linear algebra as an abstract course because students find the definitions, theorems, and proofs difficult to handle. However, Wawro (2014) believed that the shift from the learning of elementary mathematics to advanced mathematics is rather difficult for university students. Despite its importance, there are many reports in literature about students' difficulties with the learning of linear algebra concepts. Evidence of student difficulties with linear algebra has been documented over the years in several studies (e.g., Bogomolny, 2007; Hillel, 2000; Kazunga & Bansilal, 2020; Maharaj, 2015; Mutambara & Bansilal, 2021; Ndlovu & Brijlall, 2015, 2016; Stewart & Thomas, 2019). For instance, Hillel (2000) regarded the learning of linear algebra at university as a frustrating experience to both the lecturer and the student. These experiences have shown that within the study of linear algebra, the work is more abstract to the students and the lecturers, and that it is also harder to manipulate symbols. Ndlovu and Brijlall (2016) believed that students' poor performance in linear algebra is connected with the lack of conceptual understanding, and it is more of procedural understanding as the students' construction of knowledge is based on isolated facts. The finding concurs with literature in linear algebra by Triguerous (2019) who argued that when students are learning linear algebra, they struggle to manipulate the conceptual oriented questions. Ndlovu and Brijlall (2015) supported the preceding literature finding and

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Contribution to the literature

- The study explored the cognitive difficulties displayed by in-service undergraduate physics student-teachers when learning concepts on complex numbers.
- The study contributes insights on the use of APOS theory as framework in identifying in service undergraduate physics student-teachers' understanding of the complex numbers concept.
- The findings have implications on the teaching of mathematics to support the preparation of teachers who teach STEM disciplines.

said that mathematics instructors must pay attention to their students' understanding of interrelationships between concepts, rather than focusing on procedures. Widiyatmoko and Shimuzu (2018) commented that in order for students to achieve conceptual understanding in science, meaningful learning is achieved by using an integrated approach. Murphy et al. (2018) also said that deep conceptual understanding is also achieved if students engage in scientific argumentation.

Therefore, to contribute to an additional layer in the understanding of the teaching and learning processes in linear algebra, we propose incorporation of a theory of learning in mathematics called the action, process, object, schema (APOS) theory. The theory focuses on how students construct different mathematical concepts, and we propose pedagogical actions that can stimulate the learning process. One of the researchers while teaching undergraduate mathematics and physics students at a university in Zimbabwe, also noticed students' poor performance when manipulating complex numbers. There is a need to embark on a study on how Zimbabwean undergraduate student-teachers understand the concept of complex numbers. In this study, we therefore consider in-service undergraduate physics student-teachers. It is important for these teachers to possess sound mathematical content knowledge (conceptual understanding) since the learning of mathematics facilitates the learning of physics. Maths is a tool to answer physics problems. Wan et al. (2019) pointed out that complex numbers are important for the study of quantum physics, where the imaginary numbers are useful for modelling periodic motions. Wan et al. (2019) further suggested that curriculum planners should design questions that elicit students' insight about the important role complex numbers play in quantum mechanics. Foster (2014) asserted that if a teacher has a conceptual understanding of mathematics and the potential to motivate students in terms of data acquisition, they will influence classroom instruction in a positive way.

To explore in-service undergraduate physics student-teachers' mental constructions, the following research questions were formulated:

1. What are some cognitive obstacles encountered by in-service undergraduate physics student-teachers when constructing complex number concepts?
2. How do in-service undergraduate physics student-teachers' mental constructions of complex numbers link with the preliminary genetic decomposition?

The study particularly focuses on the linear algebra concept because of its relevance in a variety of situations.

LITERATURE REVIEW

Complex numbers are one of the most important concepts in mathematics education (Anevska et al., 2015). However, some researchers have considered it to be one of the most difficult concepts students encounter when learning linear algebra. Nordlander and Nordlander (2012) noted that the study of complex numbers is associated with a variety of difficulties and misconceptions that are linked to its nature and terminology. The results of his study revealed that students experienced challenges in discerning the basic properties of complex numbers. According to Anton (2010), a complex number is an expression of the form $a + bi$ or $a + ib$, where a and b are real numbers and i is the imaginary unit. Ahmad and Shahrill (2014) further elaborated that when learning concepts on complex numbers, students will encounter new terms such as imaginary, conjugate, argand diagram, and complex plane, which are difficult to comprehend. Students also encountered learning difficulties when learning the representation of the i^{th} notation for a square root of a negative number without a solution. Nordlander and Nordlander (2012) carried out a study to assess how Swedish students understand the concept of a complex number. Furthermore, Habre (2017) also noted that students had difficulties to switch from the Cartesian coordinate system to the polar coordinate system. This is because the students associate the plane with the Cartesian system. Moore et al. (2014) also noted that college students struggle to understand concepts on polar coordinate system and its application in the real-world.

Hillel (2000) believed that students encounter difficulties because they lack various proofing techniques and make hasty generalizations. Wawro (2014) in his studies posited that the content of linear algebra is highly abstract and formal for undergraduate students as compared to the computational mathematics that they were used to at elementary levels. The authors lamented, however, that definitions are very important, because they are a vehicle towards the understanding of given concepts and are rooted in the culture of the working mathematicians. Most students are asked to memorize the definitions in preparation for an exam and they perform badly if asked application questions.

It is important to note that the content on complex numbers is really embedded in definitions, where students memorize the terms without conceptual understanding attached to it in preparation for tests. In the end, if students perform badly, different stakeholders, such as the teachers, curriculum developers and subject advisors, will be worried, causing tensions and uneasiness amongst them (Siyepu, 2013).

