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Difficulties in Learning Inequalities in Students of the First Year of Pre-University Education in Spain

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We present a summary of a study carried out with students of the first year of Bachillerato (the first of two pre-university non-obligatory secondary education courses in Spain) to determine and analyse some of their errors and difficulties in learning inequalities with the aim of improving the teaching-learning process of this topic. The study was based on work on the initiation to algebra, in particular on the observed difficulties and errors relative to algebraic skills.

Keywords: Inequalities, Students' Misteakes, Secondary, Algebra, Mathematics

INTRODUCTION

Our teaching experience has allowed us to observe the difficulties that Bachillerato students have, and the errors they make, when they are studying inequalities. Many of these problems re-occur year after year. This motivated us to look for what might be some of the causes so that they could be addressed directly or at least help us to rethink how we approach the topic.

The present work formed part of the doctoral program given by the Department of Didactics of the Experimental Sciences and of Mathematics of the University of Extremadura (two-year course 1999-2001), in the second year of which a research line was developed on "Errors and Difficulties in the Teaching/Learning of Mathematics".

In this framework we elaborated a project whose main objective was: "To describe and analyse certain errors and difficulties of first year Bachillerato (note 1) students studying the options of Nature and Health Sciences or Technology in learning inequalities with the goal of improving the process of teaching-learning of that topic". The goal of the present work is to describe and discuss some of the results without it being intended as a research report.

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Theoretical Framework

We considered as referent the concept of epistemological obstacle (Brousseau, 1997). This is characterized as knowledge which has generally been satisfactory for the resolution of certain problems for some time, and which has thus become settled in the student's mind, but which subsequently is found to be inadequate when the student is confronted with new concepts and mathematical processes. According to Brousseau, its origin could be ontological or psychological, educational or epistemological (Brousseau, 1997).

We propose working on the mathematical concepts and processes that the students have studied but use inappropriately when they are dealing with inequalities. In this regard, we assume that knowing a mathematical object implies understanding and integrating definition, different systems of representation, properties, and applications (Gutiérrez & Jaime, 1996; Goldin & Shteingold, 2001; Blanco, 2001).

Likewise, we assume that "student errors are the result or the product of previous experience in the mathematics classroom" (Radatz, 1980, 16). Lochhead & Mestre (1988), citing Resnick (1983), note that "the research literature consistently indicates that misconceptions are deeply seated and not easily dislodged; in many instances, students appear to overcome a misconception only to have the same misconception resurface a short time later. This phenomenon is probably a result of the fact that when students construct learning, they become attached to the notions they have constructed" (Lochhead & Mestre, 1988, 132).

Other bases for the study were works on the initiation to algebra (Coxford & Shulte, 1988; Socas et al., 1989; Grupo Azarquiel, 1991; Kieran, 1992), specifically on the difficulties and errors that are observed with regard to the algebraic skills of Obligatory Secondary Education and Bachillerato students (Marquis, 1988; Gallego, 1995; Beltrán, 1997), as this is the field of the mathematical content of this investigation. The contributions that we have consulted point out different aspects of the teaching and learning of algebra that constitute difficulties and obstacles to learning. and which it is necessary to deal with in greater depth.

In this line, Socas (1997) and Palarea (1999) review students' difficulties and errors in learning mathematics in general, and algebra in particular. They group the difficulties into five categories: difficulties associated with:

1. the complexity of algebraic objects that operate semantically and syntactically;
2. thought processes, deriving from the logical nature of algebra;
3. teaching processes, deriving from the mathematics curriculum itself, from the educational institution, or from teaching methods;
4. the processes of the students' development; and
5. the students' affective and emotional attitudes towards algebra.

Socas (1997) classifies the main causes of error in learning algebra into two groups:

1. Errors that originate from some obstacle, such as the lack of closure, i.e., students see algebraic expressions as statements that are sometimes incomplete.

2. Errors that originate from an absence of meaning. These fall into two types:

2.1. Complexity of the objects and of the processes of algebraic thought. Examples are:

- Errors in algebra that have an arithmetic origin.
- Errors of procedure.
- Errors in algebra due to the characteristics of algebraic language.

2.2 Affective and emotional attitudes towards algebra.

Several workers have studied aspects of teaching/learning algebra that represent potential obstacles to learning. Thus Collis (1975) and Enfedaque (1990) describe the use that students make of algebraic letters and the meaning they attribute to them. Collis (1975), Behr et al. (1980), Kieran (1981), and Palarea & Socas (1999-2000) discuss the value that students attribute to the equals sign, finding that arithmetic has precedence over algebra, and Kieran (1979) discusses the use of parentheses.

