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Communication skills enabled in a pre-calculus course using dynamic geometry software

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Abstract

This article is the result of a research project that aimed to characterize the communication skills developed by students who solved problems of seven workshops related to the notions of variation and change within the framework of a pre-calculus laboratory course mediated by interactive mathematical software at a Colombian public university. Data analysis was performed considering the characterization of the skills of interpretation, explanation, and justification, which are fundamental for the communication process of variational thinking. Here we show evidence of the identification of communicative skills manifested by students when solving problems involving the use of the notions of variation, tendency, and approximation.

Keywords: communicative process, interpretation, explanation, justification, variation, change

INTRODUCTION

Educational mathematics has turned its attention to the communication process and its implications in the learning of this discipline. This has been reinforced by the *"Organización para la Cooperación y el Desarrollo Económicos (OCDE)* [*Organization for Economic Cooperation and Development*] *(OECD)"*, which within its evaluation categories includes communication skills to be assessed in international tests (OECD, 2014).

Hatano and Inagaki (1991) cited in NCTM (2000, p. 64) mention that "students who engage in discussions to justify solutions, especially when there is disagreement, will reach a better mathematical understanding as they try to convince their peers about the different points of view". This highlights the need for the appropriate use of language for the development of modeling and representation tasks.

In the PISA tests, students are mentioned with the highest level of proficiency can understand the relevant information provided by a problem and relate it effectively to solve it, in addition to being able to co-communicate their procedures by making appropriate use of mathematical language.

In Colombia, the MEN (1998, 2006) explains that the communicative process is fundamental in the mathematical education of students, helping them to establish links between their informal and intuitive notions and the abstract and symbolic language of mathematics. These links make possible the connection between different representations (physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas).

At the "Universidad Industrial de Santander (UIS)", a pre-calculus course-laboratory has been implemented since the first semester of 2013, whose purpose is to help students develop "variational thinking". The course is focused on the process of problem solving mediated by GeoGebra around the two central ideas of calculus: variation and accumulation, which allow students to establish at a mathematical level relevant to the requirements of the differential calculus course. The dynamic representations of GeoGebra favor the communication process because they provide students with visual elements that help them to explain with empirical or deductive arguments the questions and justifications requested.

The pre-calculus course made possible the development of research that sought to answer the question: What mathematical process skills are developed in a pre-calculus course-laboratory mediated by interactive mathematical software? Some answers to this question were reported by Fiallo and Parada (2018) in which cognitive skills associated with the

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Contribution to the literature

- In this article, we present some evidence of the communicative skills with which students enter the first level of the university, which can serve as an example for other universities, where we want to characterize the students in this aspect.
- This article shows some examples that can address the development of student's communication skills in the first course for incoming college freshmen.

mathematical processes exposed by the MEN (1996) are raised and explained, which have served as a guide for teachers who guide the course to promote the development of these so that students achieve greater success in the differential calculus course. In particular, for the communication process, the skills of interpretation, explanation, and justification were proposed.

The results reported in this article are extracted from the implementation of the course carried out in 2020, which was conducted remote face-to-face because of COVID-19, through the Zoom platform and GeoGebra Classroom. In the process, communication via email and WhatsApp was also favored.

This article presents evidence of the communicative process skills manifested by students who participated in the course. As a result, some descriptors, and examples of the skills of interpretation, explanation, and justification are presented, which can help to assess the learning processes in any mathematics course.

COMMUNICATING VARIATION AND CHANGE

Communication, according to MEN (1998) allows students a better understanding of mathematical objects (OMs), because it leads students to link diverse representations (pictorial, physical, graphic, symbolic, verbal, mental, tabular, through equations, and formulas) of the same object, thus combining formal notions with intuitive ones and the symbolic language of mathematics. Regarding communicating variational thinking, Fiallo and Parada (2018) mention that skills are required to interpret, explain and justify a variation. The conceptualization of each of them from these authors is presented below.

Interpret the Variation

The ability to interpret could be explained as the capacity of a person to understand, comprehend, make sense of and give coherence to a problem from a given context. According to Santos-Trigo (2007), this skill is developed when the student is able to answer the questions: what is the information of the problem?, what is the unknown?, and what are the conditions that relate to the data in the problem? highlighting the variables at stake and the mathematical knowledge to be developed. From variational thinking, interpreting a proposition, a problem, an event, a graph, a map or a scheme that represents a situation of change would involve actions such as:

- identify the variables involved in a problem situation and their relationships,
- describe the state, interactions, or dynamics of a system, in graphical or symbolic terms,
- translate information from one information system to another, and
- draw valid conclusions or hypotheses from a set of data or events.

Some strategies used to solve calculus problems may be careful reading of the statement in which the variables at stake are identified and what is being asked is understood, description of relationships between the different representations, and translation into a convenient mathematical language to answer the question, as well as providing concrete examples that give meaning to the problem statement, as also highlighted by Fiallo et al. (2021).

Explain the Variation

It is important to provide classroom spaces for students to explain the ways that have led them to find the solution to a problem because in this way they can strengthen their learning and perhaps find some procedural error, a statement with which Reséndiz (2009) agrees. NCTM (2000) standards also suggest giving students the opportunity to listen to the explanations of their peers so that they can approve or refute statements according to their criteria. In the case of elementary students, it is acceptable to use natural or pictorial language to make their reasoning known, but in secondary school, they should be required to use more formal language.

Explanatory skill is evidenced when students present convincing examples, use mathematical language to expose their ideas, and propose hypotheses, premises, and procedures to convince their peers that the reasoning is correct. Balacheff (2000) cited by Bautista (2017) adds that the explanatory skill is key in the communication process since from this stems the ability to prove and demonstrate.

Fiallo and Parada (2018), state that it is the aptitude that a student has to describe an object of co-knowledge with clear words or examples, expressing the why of a process, to make intelligible to another that object of knowledge.

Justify the Variation

During the mathematical education process, it is essential to invite students to justify their strategies and procedures, Fiallo et al. (2021). For this, it is necessary to provide an environment that stimulates them to explain the why? of a process or a statement, to verify their ideas or propositions. The teacher must also be flexible and allow his students to question him and ask him questions, and thus be able to build arguments jointly.

Jorba et al. (1998) state that when asked to justify a conjecture or process, there is the possibility of constructing coordinated reasons or arguments that can lead to accepting, modifying, or refuting a thesis or part of it. The research reported here takes the justifying ability as the capacity of a student to support an idea or a process, to accept or refute a conclusion using relevant reasons.

METHODOLOGICAL ASPECTS

According to Aguirre-García and Jaramillo-Echeverri (2012), this type of study focuses on understanding school realities in which the experiences of those who participate in the educational process prevail. For the analysis of the data, a qualitative methodology was used to characterize the skills of the communicative process achieved by first-time university students participating in a pre-calculus course. The research presented here was developed in four phases described below.

Pre-calculus Course

The pre-calculus course aims to develop variational thinking in undergraduate students of science and engineering. The course is based on the conceptual cores of variation, change, approximation, and trend through problem-solving and the use of GeoGebra. It is developed in 14 workshops that follow a didactic structure proposed by Fiallo and Parada (2018), which consists of five moments, as follows:

1. **Free exploration:** To begin with, a problem is posed with the objective that the student reviews his/her previous knowledge and solves it intuitively without using GeoGebra, the purpose is to identify conceptual difficulties and work on them.
2. **Socialization of results (sharing):** Group discussion of results is proposed, to correct errors, clarify doubts, and theoretically justify the answers.
3. **Directed exploration:** A GeoGebra file is provided for students to explore the dynamic OM and construct solutions to the problems, making conjectures and mathematically justifying the results obtained. In this phase, the teacher should guide the students' process through questions that lead them to relate the different representations (algebraic, visual, graphical, and others) of the OMs of study.
4. **Explanation:** The discussion between peers and the teacher of the solutions found for the problem is proposed; for this, the teacher promotes the confrontation between the different answers to build new knowledge.
5. **Free orientation:** At the end of each workshop, a challenging problem is proposed in which the conceptual nuclei of approximation, tendency, variation, and change are involved, so that students can use the knowledge acquired.

Experience Carried Out

The university (context of study) offers a pre-calculus course to students who are beginning their studies in science and engineering and who, according to the characterization tests applied by the university, are identified as being at high risk to take the mathematics subjects of the basic cycle. These tests are conducted by the "*Sistema de Excelencia Académica de la Universidad* [*University Academic Excellence System*] (*SEA-UIS*)" to provide support to students at higher risk.

The course has been implemented since 2013 in face-to-face mode, however, the cohorts carried out from 2020 to 2022 were developed synchronously through the zoom platform for four hours daily and asynchronously through the GeoGebra Classroom, due to the contingency experienced by COVID-19. For this article, data collected in the second academic semester of 2020 were chosen.

GeoGebra Classroom has two working modalities: group section and book section. The group section allows the teacher to design the 14 workshops so that students can directly manipulate the different simulations and constructions elaborated in GeoGebra for problem-solving, as well as allowing students to attach images as evidence of the work done, use the formula editor and receive constant feedback from the teacher through the comments section assigned for each activity. This section was used, especially for the activities that the student solved asynchronously (exploration and free orientation). On the other hand, the book section works similarly to the group section, but, with the disadvantage of not being able to attach images of the processes that the students perform with pencil and paper, and with the advantage of the teacher seeing instantaneously the students' answers to the problems performed.

Data Collection Instruments

To reach the objective of this research, some student responses to the workshops implemented in the pre-calculus course were selected. The answers and constructions in GeoGebra were retrieved from the GeoGebra Classroom, photos of the pencil and paper solutions, episodes of video recordings of the class sessions, and interactions via WhatsApp were collected. Evidence was retrieved from a student who actively participated in the 14 class sessions, being a representative case of the group, and who showed significant progress in the communication skills for the development of variational thinking.

It is important to mention that the study focused on the communication skills (interpretation, explanation, and justification) made possible by the activities (problem situations) proposed in the workshops and not on how the communication platforms used to develop the course remotely (Zoom and GeoGebra Classroom) influenced the communication and interaction processes between students and teachers. Likewise, it is valuable to propose as a future investigation the comparison between face-to-face and virtual forms of communication that account for students' understanding of OMs of study.