Dorier and Sierpinska (2001) outlined that those students had challenges with linear algebra because of its abstract and formal nature. They further argued that the nature of linear algebra (conceptual difficulties) and the kind of thinking required (cognitive difficulties) are the two sources that causes students failure to understand linear algebra. Habre (2017) noted that research on students understanding of polar coordinates and graphing was scarce. Despite the importance of complex numbers in the study of physics, it is disappointing to note that students' performance in the concepts on complex remained constantly poor, from our experience when teaching these concepts. These facts on the difficulties outlined above and the need to add new points to existing literature motivated us to use APOS theory to see how undergraduate student-teachers understand the concepts of complex numbers. We also noted that very few studies focused on division of complex especially the idea on the use of radicals and polar form of a complex number. This study thus fills a gap in this research area.

Furthermore, Nguyen et al. (2020) noted that there is the need to make a strong connection between stem disciplines. In line with this view, Thibaut et al. (2018) argued that there is the need to provide students with a strong education in STEM and this should be a priority for all students. Mustafa et al. (2016) advocated for an integrated curricular such as STEM because students had better performance. This means that an interdisciplinary approach to the integration of STEM content is important. Students should integrate concepts across different representation. This supports Lev (2006) contention that the modern quantum theory assumes that it is represented by elements of a complex number. This shows that a real-life problem in physics has been solved in support with implicit connection to the other disciplines. Thus, the fact that complex numbers in physics are also used to model alternating current, this means that the complex numbers exist in the physical world. A study by Mynbaev et al. (2008) shows that a fundamental knowledge of physics and linear algebra are needed in engineering technology courses. He further advocated that engineering, technology and mathematics are interconnected and they work in accord.

THEORETICAL FRAMEWORK

The research is embedded in APOS theory, which dates back to the work of a research mathematician, Dubinsky (1984). It is a theorem of how mathematical concepts are learned. According to Arnon et al. (2014), the development of APOS theory is stimulated by Piaget's concept of reflective abstraction in child learning. Weyer (2010) added that APOS theory is an extension of the last stage of Piaget's theory, that is the formal operational stage. The acronym APOS stands for action, process, object, and schema. Maharaj (2015) outlined that APOS theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. Arnon et al. (2014) argued that in order to apply APOS theory to describe the particular constructions by students, one needs to develop a genetic decomposition. A genetic decomposition involves hypothesized detailed descriptions of the different levels of APOS which students might need, and which gives the researcher insight of how the students might construct mathematical knowledge in order to understand a certain concept and its relationships with the concepts (DeVries & Arnon, 2004). Each of the APOS mental structures is defined below as well the way it is constructed.

Initially, a concept is first perceived as an *action*, externally requiring step-by-step physical or mental transformation of objects that need to be performed explicitly or from memory (Arnon et al., 2014; Dubinsky & MacDonald, 2001; Dubinsky & Wilson, 2013). Furthermore, the actions are based on rules and algorithms and take place without much thought, where a rule is practiced repeatedly until it becomes a routine (Brijlall, 2015). Weyer (2010) added that an individual can perform an action from memory without specific instruction so as to transform each object. Finally, an action also involves multiple-step sequence of responses. Second, Dubinsky and Wilson (2013) referred to a *process* as a mental structure that performs the same operation as the action but takes place totally in the mind of the individual. An *object*, according to Dubinsky and MacDonald (2001), is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it.

Finally, a *schema* involves many actions, processes and objects that are connected together in the mind of an individual to form a coherent structure (Dubinsky, 2004). The term *coherence* of the schema here refers to an individual's ability to ascertain whether the schema can be used to solve a mathematical problem or a particular mathematical situation, or which mental structures to use to do so. For this stage, Ndlovu and Brijlall (2015) outlined that one has to bring together various mental constructions and schemas to address the problem. An example here would be that if one needs to define the term complex number, the individual should bring together the various schemas that characterize what a complex number is, such as the real number, imaginary number, and the symbol i.

A pedagogical approach known as ACE teaching cycle (Activities, Class discussion and Exercises) is followed whenever the APOS theory is used, and this guided how teaching was carried out. Weller et al. (2003) said that in implementing the ACE teaching cycle the topic must be broken into subtopics which are then each exposed to an iterative approach of the cycle. Activities were designed in such a way that they develop mental structures as prescribed by the APOS analysis. Teacher's role was to ensure that activities were aligned to mathematical concepts under consideration. When students finished performing given tasks, they had time for feedback on the work they were doing. The second stage had discussions held amongst students themselves with teacher guiding the discussions. Students were given homework as follow up exercise to allow self-reflection on the work done to consolidate understanding of problems.

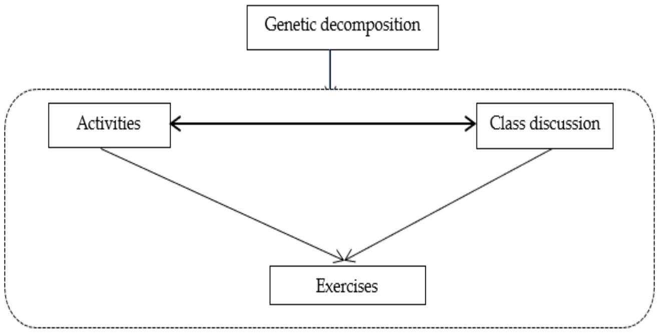

Figure 1. Relationship between ACE teaching cycle and genetic decomposition (Arnon et al., 2014)

METHODS AND CONTEXT OF THE STUDY

Research Methods

The study used the interpretive research paradigm to interpret data on how students construct knowledge when learning complex number concepts. Antwi and Hamza (2015) asserted that the interpretivist paradigm is concerned with an individual's understanding of the world around them, aiming at understanding the experiences of the individual. Antwi and Hamza (2015) further asserted that the main methods of data collection in interpretivism are interviews and participant observations, which they referred to as meaning-oriented methodologies. The researcher collects the data and later draws important information from it and makes inferences based on the patterns or themes emerging (Bertram & Christiansen, 2014). In this study, the type of data collected involves students' challenges in learning complex numbers. The methodological framework adopted for this study is a qualitative approach and a single case study was utilized.

The researchers have utilized a case study research design since involves the examination of one case in-depth and holistically. By its nature, qualitative research methodology also allows one to use different research strategies to collect data, of which structured interviews and an activity sheet were the main methods of data collection in this study. It is hoped that a diagnostic tool, the genetic decomposition, will provide insight and useful information on how the participants construct the necessary knowledge when learning complex number concepts.