Enfedaque (1990) studied students of year 8 of EGB (Basic General Education) and of years 1 and 2 of BUP (Obligatory Secondary Education) in Barcelona, putting forward some suggestions on how to introduce the use of letter symbols in algebra so as to decrease the incidence of errors, and also presenting some considerations on the teachers' attitudes to aid them in detecting those errors and, in sum, improve the students' algebraic skills.

Trigueros, Reyes, Ursini & Quintero (1996) give a design for a questionnaire to diagnose how the concept of variable is handled in algebra. They find the concept to be used with different meanings in different contexts, with the consequent differences in how it is dealt with. That this variability in how the concept is used makes it difficult to define could be the cause of many of the students' difficulties. They consider there are three forms in which a variable is usually employed in school algebra: as an unknown, as a generalized number, and in functional relationships.

MacGregor & Stacey (1997) present some difficulties in the use of algebraic notation as part of the results of a broader project denominated Concepts in Secondary Mathematics and Science – CSMS.

Kücheman (1978, 1981), within the CSMS project, notes that the considerations about understanding the algebra of numbers implies the development of skills in interpreting and handling letters and other symbols. A study of the diverse ways in which English secondary education students use letters established the following hierarchy:

"Letter evaluated: the letter is assigned a numerical value from the outset;

Letter not considered: the letter is ignored or its existence is acknowledged;

Letter considered as a concrete object: the letter is regarded as shorthand for a concrete object or as a concrete object in its own right;

Letter considered as a specific unknown: the letter is regarded as a specific but unknown number;

Letter considered as a generalized number: the letter is seen as representing, or at least as being able to take on, several values rather than just one;

Letter considered as a variable: the letter is seen as representing a range of unspecified values and a systematic relationship is seen to exist between two such sets of values" (Kieran, 1992, 396).

Usiskin (1988) points out the relationship that exists between the various conceptions of algebra and the different uses of letters in teaching, represented schematically in the following figure:

Conception of algebra	Use of variables
Generalized arithmetic	Pattern generalizers (translate, generalize)
Means to solve certain problems	Unknowns, constants (solve, simplify)
Study of relationships	Arguments, parameters (relate, graph)
Structure	Arbitrary marks on paper (manipulate, justify)

Figure 1. Relationships between conceptions of algebra and the uses of variables (Usiskin, 1988, p. 17).

Vega (1992) carried out a study whose objective was to elaborate profiles of the algebraic competence of pre-university students in Mexico City. Perhaps the most interesting finding of that study was the evidence that progress from one school year to the next was not reflected in any significant improvement in dealing with the problems that derive from algebraic manipulation.

All these contributions point to different aspects of teaching/learning algebra that constitute obstacles to its effective learning, and which require more detailed study.

METHODS

Since our intention in the present work was to make a first approach to the topic rather than an exhaustive study, we decided to use the questionnaire as the instrument for data gathering, including in it various recommendations appropriate for qualitative methods in general (Cohen & Manion, 1982; Woods, 1986) or specific aspects of the use of the questionnaire in similar studies (Enfedaque, 1990; Triguero, Reyes, Ursini & Quitero, 1996; Vega, 1995).

The questionnaire (Annex 1) was first suitably validated, and then given to 91 students from 4 different educational centres who were matriculated in the first year of Bachillerato, studying either the option Technology or the option Nature and Health Sciences (note 1). It was given after the students had received instruction in the topics that it covered. For most of the participants these were concepts that they studied for the first time in this school year: only some had prior ideas about the objects of study.

In the following section, we analyze the results for each item separately, noting the specific goals pursued with each. By way of a general picture, the following is an overall anticipation of those objectives:

- To examine the step from ordinary language to algebraic language in terms of an inequality that the students solve.
- To observe the meaning that the students attribute to inequalities.
- To analyze the use the students make of different systems of representation.
- To observe their operational abilities in solving a simple inequality.
- To observe their difficulties relative to the order relationship in the real numbers.
- To determine the difficulties the students have in assimilating different uses of letters in algebra.
- To observe their capacity to interpret the solution of an inequality.
- To check whether the students see inequalities as a tool that allows a certain type of problem to be tackled.
- To observe whether the students are able to connect the visual-geometric and algebraic languages.

