Eurasia Journal of Mathematics, Science & Technology Education, 2015, **11**(2), 251-261

Duality of Mathematical Thinking When Making Sense of Simple Word Problems: Theoretical Essay

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Received 10 January 2014; accepted 5 February 2015; published 25 March 2015

This essay proposes a reflection on the learning difficulties and teaching approaches associated with arithmetic word problem solving. We question the development of word problem solving skills in the early grades of elementary school. We are trying to revive the discussion because first, the knowledge in question—reversibility of arithmetic operations and flexibility of mathematical thinking—are the key elements in elementary mathematics, and second, because we hope to create a shift in the understanding of this knowledge development in students. Using the folk tale "The Three Little Pigs" as a metaphor, we analyze difficulties students experience while learning to solve word problems involving addition and subtraction. We formulate a hypothesis about cognitive duality of word problem-solving. This hypothesis explains some well-known learning difficulties and suggests teaching principles which would help to avoid developmental obstacles and pitfalls within the teaching/learning process.

Keywords: mathematical knowledge, development, thinking duality, problem solving

INTRODUCTION

In this paper, we use the folk tale "The Three Little Pigs" as a metaphor. This tale helped us and our colleagues to see the phenomenon of mathematical knowledge development in a new way, and we hope it will do the same for our readers.

Once upon a time there were three little pigs and their mom, who happened to be a teacher. One day, the teacher-mom gave her little students the following task: they each should build a house to protect themselves from the cold winter wind and the hungry wolf. She hoped that her students would gain valuable knowledge through the solving of this problem.

Although the three little pigs seemed to be fully engaged in the real-life situation, each proceeded in a different manner.

Nif-Nif built a cute little straw house. He was convinced that the sun would always provide warmth and that the wolf, whom he had never seen before, was far away.

Nuf-Nuf gave a little more effort and built his house out of sticks. He decided that the house would be solid and warm enough.

In contrast to his brothers, and to the great satisfaction of the teacher-mom, Naf-Naf worked hard to build a solid brick house.

We all know the story of the three little pigs. Various cultures have developed their own versions of this fascinating, horrifying, funny, and above all very educative tale. Above, we provided our pedagogical version of the story. Are there any similarities with situations in our mathematics classrooms? Faced with the same learning conditions or interacting with the

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doi: 10.12973/eurasia.2015.1325a

ISSN: 1305-8223

State of the literature

- Arithmetic word problems are differently difficult for students. The understanding of arithmetic word problems through the reading process was studied.
- Some students become successful problem solvers using various strategies in an adaptive way and others just mechanically apply learned procedures.
- The duality of human understanding of mathematical concepts is discussed in the literature.

Contribution of this paper to the literature

- We use the folk tale "The Three Little Pigs" as a m etaphor to revise how various students develop the ir knowledge and ability to solve word problems.
- We formulate a thinking duality hypothesis for the case of word problem solving. Based on this hypot hesis, we explain how developmental impasses and obstacles may be produced in traditional teaching.
- We revise known teaching strategies and propose s everal teaching principles to support effective reas oning development in students in relation to word problems.

same milieu didactique (Brousseau, 1988), some students use various strategies in an adaptive way and others just mechanically apply learned procedures (Hatano & Oura, 2003; Hatano, 1982, 2003; Verschaffel, Luwel, Torbeyns, & Dooren, 2009). Later on, some students become successful problem solvers and others not (Pape, 2003; Rodriguez, 2004; Thevenot, 2010). In other words, some students seemingly develop solid and sustainable knowledge without difficulty while others continuously struggle or fail to acquire the same knowledge. The main question we will explore in this paper is:

Why do some students build straw houses and stick houses instead of brick houses?

Using the tale as a lens, we will explore the difficulties encountered by elementary school children while learning to solve word problems with simple additive structures. We will fuel our discussion with research findings from various schools of thoughts and fields of study.

First, we will clarify the nature of difficulties that students can face in **solving** word problems. Second, we will explore the idea of cognitive duality in mathematical thinking (Sfard, 1991; Skemp, 1987; Wachsmuth, 1981), applying it to the word problem solving situation. We will propose a hypothesis to explain why some students struggle or even fail to **construct** sustainable knowledge related to word problems. Third, we will review existing teaching methods that can be used to construct this type of knowledge of word problem solving. We will demonstrate that our hypothesis about cognitive duality is coherent with many methods of teaching promoted in the research. Lastly, we will put forth questions about our teaching philosophy, thus leaving it open for critique, discussion and further research.