Participants and Context of the Study

Bertram and Christiansen (2014) postulated that a sample is a smaller group, or a subset of a population selected for a scientific inquiry. Ten second year in-service teachers studying for a Bachelor of Science Education Honours Degree in physics were selected for the study using purposive sampling. Cohen et al. (2011) asserted that participants are chosen on the basis of possessing a characteristic that is needed for the study, with in-depth knowledge about particular issues. These students were practicing teachers who were teaching physics at Ordinary level and were upgrading their qualifications so that they can teach physics at Advanced level. The participants were practicing teachers who held a diploma in education from various secondary teachers' colleges in Zimbabwe. The highest academic levels of the participants were either advanced level or ordinary level. These students were studying physics, but they also take two courses that are taught in the mathematics department. The first course in mathematics is calculus 1 (MT101), which is done in first year with other physics module and the compass wide courses. The second course in mathematics is done in part two that is a course on linear algebra (MT102) which covers concepts on matrices, vectors, and complex numbers. The design of the program was such that teachers would complete the equivalent of an undergraduate three-year degree program except that the lectures were presented during the school holidays.

Data Collection

The activity, classroom discussion and exercise (ACE) teaching cycle guides the development of instructional treatment and is an instructional approach that helps to develop the mental constructions. The diagram (**Figure 1**) summarizes the activities that are involved in the ACE teaching cycle according to Arnon et al. (2014).

The instructional design based on APOS theory was used in this study. This included activities, classroom discussions and exercises done outside of the classroom. The students were first taught the concepts by one of the researchers. After all concepts on complex numbers were taught, a worksheet was set on the concept on complex numbers. This is in line with Stewart and Thomas' (2010) study that students should first understand the basic concepts. The participants were paired up and the task was designed to develop the mental constructions suggested for by the genetic decomposition, and this formed the first step of the ACE teaching cycle.

As participants were working in pairs, the first researcher observed and took note of the grey areas that needed further explanations. During class discussions, the participants were offered the chance to present the work done in pairs on the chalkboard whilst the other participants listened and made the necessary comments where possible. The researcher offered input so as to highlight the major points on the concepts on complex numbers, and to expose participants to the various methods used to answer these questions. Exercises in the form of homework were given to strengthen and supplement the ideas that they had already constructed. Three days after embarking on the tutorials, an activity sheet was administered to the ten participants. The sheet clearly instructed them to show all their working and to provide solutions with justifications. Six items were set on the concepts on complex numbers and were written individually to see how the participants constructed their own knowledge. After the scripts were marked, some participants volunteered to be interviewed, and follow-up questions were asked depending on the work they had written. The interviews were video- and audio-recorded and transcribed. Transcripts of the interviews were analyzed qualitatively using the constructs of APOS theory. The semi-structured interviews captured insights into participants' experiences and understandings gained through the activities. The written scripts were analyzed carefully in an attempt to determine participants' understanding of the concept.

Ethical issues

The major key ethical issues that were considered in the study were informed consent and confidentiality issues. Pseudonyms were used that is participants were coded using tags A1 up to A10.

Data Analysis

Bertram and Christiansen (2014) believed that data should be presented in a way that effectively communicates as much information as possible. This concurs with Corbin and Strauss (2008) that data collected need to be analyzed so that meaning will be attached to it. Thus, data analysis was mainly based on identification of themes, patterns, similarities, and differences, and these were used for the organization and presentation of the results. We were immersed in the data, identifying emerging themes and patterns within the data using APOS theory and identifying misconceptions that the participants have from the written exercises and tests.

Data analysis was also accompanied by images of the participants' written work so as to generate rich data. The transcripts of the interviews were also analyzed to complement the written work of participants. The analysis was supported by the preliminary genetic decomposition which was part of the theoretical framework represented below.

THEORETICAL ANALYSIS OF COMPLEX NUMBERS USING AN APOS APPROACH, GENETIC DECOMPOSITION

The four stages of learning a mathematics concept used in APOS theory were derived with the help of the work from Dubinsky (1997) for a clear understanding of the genetic decomposition of complex numbers. The researchers came up with the following genetic decomposition of complex numbers:

Division of Complex Numbers

Action

The individual should be able to identify the complex conjugate of the denominator and multiply numerator and denominator by the complex conjugate in a step-by-step manner and be able to combine the real and the imaginary part. The individual should use the definition $\sqrt{-a} = i\sqrt{a}$ when working with negative radicands.

Process

The individual can imagine what the product of the corresponding elements is and is able to combine the real and the imaginary parts without carrying out the step-by-step procedures. The multiplication of a number by its conjugate can be done in one step. At this level, the individual is able to predict the result in terms of the real and imaginary parts.

Object

The individual can see the effect of division of a complex number as a totality. The individual is able to apply processes or further transformation on the geometric interpretation of division of complex numbers.

Polar Form of a Complex Number

Action

In expressing a given complex number in polar form, the individual is able to represent the complex number on an argand diagram. The individual must then be able to find the modulus and the argument of a complex number in a step-by-step procedure.

Table 1. Frequency of scores for question 1

Category	1	2	3	4
Indicator	No attempt, or totally incorrect	Use the idea of first finding complex conjugate of the complex number	Use the idea of surds	Totally correct responses
No. of participants	1	2	2	5

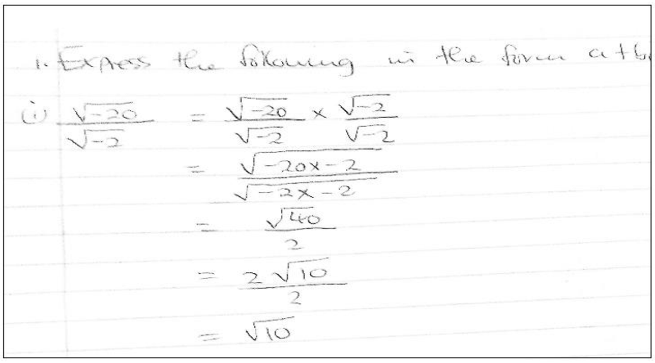

Figure 2. Written response to question 1 item a by participant A3

Process

The individual is able to visualize the effect of expressing a given complex number in polar form by finding the modulus and the principal argument of z. They do not necessarily have to illustrate it on the complex plane but must have a visual explanation. The individual must be able to find only one value of the argument that satisfies $-\pi < \theta < \pi$, that is the principal argument of z.