RESULTS

In this section we shall show some of the errors and difficulties detected, and indicate their possible causes. The basis will be the analysis of the questionnaire and the subsequent confirmation in the interviews. At no time did we set out to make an exhaustive significant analysis of the data, although Annex 2 gives some overall results.

Items 1 and 2 dealt with the passage from everyday language to the language of algebra in terms of an inequality , and with the meaning that the students attribute to these expressions and the use that they make of different systems of representation.

Many students correctly gave the expressions asked for. Some aspects of their answers stand out, however. Despite their having worked with real numbers for several years, very few students took this set as the reference for their operations. Most limited themselves to the natural numbers, which would clearly represent an obstacle to their understanding the meaning of interval. This was a constant in their resolution of various items. Likewise, although use was made of the variable to give the requested expression, its meaning was not made sufficiently clear.

For item 2, there were students who understood the requested order relationship, even giving examples, but who, in passing to the algebraic expression, wrote the relationship backwards. This problem became even greater when they tried to give the double inequality in a single expression. They had difficulty in understanding the two inequalities together. Even when they were written together, they were comprehended separately, leading to such incoherent expressions as n < -2 > -11. (A similar situation appeared in the solutions to item 11.)

The aim of item 3 was to look at the level of skill in using operations to solve a simple inequality and the ability to interpret the solution. The problem was a first-order linear inequality [5-3(2-x) > 4-3(1-x)]. We found the answers to fall into three quite distinct groups: (i) the inequality was solved and interpreted correctly, i.e. an expression of the form 0 > 2 or -1 > 1 was arrived at and it was added that the inequality is not satisfied for any value of the unknown; (ii) the correct solution was given but the interpretation was not; and (iii) not even the correct solution was arrived at.

These results bring out the difficulties the students had in interpreting the result, since some of them who solved the inequality were incapable of drawing conclusions from it. This situation was also reflected in some of the students' uncompleted exercises. We also found operational mistakes: in the use of parentheses, of the signs < and >, and of the distributive property; in operations with whole numbers; and in passing from one inequality to another that should be equivalent.

With this exercise we began to realize that the students were not differentiating conceptually between equation and inequality, since they were using either term indistinctly to refer to the latter.

The students were thus clearly finding many problems and difficulties in trying to solve an inequality. Some of these problems were due to a lack of mastery of elementary algebra, and others were characteristic of inequalities themselves. Many students understood the greater than and less than signs to be a nexus between two algebraic expressions. They then carried this nexus through the various steps in solving an inequality without attaching any meaning to it, even to the point of simply substituting an equals sign. Few students endowed the inequality with any semantic content as was clear in the failures to interpret the result even after correctly applying the algorithm to reach the solution.

Items 4-6 brought out the difficulties in handling expressions involving the sign '-' in the inequalities and the order relationship in the real numbers. Few students both chose the correct answer and gave arguments. Most simply used the same techniques they would use for equations, again showing that little semantic meaning is attached to the sign and that the aim was simply to operate and solve for the unknown without taking any account of whatever meaning the result might have.

Item 6 also showed the students' difficulty in assimilating different uses of letters in algebra (also seen in items 7 and 10), in particular that they thought that 'a' represents a positive number and 'a' a negative number.

The aim of item 5 was to see to what degree the students were able to interpret the solution of the inequality. It again showed their difficulty in reading an inequality, as well as in understanding that the result of an inequality is not a value of the unknown but an interval. Let us illustrate this with some examples of the errors that were made:

- The inequality was solved correctly, but the question posed was not answered because the student did not know what to do with the values between 3 and 5.
- Having arrived at x > 3, the student crossed out x > 5 believing that the first expression should have appeared in the statement of the problem but not the second.
- After substituting 5 and 6 in the inequality, it was argued that "Yes, it is true because there are examples that demonstrate it".

This last solution confirmed that many students think that, in order to justify the statement they are presented with, it is enough to verify it for some value.

In item 7, we again use letters as generalized numbers in order to see whether and how the students use suitable tools to prove which of the two given algebraic expressions is greater. I.e., the aim is to see whether they consider inequalities to a tool that they can use to tackle certain types of problem.