The Wolf and the Cold Wind: Why can Word Problems be Difficult?

School childrens' difficulties when solving word problems with simple additive structures are well documented (Barrouillet & Camos, 2002; Riley, Greeno, & Heller, 1984; Vergnaud, 1982). Research (Fuson, 1992; Kilpatrick, Swafford, & Bradford, 2001; Verschaffel & Corte, 1993) gives considerable importance to whole-number arithmetic knowledge in the development of problem-solving ability in students. In order to clarify a particular source of students' difficulties in developing this ability, we will first look closely at the problems themselves. We will limit ourselves to the following three problem types:

1) Pierre had 8 marbles. He then won 5 other marbles. How many marbles does he have now? Correct solution: 8 + 5
2) Pierre had 13 marbles. He then lost some marbles. He now has 8 marbles. How many marbles did he lose? Correct solution: 13 - 5
3) Pierre had 8 marbles. He then won some marbles. He now has 13 marbles. How many marbles did he win? Correct solution: 13 - 5

Several studies (Gamo, Sander, & Richard, 2009; Riley et al., 1984; Vergnaud, 1982) show that problems with structures like problems 2 and 3 are more difficult for students than problems structured like problem 1. In order to better understand where these difficulties come from, researchers (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Nesher, Greeno, & Riley, 1982; Riley et al., 1984; Vergnaud, 1982) have classified and categorized the problems. For example, Vergnaud (1982) distinguishes six categories of situations[1] that can appear in a problem. All of our problems fall under Category II: transformation linking two measures.

Vergnaud states that problems with an unknown final state (like our problem 1) are easier for students than problems where the initial state is unknown or

[1] Categories of word problems with additive structures according to Vergnaud (1982):
Category I: Composition of two measures
Category II: A transformation links two measures
Category III: A static relationship links two measures
Category IV: Composition of two transformations
Category V: A transformation links two static relationships
Category VI: Composition of two static relationships

where the transformation is unknown (like our problems 2 and 3). Moreover, problems called inconsistent (Bruno & Martinon, 2004; Carpenter, Moser, & Bebout, 1988; Hegarty, Mayer, & Monk, 1995) (like our problem 3) are the most difficult in this category. The well-known Construction/Integration model of text comprehension (Cummins, Kintsch, Reusser, & Weimer, 1988; Kintsch & Greeno, 1985; Kintsch, 2005) can explain how students may fail to solve these problems. Yet, the theory does not explain why students fail to develop the appropriate way of thinking.

Today, researchers seem to agree that "the problem is difficult to students" means that students fail to apply their knowledge to identify the appropriate arithmetic operation. In fact, it is not so much a question of the ease or difficulty of the problems themselves as it is a question of whether or not the appropriate knowledge or ability has been developed. Keeping in mind our three problems, which represent various levels of difficulty, we will look closely at the knowledge at stake and how it can be developed in students.

Different Houses: Types of Mathematical Thinking

While analyzing the empirical data about students' problem solving, Nesher et al. (1982) and Riley et al. (1984) proposed a sequence of stages and related models of thinking to explain the development of problem-solving abilities in students. In the context of the Change Problems category (transformation linking two measures), this development can be summarized as follows.

According to the authors, beginners—young students at the first stage of problem-solving knowledge development—understand a change problem as a process. This understanding is closely related to their everyday understanding of physical actions described in the problem. Starting with the initial state known, students at this stage can imagine or represent this state using manipulatives, then reproduce the known change, and thus arrive at the final quantity at the end of the representation process. We will compare this knowledge to a straw house—the easiest to construct.

Different researchers (Nesher et al., 1982; Okamoto, 1996; Riley et al., 1984) propose different numbers of intermediate stages of referred knowledge development. The main characteristic of these stages is that students can solve problems from some of the more difficult categories, but not all of them. This reminds us of the house of sticks from the original tale, a house that seems more durable than the one built out of straw.

At the final stage, students are capable of seeing the situation in its entirety and in a flexible way, without any limitation (Nesher et al., 1982). They can transform the semantic relationships given in the story into a part-whole relationship and use all relationships to come up with the necessary arithmetic operation (Riley, 1988). The ultimate proof of the final stage of development is that the student "is able to read the word 'more', and yet perform a subtraction operation" (Nesher, 1982 p. 392). This ability ensures success in solving word problems of any category—like a house made of brick protects against any danger.