Of the 14 workshops, some activities were selected from the seven workshops in which the communicative process was worked on in greater depth; the selected workshops were:

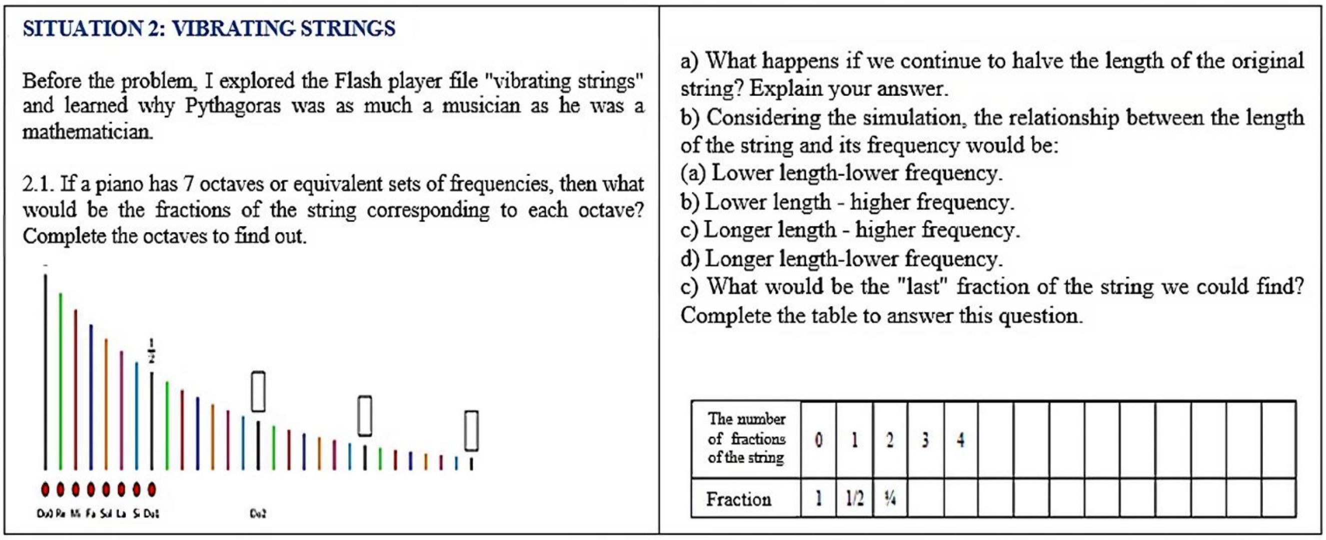
1. **Workshop 3-Trigonometric ratios in the plane:** It consisted of five activities oriented to the understanding of trigonometric ratios, emphasizing that these ratios do not depend on the length of the sides of the triangle but its angles. Also, the importance of determining the signs of the trigonometric ratios and observing their dependence on the respective quadrant in the Cartesian plane was emphasized.
2. **Workshop 4-Hypothetical situations:** Here three activities were consolidated in which students were proposed to develop a mathematical model for the problems posed. The problems were presented to the students in a descriptive way, and they were asked to interpret and translate those using mathematical structures.
3. **Workshop 5-Interdependence of variables:** Consisting of five activities aimed at strengthening the concept of function through the study of variation and change. The proposed activities were intended to bring the student closer to concepts related to variables (dependent and independent), based on the idea that a magnitude varies given its interdependence with another, thus emphasizing the functional relationship.
4. **Workshop 8-Consumption rates:** Its main objective was to guide the student in the understanding of the concept of a function and the characterization of different types of functions, which is why three activities related to the step function were proposed. This type of activity allowed introducing of a new sign to represent the idea of an integer part through different representations (graphical, tabular, and verbal, among others).
5. **Workshop 9-Functions and reality:** It consisted of three activities, whose main objective was to encourage students' understanding of patterns, relationships, and functions. It was also intended to orient toward the importance of recognizing the characteristics and differences of transcendental (non-algebraic) exponential and logistic functions through the use of resources such as GeoGebra, pencil, and paper.
6. **Workshop 11-Containers:** The primary purpose of the workshop was to work on problems related to optimization and application of the derivative. In addition, to strengthen notions about congruence and similarity of triangles, understanding of measurements, selection of units of measurement with which to work, and to strengthen students' concepts necessary to address optimization problems.
7. **Workshop 13-Box without a lid:** Its primary purpose is to build the definition of the derivative in an intuitive way and from the interpretation of this as the slope of the tangent line. It also allows orienting students on the different variations: direct, inverse, accelerated, convergent and cyclic.

It is important to highlight that to show the interpretation skill promoted in the pre-calculus course; to show the explanation skill, workshops 8 and 11 were chosen; and finally, for the justification skill, workshops 9 and 13 were selected. Since there was greater participation of the students and they strengthened their communication skills.

Systematization and Analysis of Data

The triangulation of the data is supported by the description of the skills enunciated before. The case study (whom we call David) from whom evidence was taken, volunteered information by continuously expressing to his classmates and teacher what he understood about the OMs studied in the course.

David, who entered the mathematics program when he was 16 and graduated from a public school in Colombia, was participatory, active, and interested in the work from the very first session: he constantly questioned the different opinions of his classmates. The student was characterized by interpreting and understanding the same problem from different representations (geometric, algebraic, analytical, tabular, and verbal) and being very critical.



SITUATION 2: VIBRATING STRINGS

Before the problem, I explored the Flash player file "vibrating strings" and learned why Pythagoras was as much a musician as he was a mathematician.

2.1. If a piano has 7 octaves or equivalent sets of frequencies, then what would be the fractions of the string corresponding to each octave? Complete the octaves to find out.

a) What happens if we continue to halve the length of the original string? Explain your answer.
b) Considering the simulation, the relationship between the length of the string and its frequency would be:
(a) Lower length-lower frequency.
b) Lower length - higher frequency.
c) Longer length - higher frequency.
d) Longer length-lower frequency.
c) What would be the "last" fraction of the string we could find? Complete the table to answer this question.

The number of fractions of the string	0	1	2	3	4									
Fraction	1	1/2	¼											

Figure 1. Vibrating strings of workshop N°4 (Source: Authors' own elaboration)

However, as the course progressed and the level of complexity of the workshops increased, he changed his mind, generating doubts and leading him to search for information via the web about the OMs worked on, mainly about limits and derivatives.

Characterization of the Communication Skills

This phase responds to the research objective, characterizing the communicative skills achieved by the student (case study), under the categories of analysis: interpretation, explanation, and justification.

WHAT COMMUNICATION SKILLS EMERGED FROM THE EXPERIENCE?

The categories of analysis correspond to the communicative skills (interpreting, explaining, and justifying) described in the second section of this article.

Ability to Interpret Variation

Interpretive ability is understood as the student's ability to understand and make sense of the structure of a problem (expressed in mathematical language); as well as to understand or read demonstrations, definitions, graphs, maps, diagrams, or mathematical schemes in which arguments and/or processes of an OM of study are presented.

The interpretations made by the students were expressed in different ways; some give particular examples, others use pictorial or graphic representations, and there are even those who can use the software to illustrate the situation. The following are the results of the analysis of activities that support the above statement.

In the activity N°2: vibrating strings of workshop N°4 (**Figure 1**) the students were expected to strengthen their ability to pose a mathematical model and to describe the problem by identifying the existence of the operator $(\frac{1}{2})$, as the fractions of the string are made. Finally, we wanted them to find some terms of the sequence and in other activities the general formula of the problem.

The following story shows the proper interpretation of the problem by a student who begins with a detailed reading of the problem and raises particular examples that allow him to reach a generality, this being one of the fundamental tasks to correctly interpret the statement of a problem.

> Teacher: What does the string problem tell us?
>
> David: [Student reads the problem in **Figure 1**].
>
> Teacher: Very good. Now, what is the first division?
>
> David: It is $\frac{1}{2}$ because it says dice it is divided in half.
>
> Teacher: What value is taken in the second division?
>
> David: $\frac{1}{4}$, because it says it is divided in half. Then, it will be $\frac{1}{8}$ because it is divided in half again.
>
> Teacher: What value would follow according to what you observed in the exploration?
>
> David: $\frac{1}{16}$, because we are dividing in half each time.

This strategy guided the student to answer the question: what would be the last fraction of the string that can be found?

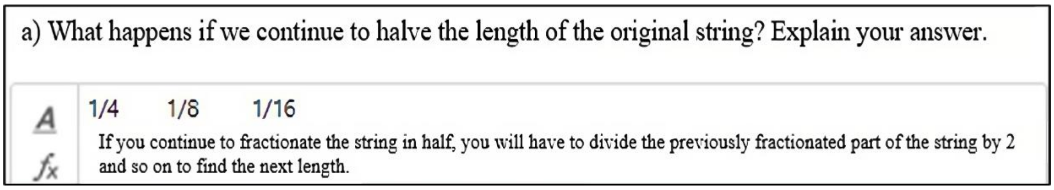


Figure 2. Interpretation of the string problem by particular examples (Source: Authors' own elaboration)

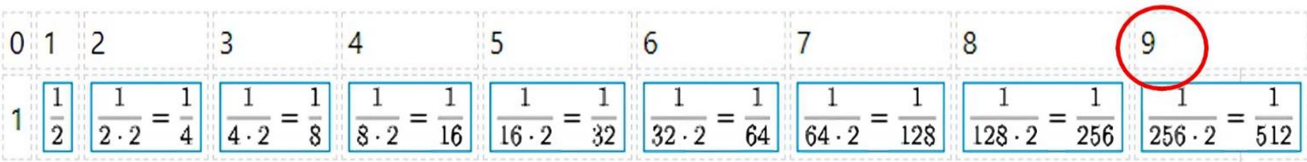


Figure 3. Reiterative string fractionation (Source: Authors' own elaboration)

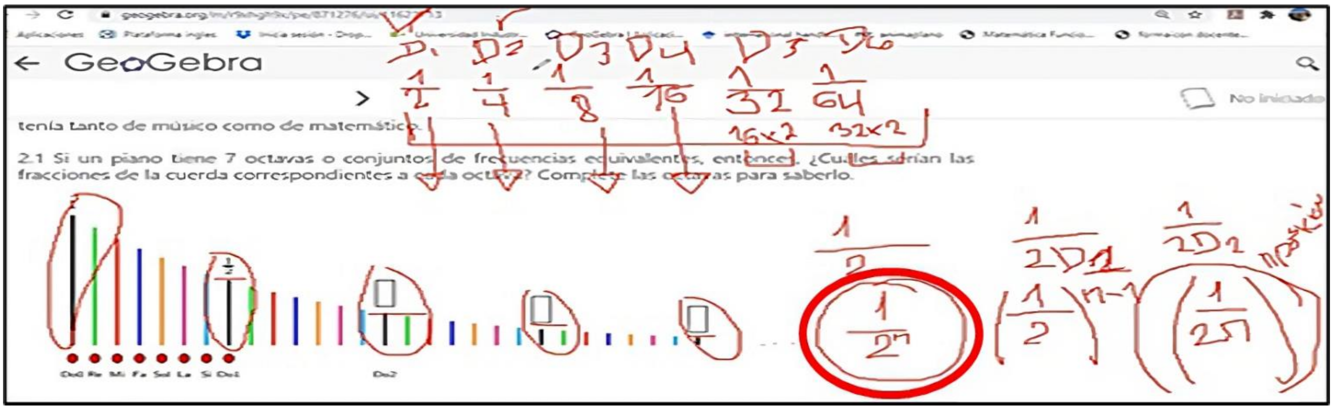


Figure 4. Algebraic representation of the string problem (Source: Authors' own elaboration)

As can be seen in **Figure 2**, David, shows his understanding of the problem, since he established that the first fractions made are: $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, values that were obtained by identifying that the string was fractionated in half, then again in half, and so on consecutively.