Only 9.9% of the students correctly reasoned their answer. Some stated that it depended on the values, demonstrating the difficulty they have in using letters as generalized numbers, and others that the result of multiplying a number by a positive quantity is greater than adding that quantity to the number, which perhaps derives from their usually working with natural numbers.

The students have not sufficiently assimilated the concept of inequality, since only a few use this tool in order to justify the answer that they gave, even though this concept was the main object of most items on the questionnaire.

Item 8 showed the students' difficulty in connecting visual-geometrical language with algebraic language. Very few used the diagram to justify their answer, i.e., comparing the area of the square of side 'a + b' with those of the squares of sides 'a' and 'b', respectively. For many, the diagram was just a drawing that at no time were they able to relate to the question in hand, and they could not even understand why it was there. It was obvious they had become accustomed in their work in algebra to using other non-algebraic tools, and that this derived not from the students themselves but from the teachers and the methods used in the classroom. Most attempted to answer by expanding the binomial sum and comparing the resulting expressions.

In item 9 too, they could have used the diagram of the preceding question, but none did. The result for this item showed the difficulty the students find in this type of question. Only one succeeded in proving the statement, seven said the statement was true after checking its validity for various cases, while the rest failed to give any argument justifying the statement.

This exercise brought out some common errors such as considering that

"if a2 > b2, then one has that a > b with no more ado than taking the square root of both sides of the inequality".

In another sense, they had difficulties in considering thesis and hypothesis. I.e., they attempted to show that a2 > b2 when a > b.

Item 10 involves letters used differently, one as the unknown and the other representing a generalized number. The idea was not for the student to give the complete range of values for 'm', but simply to find some value for which the conditions are satisfied. The underlying objective, however, was to see how the students understand and interpret a solution of an inequality.

The answers given fell into the following categories:

a. a value of 'm' was found for the conditions of the statement, i.e., such that substituting it into the inequality, the statement was found to be true for x = 0 and false for x= 2;
b. the answer was incorrect; and
c. the response was left blank.

A large group of students did not differentiate between the uses of the two letters in the inequality. This led to a deficient understanding of the statement of the problem. Also, when they came to the actual calculation of the value of a letter, they simply relied on the techniques they knew for equations to get the result, even to the point of changing the sign of the expression without seeing the need for any justification.

This aspect also carried over to the interpretation of the solutions of an inequality. Even when they reached an expression of the form m < 1, they believed that this was not determining the unknown and that it was necessary to give an expression in terms of equality, i.e., 'm' has to be equal to a single value.

Items 11 and 12 were aimed at seeing to what degree the students could perceive a functional relationship between two letters so as to establish the range of variability of one in terms of the range of variability of the other.

In item 11, many students again had difficulty in attempting to give a single expression for a double inequality, even when they had assimilated the information contained in that inequality. Thus an expression obtained from the statement " 'm' is greater than 3 but less than 10" was 3 < m > 10.

The students presented substantial differences in giving meaning to the functional relationship between the two letters. While they found no great difficulty in determining the values of 'm' from those of 'n', this was not so for the contrary process which caused them certain conceptual difficulties deriving from their concepts of dependent and independent variable.

The intervals were calculated by substituting the smallest and the greatest values of one of the letters into

1, 2. In passing from ordinary language to algebraic language in terms of an inequality.
1, 2, 4, 6, 7, 10, 11. In the use and meaning that the students attribute to letters and to algebraic expressions.
1, 2, 7. They do not take the real numbers as their reference set for their operations, but limit themselves to the natural numbers.
1, 2, 5. To understand the meaning of interval.
1, 2, 11, 12. In the meaning of the variable.
2, 3, 5, 11. To understand the meaning of the greater than and less than signs.
7, 12. To use the greater than and less than signs, and, in general, inequalities to solve exercises.
3, 5, 10. To interpret the result of an inequality.
3, 4, 6. Operational errors (in the use of parentheses, the sign " – " y the signs "<", ">", "≤", o "≥"., the distributive property, operating with integers, and in going from one inequality to another that is equivalent).
3, 4, 6. They give no semantic content to the inequality. They find no conceptual differences between equation and inequality.
4, 6. On handling expressions that involve the order relation of the real numbers.
8, 9. Difficulty of connection between the visual-geometric and algebraic languages.

Figure 2. Summary of the difficulties the students were found to have about inequalities.

the given relationship and finding the respective values for the other letter. I.e. "if $3 < m < 10$, since $m = 3 + n$, then $10 = 3 + n$ and $3 = 3 + n$, whence one has that n = 0 and n = 7, and the result is (0,7)".