Object

The individual can see the complex number as a totality and is able to distinguish the properties of argument and principal argument of a complex number. Individuals should be able to move from the trigonometric system to algebraic form.

RESULTS

In this section, the results of the study are reported. The first section discusses results in terms of the division of complex numbers, which were elicited through question 1 with its two sub-questions (a and b), as follows:

1. **Q1.** Express the following in the form $a + bi$, show all the working:
   a. $\frac{\sqrt{-20}}{\sqrt{-2}}$
   b. $\frac{1+i}{i} - \frac{3}{4-i}$.

The second section discusses results in terms of the polar form of a complex number, which were elicited through question 2.

Division of Complex Numbers

There were two sub-questions that probed the participants' ability in understanding the concepts on division of complex numbers. However, division of complex numbers is a schema that involves the coordination of one's conception of the aspect on a complex conjugate ($\bar{z}$) as well as the schema of the notation (i^2). This is included and described in the genetic decomposition. The questions and percentages of those participants who got the correct answers are discussed below.

Results for question 1

Results for question 1 item a: Item 1 was intended to provide insight into whether the participants had developed a process understanding of the concept of complex numbers. **Table 1** shows the frequency of scores for question 1 item a.

Item 1 was intended to provide insight into whether the participants had developed a process understanding of the concept of complex numbers. Out of the ten participants, only five (50%) were able to get the correct solutions. According to the genetic decomposition, this shows these participants were able to interiorize the actions of working with negative radicands, and the definition $\sqrt{-a} = i\sqrt{a}$ into a process.

The five participants had developed the necessary constructions at the process level of understanding as they provided the correct responses. Most of the participants struggled to express $\frac{\sqrt{-20}}{\sqrt{-2}}$ in the form $a + bi$, although they were exposed to the radicals and were aware that $\sqrt{-1} = i$ and that $i^2 = -1$. The responses of participants A3 and A5 were almost similarly represented. The response by A3 is shown in **Figure 2**.

The written response by participant A3 showed that she used the idea of division of surds that was learnt at elementary level, where in order to get rid of the square root sign, one needs to rationalize the denominator. The participant failed to link the questions to the idea of radical, thus showing much deviation from the demands of the question. This idea of rationalizing the denominator was used out of its domain.

The participant was not aware that the multiplicative property of radicals only works with positive values under the radical sign, that is in general $\sqrt{a}\,.\sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$. This is the rule that must be applied, and that $\sqrt{-2}\,.\sqrt{-10} \neq \sqrt{-2 \times -10} \neq \sqrt{20}$. The participant was supposed to use the definition $\sqrt{-a} = \mathrm{i}\sqrt{a}$. The solution ($\sqrt{10}$) is close to the answer; however, her working is incorrect.

Participant A5 carried out the same procedure as A3, but she also had a considerable number of errors in an attempt to express the expression in the form $a + bi$. She also proceeded to express the solutions in its lowest terms ($\sqrt{10}$), although her solution carried a ± (i.e., $\pm\sqrt{10}$). This shows that the participant lacked the correct techniques of simplifying a mathematical problem, since now is mind, the presence of square root, makes her think that she is solving a quadratic equation. Participants A3 and A5 did not even develop the action conception of what a complex number is.

Participant A8 also provided an incorrect solution, as seen below:

$$\frac{\sqrt{-20}}{\sqrt{2}} \Rightarrow \frac{\sqrt{-20}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{-2}} = \frac{2\sqrt{10}}{2} = \sqrt{10}.$$

The participant's solution here also revealed some gaps in knowledge construction of the concept of complex numbers. He was confused here with the term complex conjugate. The participant was aware that to carry out division of a complex number, you need to find the complex conjugate of the denominator, that is if $z = a + bi$, then $\bar{z} = a - bi$. He confused the term complex conjugate and thought that the complex conjugate of $\sqrt{2}$ is $\sqrt{-2}$. The participant's concept definition of division of radicals was really in conflict with normal division. He failed to unpack the structure of what a complex conjugate is. This is evidenced by his underlining errors, where he proceeded to simplify $\frac{\sqrt{-20}}{\sqrt{-2}}$ and showed that when a negative is divided by a negative, the answer is positive. His response showed that he was still operating at the pre-action level of understanding according to APOS theory.

An interview with participants A3 and A5 indicated the following:

> **Researcher:** Can you briefly explain how you can express the following expression in the form $a+bi$.
>
> **Participant A3:** Hmmm, it's when you are given a fraction, the first thing is to come up with complex conjugate. Multiply the numerator as well denominator by that complex conjugate, we call $\bar{z}$.
>
> **Researcher:** So, what is the complex conjugate here?
>
> **Participant A3:** Aah, the complex conjugate of $\sqrt{2}$ is $\sqrt{-2}$, or we say rationalize the denominator so that we get a real number on the denominator.
>
> **Participant A5:** We simply rationalize the denominator. We did this in form 3.
>
> **Researcher:** The question requires you to express the expression in the form $a+bi$. So, what must be your final answer?

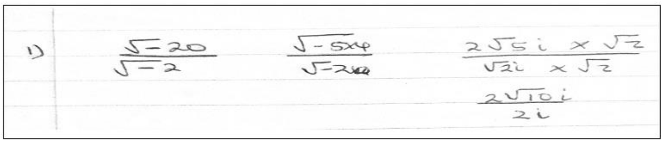

Figure 3. Written response to question 1 item a by participant A6

> **Participant A5:** The answer cannot be in the form $a+bi$, but only $\pm\sqrt{10}$, since we are finding the root, and also, it is a real number already.