From the previous reasoning, it is possible to infer that for the student it is possible to establish a general expression to find any partition, he expresses it as follows: *"and so on to find the next length"*. This expression shows that no matter the size of the string, its half will always be found, which led him to continue fractioning the string nine times (**Figure 3**).

From what is observed in **Figure 3**, it appears that the student seeks to see beyond the result of a few particular cases by seeking to identify the pattern of the sequence.

In **Figure 4**, the student takes D1, D2, D3, D4, … as the position or fractionation of the previous partition divided in half, finding that: $D2 = \frac{\frac{1}{2}}{2} = \frac{1}{4}$, $D3 = \frac{\frac{1}{4}}{2} = \frac{1}{8}$, $D4 = \frac{\frac{1}{8}}{2} = \frac{1}{16}$, and so on.

To reach the generalization of the problem and for the student to establish the function $f(n) = \frac{1}{2^n}$ some problematizing questions were posed that led to the following discussion:

> Teacher: In what way could we obtain an algebraic expression to generalize this situation, as presented in previous workshops?

From this question arose the expressions shown in red in the lower right part of **Figure 4**, where the students discussed with each of them, evaluating n in each of these possible expressions to choose the one that correctly modeled the problem. After this process, they concluded that the correct answer is $\frac{1}{2^n}$ and David, who proposed this expression, was asked to explain this result.

> David: It would be the formula $\frac{1}{2^n}$ s with n being the octave in which it is located.

> Teacher: how did you come to this expression?

> David: Teacher, by logic, I saw that the denominator of the exercise is multiplied by two each time, then the exponent of two comes out, it raises it to the octave in which it is, for example, in $D1 = \frac{1}{2}$, in $D2 = \frac{1}{2*2} = \frac{1}{2^2}$; in $D3 = \frac{1}{2*2*2} = \frac{1}{2^3}$, and so on.

From the student's response, it is inferred that he correctly interprets the problem because he manages to

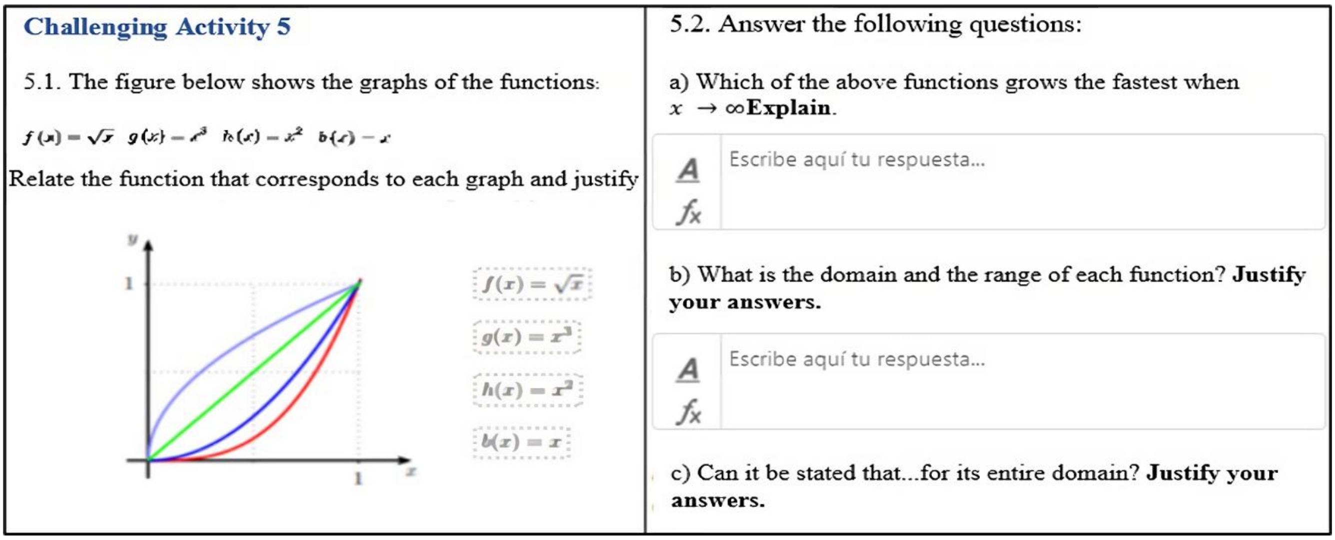


Figure 5. Challenging activity 5 of workshop 5, interdependence of variables (Source: Authors' own elaboration)

move between everyday language and mathematical language, and he also states the situation algebraically. Verbal language, according to Molina (2014), refers to everyday language expressed orally or in writing and includes specific terminology such as that of academic mathematical language.

From the previous example, it is highlighted that David found it difficult to raise the algebraic expression that related the two variables. However, the collaborative work suggested by the methodology of Fiallo and Parada (2018) in which each student externalizes his understanding of the problem and evaluates his own from the interpretations of the others, helped him to express the situation algebraically, as follows: $(\frac{1}{2^n})$, where n is the eighth or expected fractionation. The previous results show the correct interpretation of the statement that led not only to the particularization of concrete examples, but also to the generalization, which was concretized in an algebraic representation (**Figure 4**). Another evidence of interpretive skill is found in the following episode:

> Teacher: At some point does the string fractionation process end?
>
> David: No.
>
> Teacher: Why?
>
> David: Because the numbers can be divided in half, getting smaller and smaller, but still half and we can do it infinitely many times.
>
> Teacher: Will you always be able to give a value?
>
> David: Yes because we will always be able to divide in half.

From the previous dialogue, the appropriate interpretation of the question is valued because thanks to this the student can identify the information of the problem; relate the data provided (**Figure 4**) and recognizing the iterative process, an interpretation that led him to solve the problem correctly. Another strategy that David uses to interpret a problem is to resort to the numerical characteristics of the graphical representation. Thus, in the challenging activity 5 of workshop 5: interdependence of variables (**Figure 5**), he was expected to apply the knowledge acquired in previous workshops to analyze the behavior of the functions $f(x) = \sqrt{x}, g(x) = x^3, h(x) = x^2$, $and\ b(x) = x^2$ in the interval [0,1] and to identify the behavior of the functions, their domain, and range.

David shows an adequate interpretation of the problem presented in **Figure 5**, as shown in **Figure 6** (specifically in what is marked with red) he states his interpretation of the operations between the real numbers in the interval [0, 1] in the algebraic representation of the four functions. He can identify the characteristics of two functions at a glance. In addition, he states that the function $h(x) = x$ (highlighted in green), is a linear function, mentioning that it is the only one that increases proportionally. Likewise, David states that the function highlighted in purple corresponds to $f(x) = \sqrt{x}$, and that it increases gradually.

From the above, it is highlighted that the student interprets the problem based on the numerical behavior of the functions in the interval [0, 1] managing to identify the conditions of the problem, applying what was seen in the first workshop where they analyzed the behavior of the operations of multiplication, division, power, and square root in the interval [-1, 1], learning that the operations in this interval are contrary to the conceptions that the multiplication between two numbers always increases and the square root of a number always decreases. This allowed David to associate the graphical representation with the algebraic representation of the given functions, going from one representation to

The function b(x)=x is the green line. It is obviously the easiest to find since it will be the only one that increases proportionally.

Since we are taking into account numbers that are between 0 and 1, we know that with these numbers when we take out the root it will give a larger number but when we raise it to any power it will give a smaller number, so the 2 functions that involve power will be smaller than the one that includes root. The one that includes root will be the one that takes the larger values.

So, the function f(x) will be the lilac one, whose values are the greatest of all.

Between the 2 functions that involve power, the one with the smaller exponent will be greater, since in numbers that are between 0 and 1, the bigger the power is, the smaller the result will be, that is, the function.

g(x) = x^3 is the one with the red line since it takes the smallest values, this is similar to the shape of a parabola, but it is of a larger size than the quadratic function in y, and the function h(x) is going to be the blue one that has the second smallest values to the one that is building the shape of a parabola. Also, at first, it grows slowly, but then it increases in size.

Figure 6. David's response to challenge activity 5 of workshop 5 (Source: Authors' own elaboration)

b) What are the domain and range of each function? **Justify** your answers.

$f(x) = \sqrt{x}$ *dom*: $[0, \infty]$ *Rang*: $[0, \infty]$, because the roots cannot be negative...
$g(x) = x^3$ *dom*: $\mathbb{R}$ *Rang*: $\mathbb{R}$, because x can take all the numbers that form the image on the y-axis and vice versa.
$h(x) = x^2$ *dom*: $\mathbb{R}$, because x can take all the numbers that form an image on the y-axis. *Rang*: $[0, \infty]$ because x has a positive sign and indicates that the parabola opens upwards.
$b(x) = x$ *dom*: $\mathbb{R}$ *Rang*: $\mathbb{R}$, because x can take all the numbers that form an image on the y-axis and vice versa.

Figure 7. Interpretation of Dariel through conclusions of a data set (Source: Authors' own elaboration)

another, actions that show the student's ability to interpret, who through them manages to establish the relationship of proportionality and growth between the variables.