When looking at the three types of houses we refer to in the tale, it is important to note that each of them was built by a different pig. In the experiments done by Nesher et al. (1982) and Riley et al. (1984), different students (K-1-2-3) were evaluated, but the same students were not evaluated at different stages in their development. Therefore, this experiment cannot confirm that all students go through all stages of development in the same way.

Krutetskii (1976) argued that some elementary school students often see and analyze problems as a whole, in their entirety, and can thus build a sustainable holistic and flexible understanding from the very beginning. Some students may be more like Naf-Naf, who built a sustainable brick house. Naf-Naf probably thought about his choice of material (straw or sticks) before starting the construction, but no trace of such reflection is found in the tale.

Conversely, Hegarty, Mayer and Monk (1995) describe a "direct translation strategy" that some **undergraduate university students used** while solving word problems. According to the researchers, the students translated the numbers and keywords from the text into an arithmetic expression in a straightforward way. Why do some students choose to see the problem as a whole from the very beginning, while others seem to adopt straightforward sequential thinking and stay with it for such a long time? Why did Nif-Nif and Nuf-Nuf build and continue to use their fragile houses until the hungry wolf blew them down? It seems that the theory of stages cannot explain the difference in the developmental pathways in students. An alternate approach is needed.

Straw and Stick Houses vs. Brick House: The Duality Hypothesis

We briefly discussed the theory of stages in the development of word problem solving knowledge. We will now take a closer look at two different ways of thinking described in this theory.

We said above that beginners perceive change problems as a sequence of events. While reading the text, students can imagine the story as a film. They can thus represent it piece by piece in their minds as the information is given to them. We identified this type of thinking as sequential, or a straw house.

Another type of thinking seems to be performed by students who have mastered their knowledge of word problem solving. They can simultaneously consider the quantities described in the text along with their relationships and use these relationships to choose the appropriate arithmetic operation. This type of thinking can be identified as holistic and flexible, or a brick house.

Researchers have discussed the duality of human understanding of mathematical concepts such as number, function and others (Sfard, 1987, 1991; Skemp, 1987). Sfard (1991) concluded that abstract notions "can be conceived in two fundamentally different ways: structurally – as objects, and operationally – as process" (p. 1). She argues that success in problem solving implies an efficient interplay between both ways of thinking about the concepts involved (addition, subtraction). The two ways of thinking identified above correspond well to this duality, with sequential thinking being associated with understanding a situation as a process/event, and holistic thinking being associated with understanding the situation as a system of relationships or a structure.

Skemp (1987) proposes a different vision of the thinking duality: relational versus instrumental understanding. This dichotomy can also be applied to the case of word problem solving, but from a different perspective. Having a word problem to solve, some students **try** to make sense of the situation as a system of relationships, in a relational way as described by Krutetskii (1976). This leads them to a relational understanding of the problem (Skemp, 1987) supported by a holistic vision of the situation. Other students try to get directly to the operation. Sequential understanding of arithmetic operations, such as adding, putting together or taking away, lead them to a sequential understanding of the situation and an instrumental way of solving the problem. Thus, the difference in initial intentions leads to the difference in how the students understand the problem. The relational or instrumental thinking students use to understand word problems strongly determines further knowledge development (Skemp, 1987). Furthermore, sequential thinking fits perfectly with the instrumental approach to many arithmetic problems. Later in this article, we will explain how this can work for above-mentioned problems 1 and 2. For instance, in the tale, the final quality of the house is strongly determined by the initial choice of material: straw, sticks or bricks.

Similar to Skemp's ideas is the distinction between adaptive expertise and routine expertise in arithmetic proposed by Hatano (Hatano & Oura, 2003; Hatano, 1982, 2003) and further developed by Verschaffel et al. (2009). Adaptive expertise refers to the ability to choose the most appropriate solution strategy and routine expertise refers to the routine and accurate application of learned procedures without profound understanding.

There is a clear distinction to be made between arithmetic **calculation strategies** to which many researchers refer and how students **make sense of** a situation described in a word problem we refer to in this article. Using the reasoning duality hypothesis, we can propose that the holistic vision of a situation—understanding it as a structure/object which can be turned around and transformed—makes more solution calculation strategies available to the solver, thus allowing the development of the adaptive expertise. Sequential understanding makes this transformation impossible or very difficult, yielding the direct application of calculation procedures.