In item 12, the answers fell into one of four categories:

a. correct result, i.e. 'c' must take values less than 5;
b. the result given is just a single value for c;
c. incorrect result; and
d. a notably large number of students gave no response.

Again it was clear that the students generally do not see inequalities as a tool that can be used to solve certain types of problem, since only a few used the technique to respond to this item. Many of them tried all the ways they could think of to set the question in the field of systems of two equations with two unknowns. In particular, the relationship $c + d = 0$ was seen as an equation with two unknowns, and as they could find no other equation with two unknowns, they reasoned that the problem could not be solved because an equation was missing.

Finally, there is the aspect of checking the results. The fundamental goal of problem solving is to obtain a solution that is coherent with the conditions of the problem. For many of our students, however, the goal was to find a procedure to arrive at a solution, with at no time it being necessary to check the result since the procedure itself was the justification of its validity. In the figure 2 presents a summary of the principal results relative to the different items.

DISCUSSION

The analysis also shows the difficulties that the students have in assigning new meanings to concepts and mathematical processes related to inequalities. Thus, we find that the errors do not arise by chance, but rather that the students have a stable conceptual framework based on their previous knowledge – fundamentally that derived from their handling of arithmetic. We confirm that the basis of a part of the errors is in the students' prior experiences, in the sense noted by Radazt (1980).

We would like to highlight some aspects that seem to us to be significant.

Thus, a major fraction of our students have a deficient grasp of the concept of inequality. Many of them have not established any meaningful difference between this concept and that of an equation (items 3 and 12). I.e., the difference is merely in the symbol that is written between the two members of the relationship: the symbol '=' in an equation, and one of the symbols "<", ">", "≤", o "≥". The signs have no semantic value since they are used simply as a nexus between the two members of the inequality (item 3).

This absence of meaning was also manifest in the students' difficulties in reading from left to right or from right to left, i.e., difficulties in recognizing the equivalence of the expressions $x>1$ and $1<x$, or to interpret expressions of the type $0>2$ or $-1>1$ (item 3).

There were serious difficulties in passing from a statement given in words to an algebraic expression (items 1 and 2), especially if the expression involved a double inequality (item 2).

Many students had not established that there was a semantic difference between equation and inequality, and some of their conceptions of interval were as "a set of natural numbers, or at best a set of integers between some other two integers". Neither does their interpretation of the solution of an inequality seem to be the most appropriate if our intention is to endow the object of our study with semantic content (items 5 and 10). These results ratify the findings of Socas (1997) that the complexity of the objects and processes of algebra is a source of the students' difficulties.

For a good many of our students, algebra is "operating" with numbers and letters, with no other goal than obtaining their values by applying semantically empty algorithms. Thus, in dealing with an expression of the form $-7x < 5$, their objective is to leave just the unknown on one side of the relationship, and to this end they "pass the -7 to the other side of the inequality as a divisor" just as if the relationship was an equation. The goal of finding values of the unknown that make the inequality true is pushed into the background (items 4 and 6).

It was also evident that many students had still not mastered some of the difficulties of arithmetic. Thus, we found evidence of the students' difficulties in their handling of the distributive property, and in their use of parentheses, the sign rule, and the value attributed to the equals sign. These results corroborate those indicated in the second section by Collis (1975), Behr et al. (1980), Kieran (1979, 1981), Enfedaque (1990), and Palarea & Socas (1999-2000). This makes it even harder for them to acquire a new concept that requires the appropriate use of these skills (item 3). In this regard, we consider that the students' arithmetic knowledge acts as an epistemological obstacle, in the sense expressed by Brousseau (1997), to learning algebra.

The students can use algebraic letters without attributing any meaning whatsoever to them (items 5-7). We confirm that students have difficulties in the use and meaning that they attribute to letters, as was indicated by Collis (1975) and Enfedaque (1990). With respect to the different uses of letters, we consider that the students have a conception of algebra as generalized arithmetic, in the sense expressed by Usiskin (1988). We also consider that:

- A letter used as a generalized number is treated as belonging to the domain of natural numbers, or at best integers, with all the limitations that this implies, especially considering that one is working with inequalities whose solutions are intervals of real numbers (items 7 and 10).