Continuing with David's interpretation of the problem stated in **Figure 5**, we can see in **Figure 6** (in the blue box) that for him the function highlighted with the red color corresponds to $g(x) = x^3$, and that it is wider compared to the parabola, which allowed us to deduce that the blue colored graph belongs to $h(x) = x^2$. In addition, it can be observed that when he compares the behavior of the functions $h(x)$ and $g(x)$ in the interval [0, 1], he validates that as the exponent "is larger" the value of the image will "be smaller", therefore, the red graph is the one that corresponds to $g(x) = x^3$.

The mathematical activity shown above shows the student's ability to interpret a problem by verbally describing the numerical properties of the graphical behavior of the function in a given interval. The aforementioned allowed David not only to relate the graphical representation with the algebraic one, but also helped him to clarify concepts of domain and range, since by visualizing the graphical representation of these functions he identified the intervals in which the functions were defined (**Figure 7**).

The following are the results of the student's graphical interpretation of the functions that allowed him to establish the domain and range of the function.

By way of conclusion, it can be inferred that with the activities proposed in workshop 5 David privileges the interpretation of the problem from the numerical interpretation of the analytical properties of the graphical representation and the use of mathematical properties of real numbers.

Another strategy used by David for the interpretation of a problem is given from the exploration with GeoGebra in activity 2 of workshop 3, which is described in **Figure 8**. This workshop focuses on the study of trigonometric ratios by identifying the signs (+, or, -) through the quadrant in which the legs of the right triangle were located, establishing the maximum and minimum values that each trigonometric ratio could take, and at the same time making conjectures regarding the approximation and trend in particular angles (0°, 90°, 180°, 270°, and 360°).

To respond to the previous activity, the student turned to the interpretation of the situation from the interaction with GeoGebra, to identify the legs and hypotenuse of the right triangle formed, in order to establish the quadrants in which the trigonometric ratios were positive or negative. In addition, the exploration with GeoGebra allowed David to give an adequate interpretation of the situation as he varied the angle, thus emphasizing the intervals in which the trigonometric ratios were defined and how they varied.

In **Figure 9** shows the answers given by the student to the problem posed, which were directly permeated by the use and exploration with GeoGebra.

From **Figure 9**, one of the difficulties can be deduced to have presented in the inappropriate use of mathematical language since the trigonometric ratios are called sine, cosine, tangent, secant, cotangent, cosecant

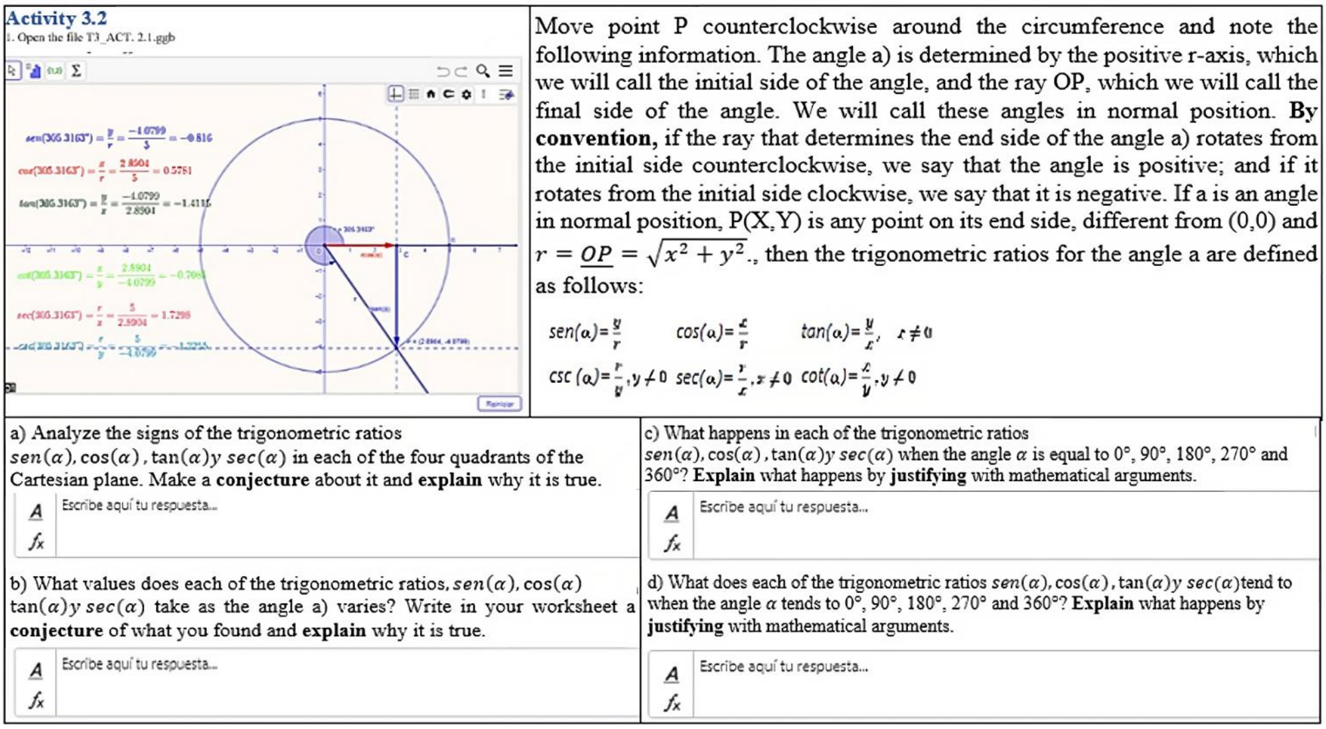


Figure 8. Activity 2 workshop 3 of the precalculus course (Source: Authors' own elaboration)

ANSWER

Sine: In the 1st and 2nd quadrants it is positive because sine is governed by the opposite side and this is parallel to the positive y-axis. In the 3rd and 4th quadrants, it is negative because sine is governed by the opposite leg and it is parallel to the negative y-axis.

Cosine: In the 1 and 4 quadrants it is positive because cosine is governed by the adjacent leg and this is parallel to the positive x-axis. In the 2 and 3 quadrants, it is negative because sine is governed by the adjacent leg and this is parallel to the negative x-axis.
Tangent: In the 1 and 3 quadrants it is positive, and 2 and 4 are negative because the Tangent is governed by the opposite and adjacent leg, (Y and X) respectively.

Secant: In the 1st and 4th quadrants is positive, because the secant is governed by the adjacent leg and this is parallel to the positive x-axis. In the 2nd and 3rd quadrants, it is negative because sine is governed by the adjacent leg and this is parallel to the negative x-axis.

Figure 9. David's answer to question a, from activity 2 workshop 3 (Source: Authors' own elaboration)

without considering the argument in each of them. On the other hand, it is inferred that there is no mathematical background in the interpretation of the positive and negative signs of each interval when analyzing the behavior of the trigonometric ratios; on the contrary, these interpretations are given thanks to the exploration with GeoGebra, which allows visually perceiving and identifying key elements (**Figure 10**).

Through the previous answers (**Figure 9** and **Figure 10**), David is perceived to focus only on interpreting the graph through the exploration with the software, but such a perception is unwarranted; therefore, it was necessary to guide and reinforce with all students the importance of answering the question why?, although it is known that it is not of interest at this early stage, given that so far the interpretive skill is strengthened.

After the exploration with GeoGebra and the collaborative work among students through the zoom group modality, significant progress can be seen in the student with the use of digital technology, since the student does not remain simply in the exploration but manages to perceive some emerging OMs that lead to the preliminary use of the appropriate mathematical language and an interpretation through it, as evidenced in **Figure 10**.

The answer in **Figure 9** also shows that the manipulated digital representation favored the student's interpretation of the intervals and quadrants in which the trigonometric ratios are defined since it transcends the idea of particular cases to general cases. What is interesting in this activity is that David interprets the problem posed with the help of the software and can account not only for the magnitudes involved in the situation posed but can also express which of them are variables and how they vary.

From the answers given by the student in workshop 3, it can be deduced that the student also uses other strategies to be able to interpret a problem, as he did in this opportunity through the exploration with GeoGebra. In addition, it not only allowed him to

b) What values does each of the trigonometric ratios $sen(\alpha)$, $\cos(\alpha)$, $\tan(\alpha) y\ sec(\alpha)$ take as the angle α varies? Write in your worksheet a **conjecture** of what you found and **explain why** it is true.

Answer
sin of alpha ranges from (-1,1).
cos of alpha ranges from (-1,1)
tan from -infinity to infinity
sec from –infinitv to one and from 1 to infinitv

c) What happens to each of the trigonometric ratios $sen(\alpha)$, $\cos(\alpha)$, $\tan(\alpha) y\ sec(\alpha)$ when the angle α is equal to 0°, 90°, 180°, 270° y 360°? **Explain** what happens **justifying** mathematical arguments.

Respuesta

sen(0)=0	sen(90)=1	sen(180)=0	sen(270)=-1	sen(360)=0
cos(0)=1	cos(90)=0	cos(180)=-1	cos(270)= 0	cos(360)=1
tan(0)=0	tan(90)= indefinido	tan(180)=0	tan(180)= indefinido	tan(360)=0
sec(0)=1	sec(90)=indefinido	sec(180)=-1	sec(180)= indefinido	sec(360)=1

d) A that you have each of the trigonometric ratios $sen(\alpha)$, $\cos(\alpha)$, $\tan(\alpha) y\ sec(\alpha)$ when the angle α tends to 0°, 90°, 180°, 270° y 360°?
Explain what happens **justifying** with arguments

Respuesta

sen(0)=0	sen(90)=1	sen(180)=0	sen(270)=-1	sen(360)=0
cos(0)=1	cos(90)=0	cos(180)=-1	cos(270)= 0	cos(360)=1
tan(0)=0	tan(90)= indefinido	tan(180)=0	tan(180)= indefinido	tan(360)=0
sec(0)=1	sec(90)=indefinido	sec(180)=-1	sec(180)= indefinido	sec(360)=1

Figure 10. Solution given by the student to question b, c, and d (Source: Authors' own elaboration)

understand the problem but also to move from particular to general situations, as shown in **Figure 9**.

Both the observations of the video recordings and the written productions of the case study students have made it possible to identify the interpretive skill as one of the skills that can reinforce through the pre-calculus course; you can see this skill in the following behaviors:

1. Identify patterns to predict specific variational behaviors.
2. Determine particular values to establish a ratio between two quantities.
3. Determine the global behavior of the ratio established between two quantities.
4. Determine variation and change that occurs in one quantity when another quantity varies.