The sequential–holistic thinking duality is in line with knowledge about how the brain functions. It is well known that different parts of the brain are responsible for sequential and simultaneous processing. Wachsmuth (1981) describes L-modal thinking as a conscious "sequentializing" of thought, for which the left hemisphere is responsible. The right hemisphere, according to the researcher, is responsible for R-modal thinking, favouring parallel and holistic thought. He insists that an efficient interplay of the two modes of thinking is at the core of successful mathematical thinking. This brain-related duality supports our hypothesis.

From this perspective, it seems that solving problems with different levels of difficulty can be predominantly performed by different brain processes. Sequential L-modal thinking can successfully support the solving of easy word problems. It can include thinking about an action or procedure and a one-to-one representation of numbers using physical blocks or by drawing circles. More difficult word problems require R-modal thinking: simultaneously thinking about all known and unknown quantities.

Some recent results from research in neuro-education support the idea of this duality. Stavy and Babai (2009) use the method of brain imaging to analyze the process of solving certain geometric problems with similar questions. In some problems, a *congruent condition*—cases where figures with smaller perimeters have smaller areas—was used. In other cases, *incongruent conditions,* the relationship between the change in the perimeters and the change in the areas was opposite. Results show that the parts of the brain that were most active during the problem solving process differed when it came to correct thinking with "simple" (congruent condition) and "difficult" (incongruent condition) problems (Stavy & Babai, 2009). The simple/complex or intuitive/reflexive thinking duality supported by Stavy and Babai's (2009) research is different from the holistic/sequential dichotomy. However, their example confirms that solving two **mathematically similar** problems can involve the activation of **completely different brain regions**. We hypothesize that this main idea can also be applied to word problem solving.

Returning to our list of arithmetic problems (page 214), we can say that to understand problem 1, one can rely on a sequential unidirectional semantic analysis, while problem 3 **requires** more control and the non-sequential coordination of data elements and their roles and relationships in the situation. This holistic vision of the situation as a system/object allows the structure to be transformed into an appropriate arithmetic operation/process. Thus, solving these two problems could require completely different brain processes, the second likely also requiring efficient coordination of different types of thinking.

To summarize, we propose that efficient problem solvers should be able to transform the situation described in the word problem into a systemic holistic vision of the additive relationship, even though the story describes an action or change. They should also be able to derive an action or mathematical operation from this object-like view of the situation. In short, constant and efficient coordination between sequential and holistic vision is needed to produce truly flexible thinking.

We do not know which particular brain regions and centres are involved in solving word problems of different types. However, if our hypothesis about the fundamental cognitive difference between the two modes of thinking in problem solving is correct, sequential thinking about a problem cannot be transformed or developed into holistic thinking. The use of the "sequential centre" will not by itself affect the development of the "holistic centre" or "coordination centre." Special conditions may need to be created to promote holistic thinking development in some students. Special conditions are also needed to develop students' ability to coordinate both sequential and simultaneous (instrumental and relational) thinking while solving word problems. In our tale, Nif-Nif and Nuf-Nuf were able to build a house, but appropriate material was not available to them. Only the appearance of the wolf forced them to abandon their non-sustainable houses and look for new solutions. Even then, they did not immediately retreat to the brick house. Why?

According to our hypothesis, students' performance in problem solving should look quite stratified: considerable success in problems with an unknown final state and limited success in other problems. Yet, some intermediate results in problem solving are reported in research (Nesher et al., 1982; Vergnaud, 1982). The results could be explained by the intermediate stages of knowledge development, as discussed above. They may also be explained by some adaptations of linear thinking that students can develop through the practice of problem solving in some conditions. When the wolf destroyed the straw house, Nif-Nif found took refuge in his brother's stick house. We will try to show that intermediate performance in problem solving does not necessarily confirm that a student is on the path to acquiring advanced thinking.

The Stick House: Developmental Impasse or Didactical Obstacle

In the previous section, we formulated a hypothesis about cognitive duality in word problem solving. If we follow the position of the stages theory and consider our hypothesis, the development of word problem solving knowledge should be seen as a transformation of sequential thinking into holistic and flexible thinking through the practice of problem solving. However, many students struggle with this transformation. Why can some students continue to use sequential thinking even when more difficult problems are asked of them? There are certainly several ways to succeed in problem solving without conducting a holistic analysis of the situation.