Here, we can see that the participants were confident in applying algorithms wrongly. Participant A3 showed that she lacked the background knowledge of the concepts of complex conjugate as well as negative radicand, which impacted negatively on the construction of the necessary mental structures at the process level. Participant A5 displayed a misconception about what a complex number is, thinking that $\sqrt{10}$ cannot be written in the form $a + bi$, which thinking hampers her from developing a mental construction at the action level.

Another participant (A6) also showed some lack of algorithmic skills in an attempt to express the given problem in the form $a + bi$. The participant obtained the following expression (**Figure 3**).

Looking at the written response of A6, so many mathematical errors are evident. In the first instance, the participant was supposed to put an equal sign between the given fractions. In addition, the participant seemed to confuse the definition of i. He was supposed to get $\frac{2i\sqrt{5} \times \sqrt{2}}{i\sqrt{2} \times \sqrt{2}}$ instead of $\frac{2\sqrt{5i} \times \sqrt{2}}{\sqrt{2i} \times \sqrt{2}}$. He displayed a misconception on the definition of $\sqrt{-a} = i\sqrt{a}$, showing that he did not conceptualize the idea which he had learnt as isolated facts. Though this seems like a minor notational error, it signals a deeper confusion between carrying out the procedure to a process level of understanding. An interview with A6 indicated the following:

> **Researcher:** Can you briefly clarify how you solved the first problem.
>
> **Participant A6:** I first expanded the first square root, since I am able to find the square root of 4. So, I got (writing down) $2\sqrt{-5}$ on the numerator.
>
> **Researcher:** How then did you proceed?
>
> **Participant A6:** I used the definition of i since I am given a negative radicand to get $\frac{2\sqrt{5i}}{\sqrt{2i}}$, since $\sqrt{-5}$ = $\sqrt{5i}$.
>
> **Researcher:** What did you do next?
>
> **Participant A6:** I now rationalize the denominator.

Table 2. Frequency of scores for question 2

Category	1	2	3	4
Indicator	No attempt, or totally incorrect	Attempt to find the complex conjugate with various flaws	Attempt to find common *denominator* with incomplete solution	Totally correct responses
No. of participants	1	4	1	4

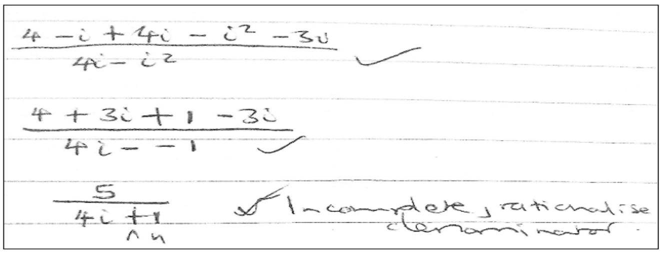

Figure 4. Written response to question 1 item b by participant A6

> **Researcher:** So, why didn't you reduce the expression to lowest terms and express it in the form *a*+*bi*?
>
> **Participant A6:** Uhh, I do not know how to proceed from there.

From the discussion the participant was not able to express $\sqrt{-5}$ as a complex number in terms of i. The i must not be under the square root sign. However, with continued probing, he showed that he did not understand the knowledge on how to express the radical in the form $a+bi$ and continued to cling to his misconception that $\sqrt{-a}$ = $\sqrt{ai}$. The participant was not able to simplify the given expression. This shows that he had failed to interiorize the procedure into a process level of understanding.

Results for question 1 item b: Question 1b was aimed at exploring the participants' conceptual understanding of knowledge of evaluating and understanding division of complex numbers and its relationship to other concepts such as complex conjugate, the symbol i^2, multiplication of complex numbers and subtraction of fractions involving complex numbers. The question addresses the object understanding of division of fractions in the genetic decomposition. **Table 2** shows the frequency of scores for question 1 item b.

The results showed that four participants (40%) constructed the correct concept image of the concept. All of them included all the necessary procedures and were able to apply them in a new context, as they solved problems easily of the form $\frac{a+bi}{c-di}$ during the class discussions. They proved to have a clear understanding of the concept of conjugate since they were able to find the conjugate of i, which is $-i$, as well as treated the square of a negative number correctly. These four participants were interviewed, and their responses showed that they had really interiorized the process into an object understanding of division of fractions.

Those participants who failed to obtain the correct result were divided into two groups. The first group included those who first found the common denominator, and the second group those who attempted to find the complex conjugate of the separate fractions. Participant A6 belonged to the first group and struggled expressing the expression in the form $a+bi$. The participant was able to carry out some of the procedures in a step-by-step manner, as illustrated in **Figure 4**.

The participant was able to find the common denominator and simplify the given expression up to $\frac{5}{4i+1}$, which is incomplete. This showed that the participant was able to make a link between the two fractions and to add them. However, his response revealed that he was unable to perform all the necessary procedures in an attempt to express the expression in the form $a+bi$, showing an inadequate conception involving division of complex numbers. This shows that the participant possessed an action conception, since he performed the calculation in a step-by-step manner and could find the common denominator to evaluate the symbol i^2. During the interview with the participant, he struggled to see any relationship between his solution, the intended result and could not figure out what he had to do next. He only pointed out that he had already expressed the expression in the form $a+bi$. He pointed out that if he simplified the expression further, he would obtain the solution $\frac{5}{1}+\frac{5}{4i}$, which was the intended result and was already in the form $a+bi$. The participant could not engage constructively with the concept and had hence failed to interiorize the concept of division of fractions into a process level of understanding.

Participants A9, A5, A10, and A3 fell into the second group of failed responses by attempting to first find the complex conjugate of the separate fractions. For example, A5 seemed confused and revealed a series of gaps in knowledge construction when simplifying the expression, as shown in **Figure 5**.