The interpretive skill, as mentioned above, is a skill that is worked on in all the workshops of the pre-calculus course. Students are expected to express a supported answer, that is, to make use of mathematical language and theoretical elements such as a proposition, and a theorem, among others; so, to achieve this, a correct interpretation must be made of all how the activities are presented, whether in verbal, pictorial, algebraic language or with the help of the software.

Ability to Explain Variation

As mentioned above, the explanatory skill is related to the student's ability to describe an OM of study. For such a description, one can use particular examples, explain the reason for each action performed to account for the solution of a problem, use mathematical language, and generalize patterns that lead to a conjecture or premise, among other strategies. Interpretation and explanation cannot be seen as separate skills, since for the student to find a correct explanation to the proposed problem, it is essential to have clarity about the problem to be addressed, that is, to have given an adequate interpretation to it.

Activity 8.1
1.1 In Bucaramanga, in a resale of minutes, $100 per minute or fraction thereof is charged.

a. How much does a call of 25 s, 1 min, 1min 1s, 1 min 30s, 2 min 15s, 2 min 59s, 4 min, 5 min 36 s, 9 min 50s, 23 min 48 s, 1 h 15 min 32 s?

Escribe aquí tu respuesta...

b. Find the function that represents the interdependence between the cost of the call and the time.

Escribe aquí tu respuesta...

1.2 Communicating and sharing
Discuss with your classmates and the teacher the results obtained. Write your conclusions on the worksheet.

Escribe aquí tu respuesta...

Figure 11. Workshop 8: Consumption rates, activity 8.1 (Source: Authors' own elaboration)

Below are some of David's responses to the proposed activities and transcripts that show the acquired explanatory skills. Some of these results were influenced by the interactions with his classmates and their interest in sharing their solutions.

In activity 8.1, in sections 1.1 and 1.2 of workshop 8 called consumption rates (**Figure 11**), the student is proposed to model a situation of consumption of minutes, which is related to the integer part function; however, most students associate this problem with the linear function.

David is one of the students who gives an adequate interpretation of the problem, starting with concrete examples in which he identifies how much he must pay for a certain number of minutes, as shown in **Figure 12**. It is important to emphasize that the student manages to identify the behavior of the model, through the

Activity 8.1
Homework 1
1.1 In Bucaramanga, in a resale of minutes, $100 per minute or fraction thereof is charged.
a. How much does a call of 25 s, 1 min, 1min 1s, 1 min 30s, 2 min 15s, 2 min 59s, 4 min, 5 min 36 s, 9 min 50s, 23 min 48 s, 1 h 15 min 32 s cost?
Calls of 25 sec and 1 min cost 100
1 min 1s, 1 min 30s, cost 200
2 min 15s, 2 min 59s, cost 300
4 min cost 400
5 min 36 s cost 600
9 min 50 s cost 1000
23 min 48 s cost 2400
1 h 15 min 32 s = 75 min 32 s cost 7600.

Figure 12. David's answer to activity 8.1 of Workshop 8 through concrete examples (Source: Authors' own elaboration)

Homework 2
b. Find the function that represents the interdependence between the cost of the call and the time.

f(x) = 100.x
{ 1   0 < x < 1
2   1 < x < 2
3   2 < x < 3
4   3 < x < 4
5   4 < x < 5
6   5 < x < 6
7   6 < x < 7
8   7 < x < 8
and so, on until it is spoken by person D:
f(x) =110.x {If x does not belong to the integers. x is taken to be the nearest larger integer.

Figure 13. Solution using mathematical and everyday language to activity 8.1 of workshop 8 (Source: Authors' own elaboration)

particularized examples, being a very important moment in the resolution of the problem, since it allows inferring the correct interpretation given by the information provided, a fact that led David to the use of mathematical language and to raise some conjectures that he was justifying.

The previous situation led the student to identify that the behavior of the function modeling the problem was not a linear function, although, in **Figure 13**, it is exposed as a linear model when writing $f(x) = 100x$, David clarifies that certain restrictions must be met. From this answer it is important to highlight how the student tries to use verbal and mathematical language despite not knowing the function, mentioning that: "$f(x) = 100x$ (if x does not belong to the integers, x is taken by the nearest greater integer)", this shows that the student has an approach to the integer function, however, he does not manage to express it through an algebraic language.

It is noteworthy that in the student's interest to understand his answer and that his classmates understand his solution, he uses different ways to represent the information, which leads to the same idea, the integer function. To arrive at this terminology, the student relied on his classmates and other media such as the Internet, because he had clarity in the behavior of the function, mentioning the need to approach the nearest integer, however, David makes it known that it is a "rare" function for him, because he had never worked with it. This shows a new learning process in the student.

From the previous event, it is also important to highlight that the interaction between students to explain their solutions led to a discussion about the integer function of the floor and ceiling function. The following is a fragment in which the student explains his solution to his classmates, in detail, and through different representations, he can explain how he obtained the previous answers.

> Yurley: My function is *f(x)=100 E(x)*.
>
> Teacher: Who is *E(x)*?
>
> Yurley: Teacher, as we were saying that it has to approximate the integer, I looked for a function that did that, and I found the integer part function, floor and ceiling function, to make that approximation.
>
> Jeremías: I think that is the function, at least I had not heard about *E(x)*. It gave me *f(x)=100x*, but x always has to be approximated, for example, in 25 seconds we already know that we are going to charge one minute, that is, *x*=1. But the problem is that x is equal to the time, but approximating it, we would have to do it manually or as the classmate says with that function, it is already opening my mind, and I can see with that how to go to an exact number, not manually.
>
> David: Teacher, I have the same idea as my classmate, but I do not know very well how to call that function, as I did not know what my classmate said about *E(x)* I put a condition, and

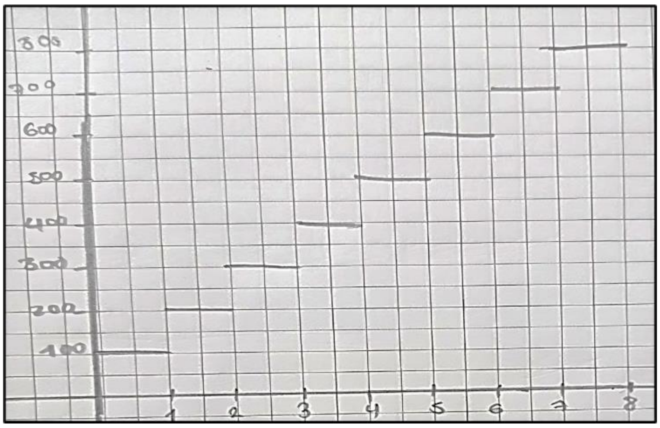


Figure 14. Solution to activity 8.1 of workshop 8 by means of graphical representation (Source: Authors' own elaboration)

> my solution is *f(x)=100x* (if *x* does not belong to the integers, *x* is taken by the nearest integer) (**Figure 13**). For example, if I have 1.5 the nearest greater integer will be taken, i.e., in this case, two. If I have 2.3 the nearest greater integer must be taken, i.e., three. If I have 4.1, then five, and so on.
>
> Cristóbal: Teacher, precisely the function that David tells us about is what the integer part function does, put an integer number less than, greater than, or equal to *x*, which is what the integer part function does, in this case, we approximate the greater one.
>
> Teacher: Why do you take the x as the closest major integer and not the closest minor integer?
>
> David: Because we are always going to approach the next number, it is like in real life, if one speaks one minute with five seconds they charge two minutes, that is why we must approximate the decimal to a larger natural number, because we must pay more and not less, because we do not want to approach the smaller one, because, for example, if it is 1.5 and we approach the smaller one, it gives us one and we are interested in two. Look, teacher, it would look like this representation, that's what I mean (**Figure 14**).

The previous interaction allows showing how the explanation given by the students is fundamental to generating spaces for discussion and learning, besides strengthening the mathematical language, since David had his intuitive idea of the function, however, they did not know the mathematical symbology to refer to it, a situation that was favored when another of his classmates gave the appropriate name to this function. In addition, the previous discussion allowed David to show different representations of the same situation, which led to the conviction of his classmates that what he had done was correct.

Another of the proposed activities that show the explanation skills developed by the student is presented in the activities of workshop 11 (**Figure 15**) in which the student, through interaction with the software and particular examples, can interpret the problem and thus establish some conjectures. However, it is important to highlight that in workshop 11, most of the students were not satisfied with giving explanations only through particular examples or "because it was seen in the software", on the contrary, they tried to give more explanations from the mathematical foundation, which was a great advantage because it allowed them to convince their classmates more easily. Such is the case of David, who resorts to concepts of similarity and proportionality to give an answer to activity 11.2 of workshop 11, as described below.

Through the interaction with the software David manages to perceive the magnitudes that are varying, he mentions that: "as the cone is filled with water, the

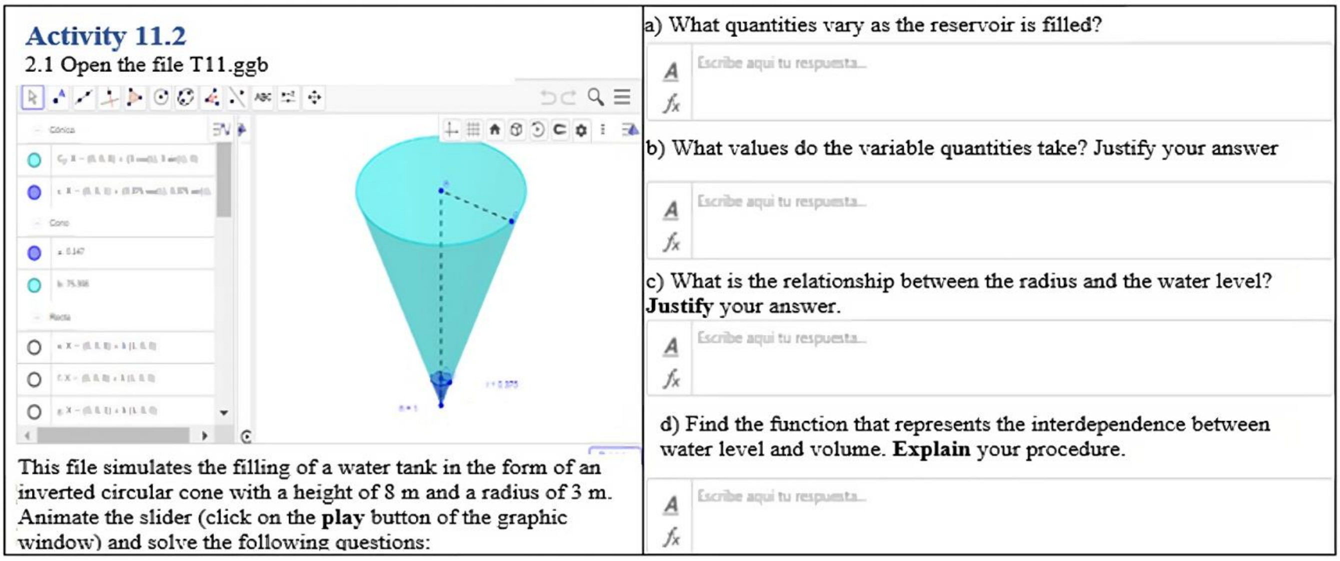


Figure 15. Activity 11.2 of workshop 11 (Source: Authors' own elaboration)

c) What is the relationship between the radius and the water level? **Justify** your answer.
Answer
The larger the radius, the higher the water level, which means they are directly proportional.