- A letter used as an unknown is endowed with the greatest meaning and recognition by our students. Nonetheless, the need the students feel to find specific values for the letter deriving from its use in equations represents a major barrier to their interpretation of the solution of an inequality (item 10).
- Lastly, when a letter is used in a functional relationship, the way in which this relationship is presented becomes very important, since the students have deeply rooted ideas of dependent and independent variables with all that this implies for the reversibility of the relationship (item 11).

With respect to the use of variables, we note the difficulties that the students have relative to the three meanings described in Trigueros et al. (1996). They show greater facility in using a variable as an unknown, but greater difficulty with its use in functional relationships. In this regard, we consider it necessary to work on the three given uses of the variable, and on the possibility of flexibly passing from one to another according to the demands of the task that has been set. This last aspect presented many difficulties for the students that we studied.

With respect to the different systems of representation, ideally the use of more than one system would favour the understanding of algebra since different systems provide alternative and complementary strategies (Palarea and Socas, 1999-2000). Our students, however, use nothing but algebraic language to approach the different problems they had to answer (items 8 and 9). In most cases this was a consequence of the view that many of us as teachers have, and that we carry over to our classrooms, of the teaching-learning of algebra. In developing the content of algebra, we only use algebraic language and do not provide our students with other tools to represent concepts and thereby make them easier to learn.

The absence of meaning is one of the principal problems arising in working with inequalities. If our intention is for the students' learning of inequalities not to be reduced to mere mechanical tasks, it is important to give them a clear idea of the concept of equivalent inequality since it is this that endows the techniques of solution with semantic content.

Finally, the study induces us to assume that part of the difficulties presented by the students could be understood as corresponding principally to two points of the classification schemes given by Socas (1997) and Palarea (1999) – difficulties associated with the complexity of the objects and processes of algebra, and difficulties associated with the processes of teaching. In the former case, we see that the students have not managed to understand the mathematical objects involved in the inequalities with respect to integrating definition, different systems of representation, properties, and applications. In this regard, arithmetic proves to be an epistemological obstacle in a general sense, as well as in relation to certain specific concepts and processes. This situation is the consequence of traditional teaching methods that are based on developing algorithmic procedures, and which at times neglect to deal with the meaning of the objects that are being used.

CONCLUSIONS

The results of the work show that students find two types of difficulty in dealing with inequalities. On the one hand, arithmetic is still the fundamental referent for those students who make errors in the algebraic procedures, and, on the other, the absence of meaning is the underlying cause of the failure to understand the concepts and the algebraic process.

Given that the work is meant to be a first approximation to the topic, we wish to conclude by noting the need for a more detailed investigation of the difficulties in the teaching/learning of inequalities, with the problem being approached from different contexts, such as arithmetic, algebra, and geometry.

Educational Implications

The absence of meaning is one of the main problems that arise in working with inequalities. For that reason, greater attention will have to be paid to how the concept is introduced if one wants to avoid the learning of inequalities being reduced to merely mechanical tasks. Any mechanism of solution must allow students to understand the meaning of the process they follow to arrive at the correct solution of an inequality. Otherwise the mechanism they learn will be a source of error and one that they will not use unless the teacher or the textbook specifically tells them to.

Given the difficulties deriving from the complexity of the elements of the algebra, as teachers we should keep the following in mind when teaching inequalities:

- Not to introduce the concept of inequality or the techniques for their solution too rapidly.
- Ensure that the symbols used are clearly differentiated and that they have semantic value for the students.
- Establish with clarity the differences between the concepts of equation and inequality, with the clear implications that this entails especially when it comes to interpreting their solutions.

- Use different languages: 'everyday' language, visual-geometric language, and algebraic language. Translation from one to another favours a better understanding of the concept. The visual-geometric language in particular needs to be treated in some depth.
- The introduction of the formal notation can not be disconnected from the acquisition of the meaning of the concept and the processes needed to solve inequalities.
- The use of different strategies to approach questions related to inequalities both enriches the learning process and allows more students to acquire the concept.

The absence of meaning of mathematical objects is one of the main problems that we face in our classes. All our work must be oriented towards the search for educational alternatives, the more diverse the better, aimed at providing the meaning which will constitute the principal basis for learning mathematics

NOTES

Note 1. Spain's educational system is organized into three stages:
Primary from 6 to 12 years old.
Secondary from 12 to 16 years old.
Pre-university (Bachillerato) from 16 to 18 years old.
The student participants in the investigation were 16-17 years old.

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