The response above shows that the participant was able to find the complex conjugate of the complex number i and $4-i$. However, she missed the first point when she multiplied i by $-i$. The participant further showed that she lacked background knowledge in simplifying an expression in index form effectively. She proceeded to multiply $-i$ by $-i^2$ and obtained i^3 instead of subtracting and went further to express i^2 as -1. She continued by treating $-7i^3$ and $-3i$ as like terms and obtained the following expression as her final result:

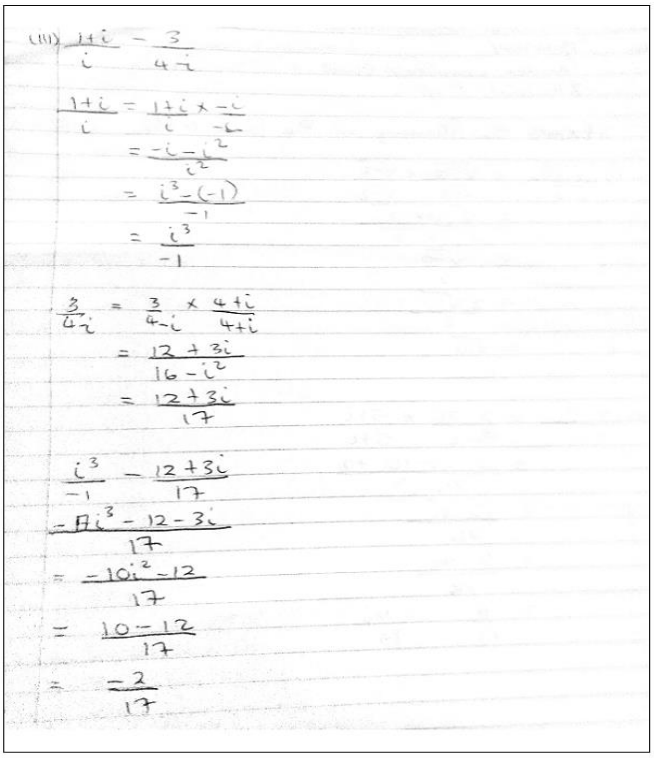

Figure 5. Written response to question 1 item b by participant A5

$\frac{-10i^2-12}{17} = \frac{10-12}{17} = \frac{-2}{17}$. This participant had not grasped the background knowledge on simplification of algebraic expressions. During her interview, A5 failed to provide a meaningful explanation to support her procedures. This shows that this participant had not made the necessary mental constructions according to APOS theory.

Participant A9 showed a greater understanding of the procedures and obtained the following expression: $\frac{17(1-i)}{17} - \frac{12+3i}{17} = \frac{17-17i-12+3i}{17} = \frac{5}{17} - \frac{14i}{17}$. The participant, however, failed to multiply $3i$ by the negative value, indicating that she lacked the basic schema of expansion of brackets learnt at O level. This hindered her to develop her understanding at the object level according to APOS theory. During her interview, she could not figure out where she went wrong. All these participants who got it wrong revealed some gaps in knowledge constructions and had an action conception regarding the division of complex numbers.

Polar Form

There were three sub-questions that probed the participants' ability in understanding the concepts on expressing given complex numbers in polar form and one of them is discussed here. The question focused on uniquely determining the argument and the modulus of *z*. This is indicated in the preliminary genetic decomposition. The questions and percentages of those who got the correct answers are discussed below.

Question 2

Results for question 2

Question 2 that was asked to participants tested their knowledge on concepts on expressing given complex number in polar form, as follows:

2. **Q2.** Express the following complex number in polar form, explaining your result: *-2+2i*.

The question tested the object understanding of expressing a given complex number in polar form. The frequencies for question 2 are shown in **Table 3**.

Participants A1, A2, A6, and A9 answered the question correctly. They were able to find the modulus and argument of *z* without carrying out all the steps. They drew the argand diagram so that they could locate the principal argument. This suggests that these participants were more likely to interiorize the process of expressing a complex number into polar form and into an object level of understanding. Participants A1 and A6 were interviewed and were able to explain explicitly how they expressed the given complex number in polar form. These participants displayed a complete understanding and also showed all the aspects of the mental constructions proposed in the genetic decomposition. By expressing the given complex number into polar form, the participants encapsulated the processes into an object understanding of the concept.

The other participants displayed quite a number of misconceptions in an endeavor to show their understanding of the concept of polar form of a complex number. An example is Participant A8's response, which can be categorized as being at the action level according to APOS theory. His solution showed some step-by-step procedures and showed that he conceptualized the idea of argument and modulus, as illustrated in **Figure 6**.

Participant A8 followed the correct procedures in that he was able to write down the general expression for expressing a complex number in polar form and find the absolute value of the complex number $-2+2i$. However, he totally missed the point on how to find the principal argument, and this hampered him from developing his understanding at the object level of a complex number. The participant here was supposed to

Table 3. Frequencies of scores for question 2

Category	1	2	3	4
Indicator	No attempt, or totally incorrect	Failure to find the principal argument and the value of *r*	Failure to find the principal argument	Totally correct responses
No. of participants	2	2	2	4

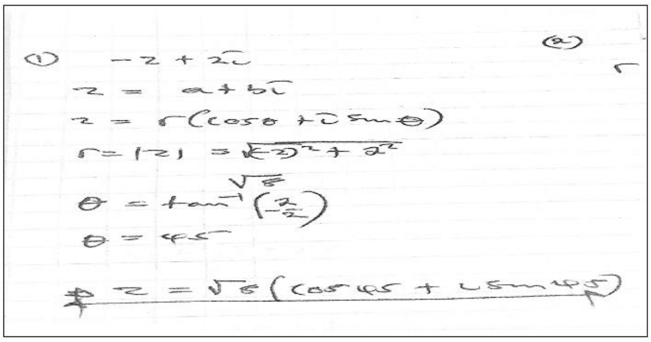

Figure 6. Written response to question 2 by participant A8

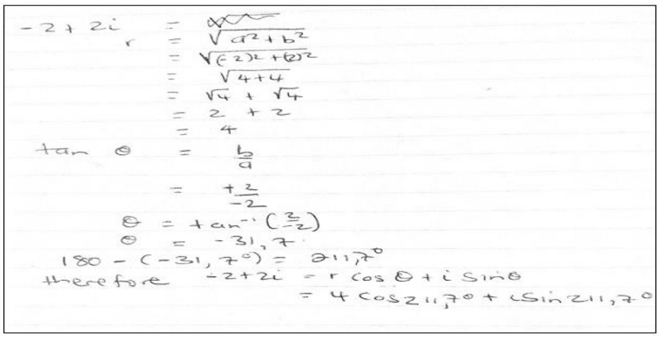

Figure 7. Written response to question 2 by participant A4

locate the complex number in the complex plane (by illustrating it in an argand diagram) and specify the direction of the complex number. Participant A10 also failed to find the principal argument.