Figure 16. Answer given by David to activity 11.2 (Source: Authors' own elaboration)

Answer

$3h = 8r$
This relation is drawn by means of similar triangles.

Figure 17. Radius to height ratio found by David (Source: Authors' own elaboration)

height, the radius, and the volume increase", he also establishes the values that each of the variables can take, mentioning that: "The radius varies from zero to three, the height varies from zero to eight, the volume varies from zero to 24". This interaction with the software also allows him to conjecture that the larger the radius, the higher the water level, erroneously arguing (like most of the students) that if one magnitude increases and the other increases, the magnitudes are directly proportional (**Figure 16**).

Regarding question c (**Figure 16**) in the first answer David, gives a wrong argument (most students assume that, if one magnitude increases and the other increases, the magnitudes are directly proportional, and if one increases and the other decreases they are inversely proportional). By asking more complex questions, the student poses another answer and manages to establish the conjecture shown in **Figure 17**.

The previous conjecture was given by the student through the interaction with the software and particular examples, however, in clarifying mathematically and not to remain only in the examples, he turns to properties of the similarity, as shown below:

David: I moved the t (the slider) and saw two similar triangles. We know that the largest radius is three (given information) and the largest height is eight. Then here (**Figure 18**) are two triangles, and these are similar.

Teacher: What are similar triangles?

David: Similar triangles tell us that: two triangles are similar if they have the same internal angles; that is, the three angles are equal to those of another triangle regardless of their size. I do not know how to put it, teacher (he thinks for a few minutes), mm I know, if it has proportional sides and equal angles. For example, pointing to **Figure 19**, x that is the leg, over the same side of the other triangle, which is b, and then the other side over the other side of the other similar triangle. This equality will always hold.

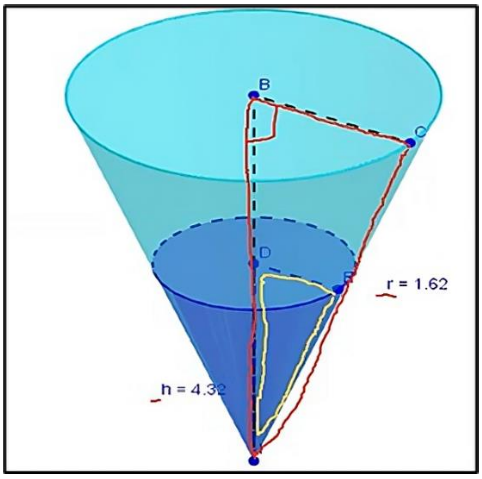


Figure 18. Solving the problem using similar triangles (Source: Authors' own elaboration)

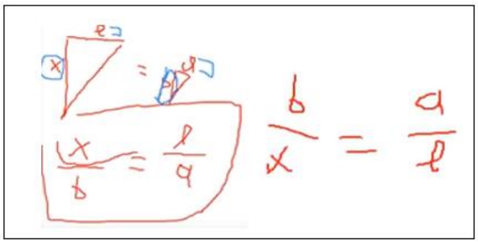


Figure 19. David's explanation of what a similar triangle is (Source: Authors' own elaboration)

Teacher: Going back to what David mentioned, he says that you can take two similar triangles and that the idea was to find a proportion (triangles drawn on the left side of **Figure 20**).

David: Yes, teach, here we see that the radius and the height are varying as time progresses. Now, applying similar triangles, we would get the radius of the cone as three, the height of the cone, which is eight. Now, we name those of the triangles that give the water level: it would be r and h, so applying similarity, we place the major radius over the minor radius $\frac{3}{r}$ and the greater height over the smaller height $\frac{8}{h}$, and then it would be $\frac{3}{r} = \frac{8}{h}$ (**Figure 20**). That would be the same as writing $3h = 8r$ and there we find a proportion. Each time the radius grows, the height will also grow in a higher proportion.

Yadira: Oh, teacher, it has to do with what we said just now that h/r=2.6 and r/h=0.375, as we saw in the example because 3/8=0.375 is the proportion that we had found numerically; so if the process is correct.

The previous fragment shows that the student is no longer satisfied with taking particular cases or giving answers attributed only to the software; he requires

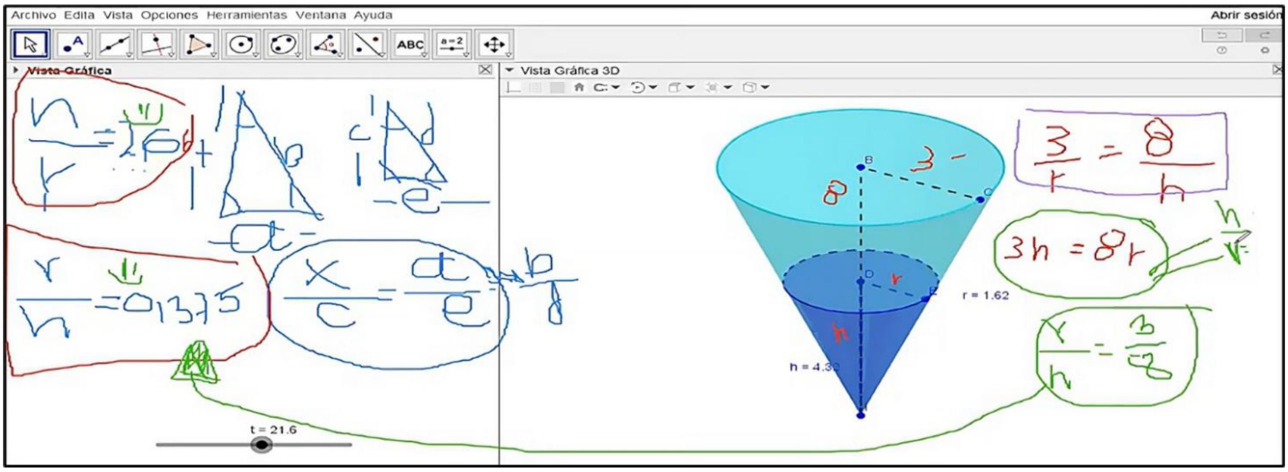


Figure 20. Radius and height ratio of the cone (Source: Authors' own elaboration)

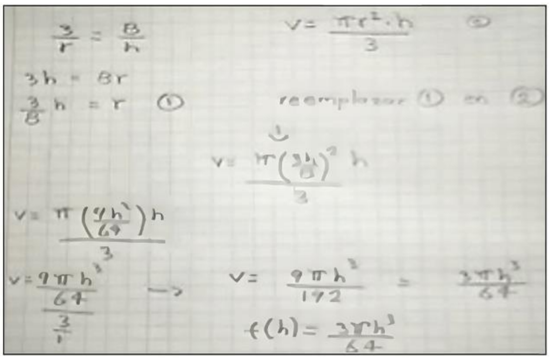


Figure 21. Solution to the cone problem (Source: Authors' own elaboration)

mathematical arguments that allow him to make his answers credible to his classmates. Through the dialogue, you can see how the classmates realize that the process carried out by David is indeed correct and that starting from specific cases, reaching a generality and a more solid argument is possible.

David continues with this same reasoning that allows him to provide an accurate answer to the problem posed, as shown in **Figure 21**. It is important to highlight the student's ability to test the different concepts worked on and mathematical formulas, such as the volume of the cone, as well as to use other OMs as a method of substitution. In the student's answer, it follows David to manage to explain the method used clearly, precisely, and concisely to find and validate the conjecture, a fact that favored the understanding of the problem to some students who had not understood it and did not have a way to follow.

Below are some examples of actions that characterize this skill in terms of the objects involved in situations of change and variation worked on in the pre-calculus course.

1. Express in natural language the ratio between two or more quantities that vary.
2. Express in natural language the functional relationship between two varying magnitudes from the tabular register and its algebraic expression.
3. Construct different representations (graphical, tabular, and algebraic) from a statement in the natural language of a functional situation in which the ratio between two or more quantities that vary is presented.
4. Create a graph that shows the variation of two or more magnitudes, either with a graph in the Cartesian plane, a static graph, or a digital representation.

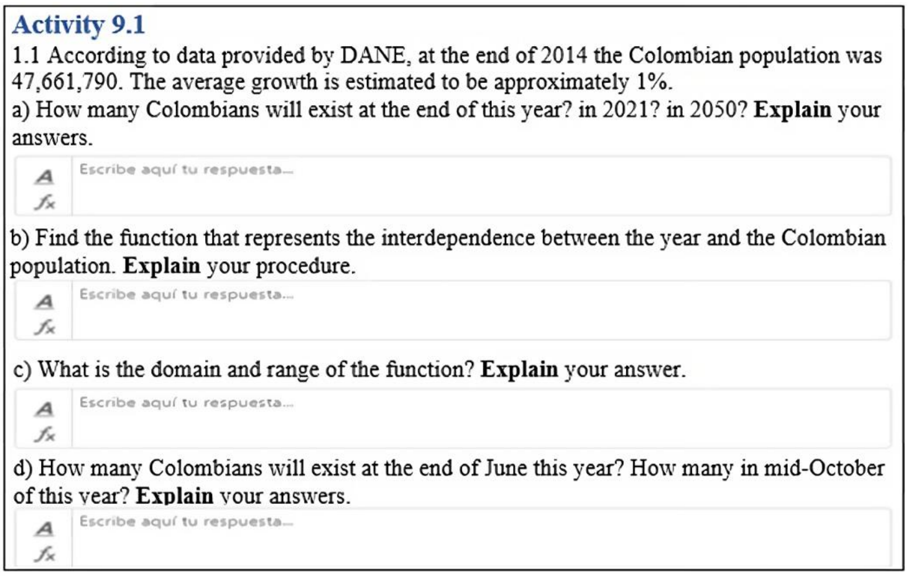
Ability to Justify Variation

Different authors such as Parra et al. (2010) agree that justification can be observed in those actions performed by the student to clarify, give reasons, develop ideas, and prove or disprove hypotheses using different representations and compelling mathematical arguments.