Students in their first stage of problem-solving knowledge development can successfully solve problem 1 (from our list above) through sequential thinking. Since the solution obtained this way is correct, learners will judge the thinking itself as appropriate and generalized for any problem solving situation. Little Nif-Nif who built a cute little house out of straw would consider this material to be good enough.

Carpenter et al. (1993,1999) state that young children can solve word problems of different categories, even those considered to be more difficult. However, according to authors, the strategies children use to solve those problems often **directly and sequentially** model the action or event described in the problem. These strategies are applicable if numbers are small and represented by physical objects or if a detailed drawing of said objects is available for students. Therefore, the use of manipulatives can potentially further promote sequential thinking in some students. Authors also stress that:

> *…the modeling and counting strategies that children use to solve simple problems with relatively small numbers are too cumbersome to be effective with more complex problems or problems with large numbers. For these problems, mathematical representations are needed so that algorithmic procedures can be applied. (Carpenter, Moser, & Bebout, 1988, p. 345)*

When the problem has big numbers or requires a formal mathematical expression as a solution, other behaviour can be observed in students.

One possible application of sequential thinking in more difficult problems is the direct translation strategy mentioned in the previous section (Hegarty et al., 1995). Here is one possible way to solve our problem 2: *Pierre* *had 13 marbles. He then lost some marbles. He now has 8 marbles. How many marbles did he lose?*

Students can translate this problem as follows:

13 marbles → 13

lost → minus

8 marbles → 8

Correct mathematical expression: 13 - 8 = 5

Using this strategy, students move too quickly to the arithmetic operation without carefully studying the underlying mathematical concepts and relationships (Hegarty et al., 1995).

In our previous research (Polotskaia, 2014), we observed another interesting phenomenon. In the problem where the unknown is not the final state, students can intuitively substitute the actual structure of the problem for a well-known *unknown-final-state* semantic structure. Here is another way to solve our problem 2: *Pierre had 13 marbles. He then lost some marbles. He now has 8 marbles. How many marbles did he lose?*

Some students may interpret this problem as follows:

Pierre had 13 marbles. He then lost 8 marbles. How many marbles does he have now?

In this interpretation, *lost marbles* and *current marbles* switch roles. However, this erroneous interpretation allows students to construct the correct mathematical expression: 13 - 8 = 5 and get the right answer. In this case, the incorrect perception of the semantic structure of the problem—*structure substitution*—can help students to easily solve a difficult problem via sequential thinking.

The fact that inappropriate thinking results in a correct answer is often hidden and can be only discovered in individual conversations with students. For example, DeBlois (1997, 2011) observed similar behaviour discussing word problems with students who were experiencing difficulties in math. In her experiments, students were able to produce a correct mathematical expression, but failed to explain the relationship between quantities.

The structure substitution phenomenon and the direct translation strategy may create the impression that a student has reached the intermediate stages of word problem solving knowledge development. However, these ways of thinking cause students to stray from the knowledge development road as they do not lead to holistic and flexible thinking. The stick house seems to be better than the straw house, but it can still be blown down by the wolf.

From our three typical problems, two can successfully be solved based on "stick house" strategies, which thus confirms the "success" of the strategy for learners. As has been already discussed in research (Nesher, 1980; Xin, 2007), textbook content—the choice of problems—can considerably contribute to creating an impasse in students' knowledge development. The stick house can be comfortable enough if the wolf is far away. Yet, once built and occupied, it can become an obstacle to the idea of reconstruction.

Structure substitution or the direct translation strategy may not help in inconsistent problems (Lewis & Mayer, 1987; Pape, 2003) where one reads *more* but needs to subtract. In these problems, holistic and flexible thinking is really required (Nesher et al., 1982). Only a brick house can really protect little pigs from the hungry wolf.

In Figure 1 we present our global vision of the stages of knowledge development discussed above. The first stages of development observed and described by researchers can be more or less just effective adaptations of the same sequential thinking. We should seriously question the existence of the developmental link between these stages and the holistic and flexible thinking required at the final stage.

Building a Brick House: Thinking Duality and Teaching Strategies

To solve difficult word problems, students should be able to see the mathematical structure of the problem in a flexible and holistic way. How can students develop this type of thinking when solving problems? How can we avoid building stick houses? What kind of bricks should the brick house be built with?