Another participant (A4) knew the procedures to be followed in expressing the complex number in polar form. However, she lacked the basic knowledge in finding the absolute value of the complex number as well as the principal argument, as illustrated in **Figure 7**.

The participant failed to simplify the expression $\sqrt{4+4}$. This error is really a cause for concern because this conception is encountered at high school. The participant displayed confusion of basic algebraic manipulation. The participant further lacked the techniques to carry out procedures correctly. For example, she knew that she needed to find the argument of z but made so many evident mathematical errors. A wrong value of θ was obtained. She was unable to make the correct mental manipulation and failed to make a link of the type of solution that she was going to have. She failed to react to correct external cues as to the principal value of z, showing failure to engage constructively with the concept. She obtained a reflex angle. However, had she represented this in a complex plane (argand diagram) showing the polar coordination of real and imaginary in the coordinate system, she would have seen that the value of z was in the second quadrant.

Participant A7 obtained the following expression when calculating the value of $|z|$:

$$|z| = \sqrt{(-2)^2 + (2i)^2} = \sqrt{4+4i^2} = 0.$$

Here, the participant missed some points and did not have the conceptual understanding of the concept on the absolute value. The participants' answers showed that they had not made the necessary mental construction of the concepts. Excerpts of individual interviews with Participants A4 and A8 are represented below:

> **Researcher:** May you explain the procedure for expressing the given complex number in polar form (pointing to the expression $-2+2i$).
>
> **Participant A8:** We first find the value of r and then we find the *arg z*.
>
> **Researcher:** How do you determine the principal argument of z?
>
> **Participant A8:** You find the tangent of θ using the calculator, then you are done. The value that you get, you write it in the form $z = r(cos\theta + isin\theta)$.
>
> **Participant A4** (in response to the researchers question he said): The equation is given by $z = r(cos\theta + isin\theta)$, so we find r and the angle. The answer that we get, we must subtract it from 180^0.

The excerpts show that these participants had some knowledge gaps in the construction of the polar form of a complex number. Participant A8 was supposed to explain the need to first represent the given complex number in an argand diagram, so that is aware about the type of solution he must get. In this case the angle is in the second quadrant and is obtuse. They were also unable to explain how to determine the value of z, which is uniquely determined by either adding or subtracting any multiples of 2π to or from π, and that there is only one value of the argument that satisfies $-\pi < \theta < \pi$. The participants thus displayed lack of procedural understanding (Siyepu, 2013). This further indicated that these participants had not even interiorized the procedures of determining the polar form of a complex number into a process level of understanding.

DISCUSSION AND CONCLUSIONS

The study has provided answers to the research questions. It attempted to unpack the cognitive difficulties possessed by undergraduate in-service teachers when learning the concepts on complex numbers, presented with different modes of representation. The study revealed that the majority of the participants struggled to understand the basic concepts on division of a complex and polar/trigonometric form of the complex number. Ten participants' written work was analyzed and scrutinized, and interviews were conducted with some of them to determine the areas that needed urgent attention. We noted the following weaknesses as student attempted to carry out division of complex number.

Participants struggled to remember the key concepts when carrying out division involving radicals. Sixty percent of the participants could not figure out the need to use the definition $\sqrt{-a} = i\sqrt{a}$. Some of the participants revealed during interviews that they rationalized the denominator by multiplying it by $\sqrt{-2}$, with some arguing that the complex conjugate of $\sqrt{-2}$ is $\sqrt{2}$. This coincides with Nordlander and Nordlander (2012) asserted that the word *imaginary* has a kind of built-in negative connotation, as shown by some of the participants who thought that the imaginary part must be negative. These are some of the concepts learnt at elementary level. We refer to these concepts as the "met befores", with De Lima and Tall (2008) asserting that if these concepts are not properly understood and taught, this may cause a barrier in the development of new knowledge. The participants were also taught during the same semester that when dividing by a complex, you find the complex conjugate. Here can be seen that learning on prior experiences can greatly impact learning. De Lima and Tall (2008) referred to these infringing factors as the "met afters", which can affect learning in a negative way only if they are used outside their domain. They failed to relate the structures that make up the concept complex conjugate. They thought that a complex conjugate of a complex number must have a positive and negative number. The other difficult was failing to express $\sqrt{10}$ in the form $a + bi$ and thought that the root was the final answer, thus making the solution incomplete as well as thinking that the result was the final answer. Some obtained correct solutions from wrong working this supports Naidoo's (2007) connotation that the correct answers obtained by students does not prove that these students understand the concept.

Some students also failed to simplify the expression $\frac{5}{4i+1}$ arguing that it is already in the form $a + bi$ since we obtain $\frac{5}{4i} + \frac{0}{1}$. Students struggled to manipulate the symbol i when trying to rationalize the denominator by multiplying by the complex conjugate. This study coincides with the study by Egan (2008) who said that students struggle to understand the concept imaginary numbers, hence he advocated that students must be introduced to the square root of -1 as i^2 before embarking on any other concepts involving complex numbers. Ahmad and Shahrill (2014) also revealed that students make errors in failing to replace $\sqrt{-1}$ by j as well ignoring the sign of the negative numbers and powers of j. This concept of -1 as i^2 makes the backbone of what a complex number is. Students also manifested a number of errors which included simplification of algebraic expression, failure to manipulate basic algebraic expression as well as struggled with the concept of simplifications of directed numbers when manipulating division of complex numbers.