In this section, the activities of workshop 9: functions and reality and, workshop 13: box without a lid in which the student's justifying skill is evident, are retaken. This time, emphasis is placed on the mathematical arguments used by the student, and the different representations and examples used to prove or disprove the hypothesis.

The justifying skill is evident in the previous workshop (**Figure 22**), since it implies that the student performs a correct interpretation of the problem, achieves from particular examples, the exploration and verification to find a function that represents the proposed situation, provides explanations from the mathematical point of view on the conjectures raised, is able to explain the reasoning behind the reasoning performed, and can convince his peers and the teacher that the reasoning performed is correct.

In general, this activity was very hard since they did not understand the meaning of "1% growth of the population" and did not associate the interdependence between the year and the population since they did not consider the previous population to find that of the


Activity 9.1

1.1 According to data provided by DANE, at the end of 2014 the Colombian population was 47,661,790. The average growth is estimated to be approximately 1%.

a) How many Colombians will exist at the end of this year? in 2021? in 2050? **Explain** your answers.

Escribe aquí tu respuesta...

b) Find the function that represents the interdependence between the year and the Colombian population. **Explain** your procedure.

Escribe aquí tu respuesta...

c) What is the domain and range of the function? **Explain** your answer.

Escribe aquí tu respuesta...

d) How many Colombians will exist at the end of June this year? How many in mid-October of this year? **Explain** your answers.

Escribe aquí tu respuesta...

Figure 22. Activity 9.1 of workshop 9 (Source: Authors' own elaboration)

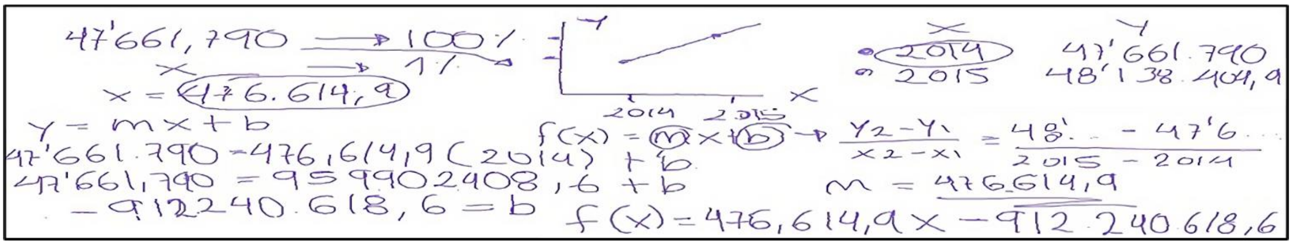


Figure 23. Misinterpretation of the problem of activity 9.1 (Source: Authors' own elaboration)

following year. As a consequence, this led some students to interpret the problem as a linear function (**Figure 23**).

In the case shown in **Figure 23** the student finds 1% of the population (of the year 2014), and then the result obtained is added to the population of the year 2014, i.e., in the year 2015 the population would be $P(2015) = 47{,}661{,}790 + 476{,}614.9 = 48{,}138{,}404.9$.

Simultaneously, the student finds the population in subsequent years; however, the 1% that she considers is constant, i.e., it is the population in the year 2014, which is one of the first difficulties in the interpretation of the problem. Therefore, having two respective values for the Colombian population and time, the student turns to the point slope equation to find the linear function according to her modeled situation, as shown in **Figure 23**.

This led some students, such as David, to refute this solution, since they had another interpretation of the problem, a fact that emerged after the interaction among the students and that gave clarity to the problem. David, in response to his classmate's answer, mentions that he found another different function, which for him is not a linear function because it depends on the previous population, which he manifests by saying: "it is like the problem of the medicine" (another workshop worked on), I followed the same idea.

> David: First of all, I took $P0$ to look at the population, because it said that it increased by 1%, then it is necessary to take $P0$ and get $P0(100\%) + P0(1\%)$, this would give in the following year, this would be $P0$ plus the other value. Here we can take a common factor to $P0$ and it would be $P0(1 + 1{,}01)$, this would be for the year 2015, then for the year 2016, it would be $P1$ multiplied by $100\% + 1\%$, which is what increases, and we replace $P1$ and it would be $P0(1 + 0{,}01)(1 + 0{,}01)$ (**Figure 24**).
>
> We repeated this process quite a few times, and we realized that for each year that increases, we found a pattern, for example, it would be $P0(1 + 0{,}01)^2$ and we realized that two (the exponent) represented the number of years, so we will always have that $P0$ is the initial population in 2014, multiplied by (1+0,01) raised to an exponent that represents the number of years; for

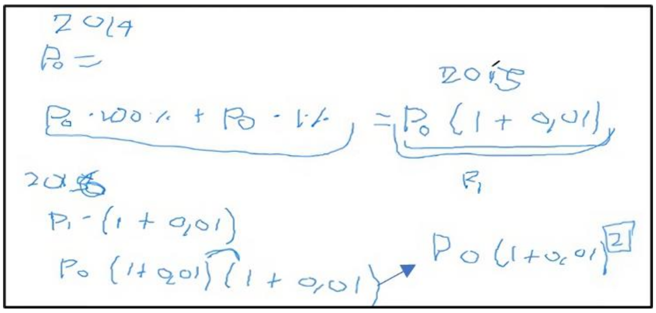


Figure 24. Explanation of the conjecture proposed by David (Source: Authors' own elaboration)

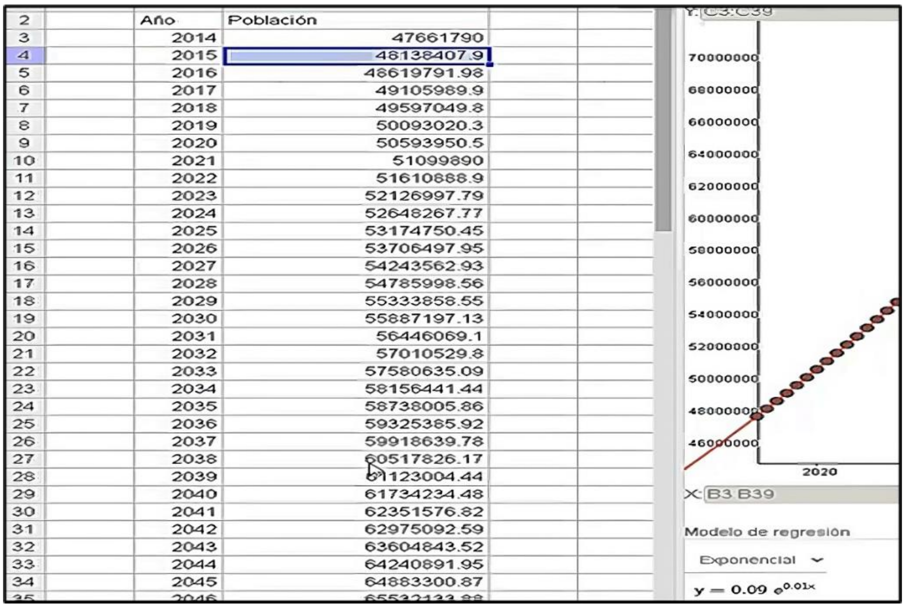


Figure 25. David’s solution to activity 9.1 by digital technology using particular data (Source: Authors’ own elaboration)

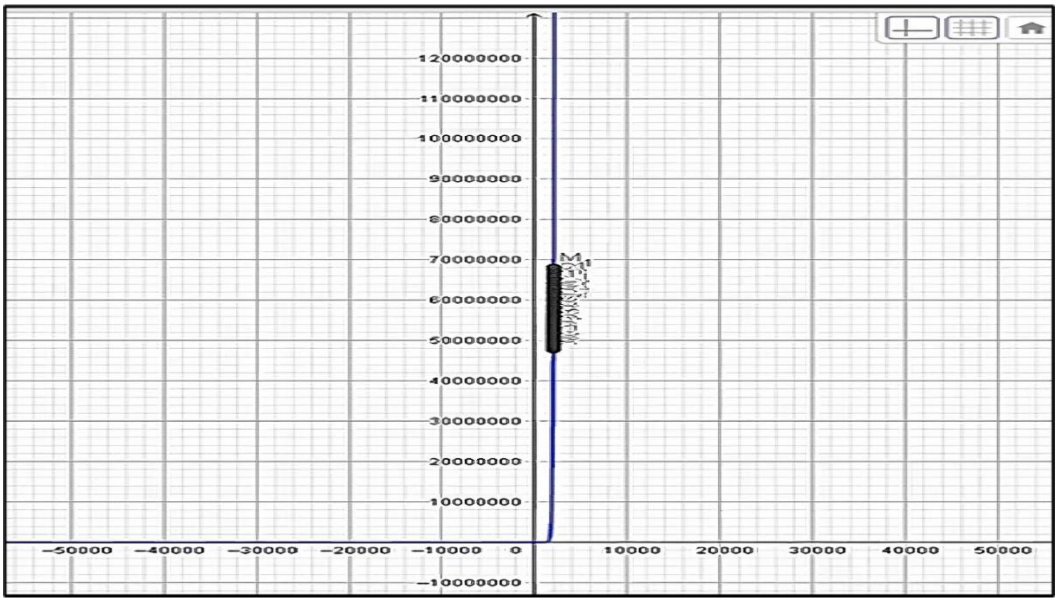


Figure 26. David’s solution to activity 9.1 by digital technology using particular data (Source: Authors’ own elaboration)

> example, to find 2020, then we find how many years there are from 2014 to 2020, that is to say, it gives six, which would be the exponent and from there the function comes out. I checked it with five values, and it worked.
>
> José: Teacher, it was very similar for me, but I made it simpler, I just made a spreadsheet and that’s all, and the results came back. So, what my colleague did is fine (**Figure 25** and **Figure 26**).