We also noted that students showed a lack of conceptual evolution of the concept on polar form of a complex. The students displayed a number of errors and they failed to express their reasoning capabilities but were confident in applying rules. Students struggled to find absolute value of given complex number. Students failed to apply the formular $a + bi = \sqrt{a^2 + b^2}$ whereby they were squaring bi instead of b such that they end up getting a negative result. That was a serious misconception which shows that the student has learnt these concepts by rote. Some confusion of basic algebraic manipulation was also seen for example in order to simplify $\sqrt{a^2 + a^2}$, separate roots were found instead of adding the two and then find the square root. Another challenge that was displayed was failure to find the required argument. The aspect on first representing the given complex number on the complex plane was not mastered, so that they can find the principal value. Haber (2017) argued that students had challenges with the complex number when it comes to represent them on the cartesian coordinate system. This is in line with Borji et al. (2020) and Tan and Toh (2013) who noted that student had major difficulties with the polar coordinates especially when plotting the points on the complex plane when (r, θ) is negative.

The APOS analysis in this study was guided by the genetic decomposition described before. The APOS theory together with the preliminary genetic decomposition provided an important root of discovering how the undergraduate students understand the concept of complex numbers. The mental constructions in the task link to the preliminary genetic decomposition. Considering the concept of division of complex numbers involving negative radicand, it shows that 50% of the student did not even develop their understanding at the action level because they were not able to apply the rule for multiplication by radicands. Most of the students tried to find the complex conjugate which was incorrect. For the second concept on division, the responses from the content analysis and interviews suggested that most of the participants had developed at least an action conception. This was evidenced as the students were able to find the complex conjugate of i and $4 - i$ or first finding the common denominator and simplifying the expression. However we noted that an obstacle that hampered the full development of the mental constructions at the process level was that many of the participants lacked background knowledge of simplification of expression in index form as well as the manipulating i^2. We also noted that some participants were comfortable engaging with algorithms for expressing the complex number in the form $a + bi$ without having constructed the meaning of the concepts and without really understanding them. An example is that after combining the two terms, the result expression obtained was $\frac{5}{4i+1}$ but students failed to simplify it further and thought that it is the final answer. These omissions showed that the participants' understanding of the concept is externally directed, since they did not engage with the meaning of the expression, indicating that they simply learnt it by rote. DeVries and Arnon (2004) explained that students face difficulties in mathematics because they rely much on memorized rules without conceptualizing the concepts taught.

We also noted that if concepts are understood as isolated facts, they hamper students to develop their understanding at the object level, and it delays the development of the necessary schema. The students who tried to simplify their work by finding the complex conjugate of i made multiple errors such a failure to simplify general algebraic expression resulting in their work conceptions being limited to the action level. Kazunga and Bansilal (2017) asserted that students who demonstrate a misunderstanding of prerequisite concepts of basic algebra cannot interiorize actions into a process. This is because they could not interiorize the actions into a process, nor encapsulate the process into an object. This really showed that prerequisite knowledge plays an important part in knowledge acquisition (Bansilal et al., 2017). We also noted that only 40% of the participants were able to interiorize the processes of the polar form of a complex number into an object level understanding. We noted some students failed to find the absolute value when expressing the given complex number in polar form had their reasoning which had not moved past the action stage. They failed to apply the correct rules thus ended up squaring $2i$. Also, the main obstacle that hampered the full development of the correct mental constructions at the object level was that many of the participants failed to distinguish the argument in polar form of a complex number and the principal argument. In addition, they failed to visually represent the complex number in coordinate form. The interviews revealed that some of the participants were simply using rules to answer the questions without fully engaging with the concepts. We also noted that the mental constructions in the task link to the preliminary genetic decomposition.

Implications for Teaching

In this study, we suggested a genetic decomposition showing how undergraduate in-service teachers may construct the concepts on complex numbers. APOS theory was used as a lens to understand how student-teachers construct their mathematical knowledge. The findings of this study have confirmed that student-teachers find it difficult to understand the concept of complex number which has been introduced using the formal definition. Many of the participants did not have appropriate mental structures at the process and object conceptions. Maharaj (2010) noted that more time is needed or should be devoted to help students develop the mental structures at the process and object levels. This concurs with Mutambara and Bansilal (2021), who asserted that instructors should spend more time so as to engage more deeply with the material. Tall (2014) and Kazunga and Bansilal (2020) noted that it is important to take into consideration the students' prior knowledge so as to enhance the learning of new knowledge. There is the need to engage more deeply with prerequisite concepts on simplification of algebraic expression, expansion of brackets, multiplication by a negative number, use of the definition $\sqrt{-a} = i\sqrt{a}$ as well as the aspect on complex conjugates. It is also important to dwell much on the use of an argand diagram so that the students will not struggle to find the principal value. This will enable students to engage more comfortably with the work on division of complex numbers as well as the polar form. Tall (2014) also added that instructors need to look closely at the "met befores" and the "met afters" so that they can influence learning in a positive manner. Participants need to be taken beyond the action and process stage understanding of concepts by giving them more structured opportunities so that they can discover the relationships between the cartesian coordinate and the argument of the given complex number. We also recommend that educationists include examples in their teaching that encourage students to focus on discovering errors and also to develop other forms of mathematical reasoning. Further studies can concentrate on the error and misconception that students reveal when studying tasks on complex numbers presented in algebraic and polar form leading to de Moivre's theorems. This further supports Chauraya and Brodie's (2018) contention that the teachers who focus on understanding student errors will also improve their own mathematical knowledge. Another concern is that these teachers as they go to teach, they need to participate in some professional developmental programs in their districts that help develop their understanding of these concepts. In order to strength the application of complex number in the teaching of physics, it is recommended that the model of teacher training program should not teach the mathematics modules component separately.

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