The digital representation can be perceived to have been a fundamental resource for the students to justify their answers since they interpret the statement and understand the dynamics of the situation to pose the population in the following years, although they do not express their interpretation in an algebraic expression, they manage to do it graphically taking advantage of the digital technology, showing directly how one magnitude changes concerning the other, and also to obtain the population for more than a few particular cases. This type of activity favors the students’ justifying skills since an association exercise is done between the visual that allows the digital and oral representation in terms of the communication skill.

The above situation also shows that the students were not satisfied only with particular cases; on the contrary, they used previous knowledge, workshops, and mathematical properties in their interest in generalizing to a function. The mathematical language used is more formal, and they manage to perceive a solution through different representations, i.e., they do not stay with a single solution but transcend between the numerical,

Workshop 13. Box without a lid

Activity 1

1.1 From a rectangular sheet of size 6 dm by 4 dm, construct a box without a lid by cutting out squares of equal size at the four corners, so that it stores the largest volume. What are the dimensions of the height, width, and depth of the box with the largest volume? **Why? Explain your procedure and your answer.**

1.2 Communicating and sharing results

Discuss your results with your classmates and your teacher. **Write** your conclusions on the worksheet.

1.3 Open the GeoGebra file T13.ggb and animate point P.

a) On which variable quantity or quantities does the volume of the box depend? Why?
b) What values can the height take? **Why?**
c) What values can the width take? **Why?**
d) What values can the depth take? **Why?**
e) What values can the volume take? **Why?**
f) What is the relationship between width and height? **Why?**
g) What is the relationship between depth and height? **Why?**
h) Represent algebraically the volume based on height.
i) What are the dimensions of the height, width, and depth of the box with the largest volume? **Why?**

Figure 27. Workshop 13-Box without a lid (Source: Authors' own elaboration)

algebraic, and graphical. What has been described shows that students explore, verify, relate solutions, and contribute significantly to the resolution of problems from a mathematical perspective; their arguments are more solid and do not remain in mere particularization.

Another activity that shows the student's justifying skill is found in workshop 13 in which knowledge about the volume of a box, algebraic expressions, variation and change, the interdependence of variables, and notions of the derivative of a function are tested (**Figure 27**).

In this activity, it is interesting to see how David uses a more solid mathematical language and no longer uses random values to assign the length of the side of the square to be removed at each end, on the contrary, he is clear that he needs to solve the problem based on more solid arguments and theoretical tools of the same discipline, For this reason, he decides to understand the problem from a graphical representation and based on this, identify some algebraic expressions that can be related to each other, to leave the problem expressed in a single variable and thus use the formula of the volume of a box, as shown in **Figure 28**.

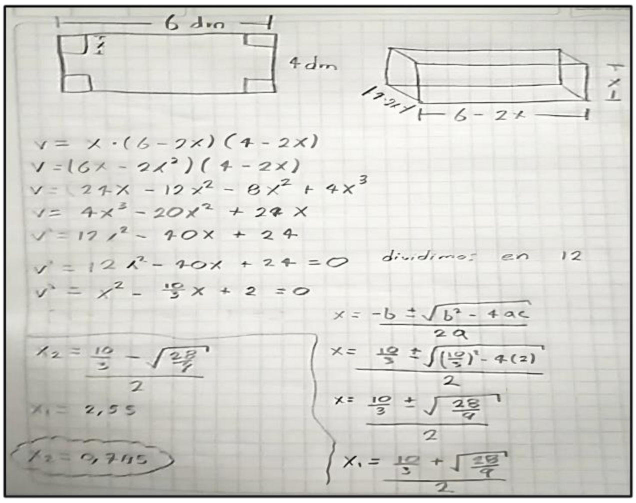


Figure 28. Student's justification of the problem of the box without a lid (Source: Authors' own elaboration)

For the solution to the problem, the student decides to call x the side of the square that is being removed and based on this, he finds the lengths of the rectangular sheet, i.e., $6 - 2x$ y $4 - 2x$. The correct interpretation of the student is observed to the problem to lead him to raise these expressions; later, he uses these expressions to find the volume of the box, which shows the use of mathematical arguments used by the student for the solution to the problem.

Regarding the values that each measure of the box can take, the student also goes to the correct interpretation of the problem and observes the minimum and maximum value that could be found considering the context of the problem, for this reason, the questions in **Figure 27** are answered as shown in **Figure 29**.

With the previous answers, the student manages to question himself about the maximum and minimum values that can take the width, height, and depth of the box, giving justifications from the mathematical point of view of why to take some values and not others, this when he mentions the impossibility of taking negative numbers and when he establishes that these also depend on the maximum and minimum value of the lengths of the rectangular sheet.

Subsequently, the student establishes the intervals in which each length of the box varies, to proceed to check his answers by interacting with GeoGebra and thus determine if the established values were correct since the digital representation directly shows the variant and invariant magnitudes just by manipulating a point of the representation, as shown in **Figure 30**.

We discard as answer x=2.55 because we know that the smallest side measures 4-2x, and if we replace 4-2(2.55) = 4-5.1 =-1.1 and it is automatically discarded because there are no negative distances.	
It depends on the side of the squares formed at the corners since the 3 magnitudes of the box can be written based on that side.	The height can take the values (0,2) this is because the height is going to be the x-side of the squares to be cut out of the sheet and the shortest side of the sheet must measure 4.
The depth can take the values of (2,6) since 6 is the maximum length of the sheet and, knowing that the minimum value is 2 if we cut the squares as large as possible with side = 2 gives us that the two sides of the two	The width can take the values of (0,4) since the problem tells us that one of the sides (in this case the one taken for the width) must measure 4 so it can not exceed that value.

Figure 29. Magnitudes that the box dimensions can take (Source: Authors' own elaboration)

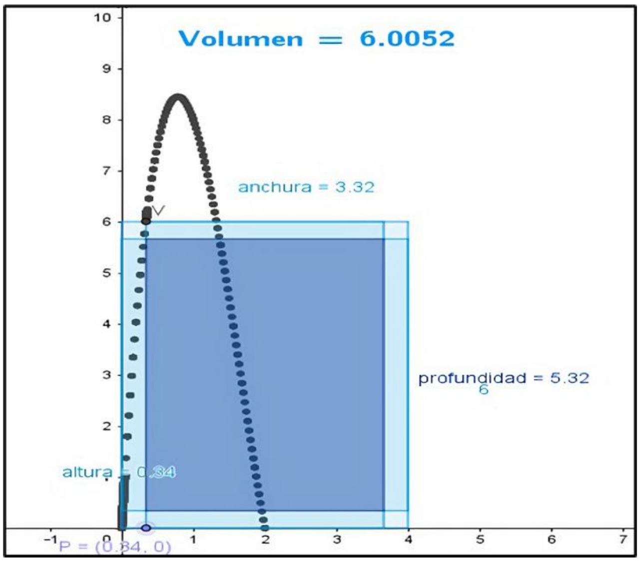


Figure 30. Maximum volume of the box by using digital graphic representation (Source: Authors' own elaboration)

At the moment there was no difficulty for the student; however, when inquiring about the lengths of the box to find the maximum volume, there is already a complexity by the use of new OMs such as the derivative, an object of study that was remembered by the student by conducting previous workshops. Therefore, when David uses GeoGebra properly, he can confirm with his classmates that the reasoning followed using the derivative of the function was the correct one to find the maximum volume of the box, thanks to the analytical interpretation of the derivative through the manipulation with the software, since its behavior is clearly seen and, the maximum and minimum points are established (**Figure 30**). Finally, this type of activity is highlighted to favor the students' justifying skills since an association exercise is done between the visual that allows the digital and oral representation in terms of the communication skill.

In this skill, the process of learning to write mathematically is similar to learning to write of any kind. This process is gained through practice with expert guidance. When students practice communicating they should express themselves more clearly and coherently and, also, acquire and recognize the different mathematical styles of dialogue and justification. Through the stages, their justifications should become increasingly complete, and students should be able to state the properties used to support their justifications.

This research allowed us to identify that the justifying skill can reinforce through the activities of the pre-calculus course; this is by experimentation and exploration of situations that involve change and variation. As previously mentioned, justification is observable in actions channeled in the coherent evidence of ideas. Some of the actions identified for this skill are, as follows:

1. To make a graph that shows the variation of two or more quantities, whether it is a graph in the Cartesian plane, a static graph, or a digital representation.
2. Use hand or body movement to model the behavior of two or more varying quantities.
3. Determine an algebraic expression from a natural language statement of a functional situation.
4. Construct different representations (tabular, graphical, and algebraic) from a natural language or mathematical statement of a functional situation.
5. Express in verbal or mathematical language the functional relationship between two quantities that are changing and varying.
6. Make a table that presents the ratio between two or more quantities that vary, either in a numerical or symbolic formulation.

CONCLUSIONS

The characterization of the communication skills was raised in the list of descriptors listed in the previous section, which account for specific actions to evaluate the interpretation, explanation and justification of variation skills, by students. From the research emerges the description of communication skills around variation and change, as follows:

- Students develop the ability to interpret statements in verbal language when they can identify the conditions of the problem, the unknowns, the data present in the statement, and the data that are not explicitly found. It is considered here that a student who correctly interprets a statement and has no prior knowledge will not be able to solve a problem.

- Explanations can become a tool for understanding, as Duval (1995) puts it: an explanation provides one or more reasons to make a fact, phenomenon, or result understandable. NCTM (2000) document states that when students are encouraged to think and reason about mathematics and to communicate the results of their thinking to others, orally or in writing, they learn to be clear and convincing. Listening to others' explanations gives them opportunities to develop their understanding.
- Express in verbal or mathematical language justifications of the problem posed. Make appropriate use of mathematical language in problem solving.

Finally, it is worth emphasizing that the skills mentioned above can only be achieved if discussion and confrontation of ideas among students are promoted, where the need to convince themselves and others of the truthfulness of their statements must be stimulated and insisted upon. In this process of conviction and validation, the use of arguments expressed in an increasingly formal and generalized mathematical language and the use and coordination of different systems of representation are necessary.

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