In contemporary writings on mathematics education, some studies (Fagnant & Vlassis, 2013; Gamo et al., 2009; Neef, Nelles, Iwata, & Page, 2003; Ng & Lee, 2009; Xin, 2008) are concerned with the use of various graphical and schematic representations of word problems. According to Xin (2008), schema-based instruction can help students with difficulties to develop a more profound knowledge of multiplicative relationships and become better problem solvers. Gamo et al. (2009) demonstrate that comparing problems and using different representations can help students develop effective problem solving strategies. Any graphical or schematic representation potentially gives students rapid visual access to the entire system of quantitative relationships described in the problem. Therefore, using diagrams and schemas should encourage students to form a holistic vision of the problem.

Other studies propose particular didactic management and class work organization. Neef et al. (2003) have shown that learning about the roles of each data element in a problem significantly improves success in problem solving among students with developmental disabilities. DeBlois (2006) suggests that a request for feedback on the solved problem may provoke coordination between representations and procedures and thereby lead students to reorganize their thoughts.

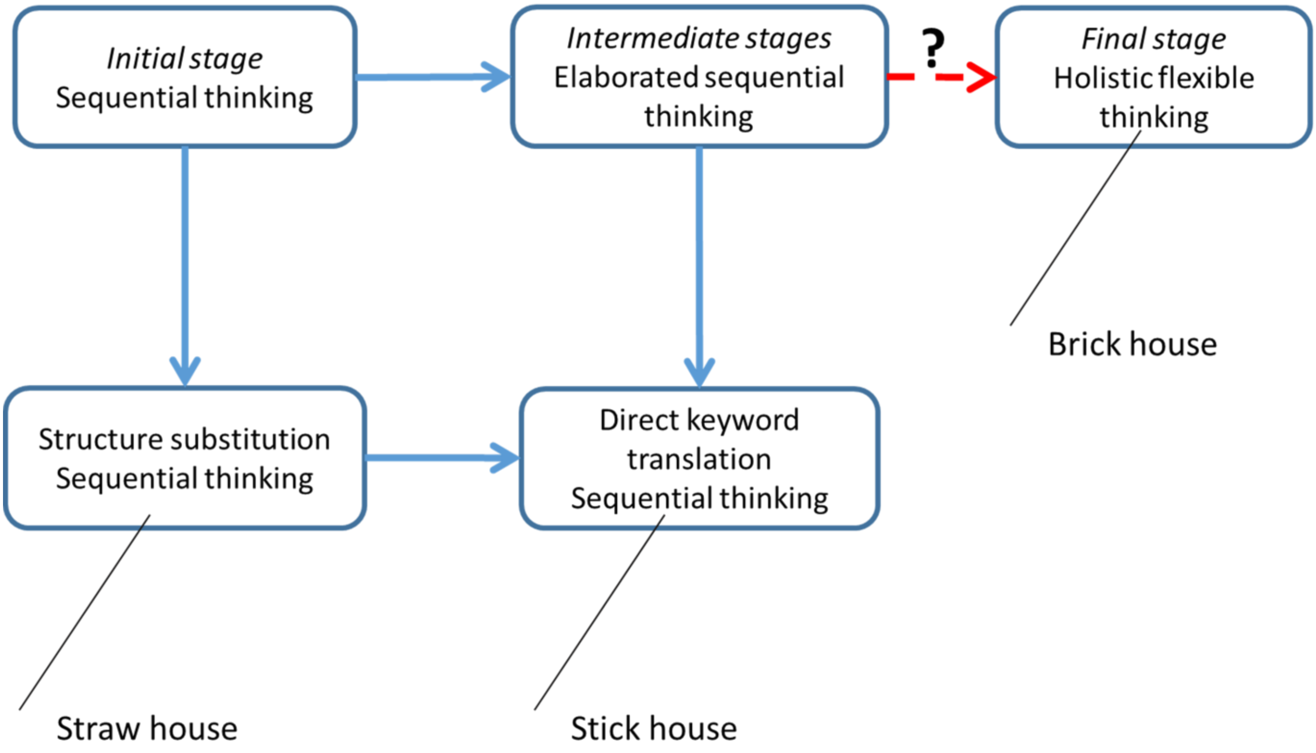

Figure 1. Summary of stages of knowledge development

Erdniev (1979) proposed that the direct problem should be solved in parallel with the inverse problem (the same additive situation with the unknown in a different place). Zaitseva and Tselischeva (2010) propose that students be asked to compose an inverse problem after having solved the direct one. All these approaches clearly reflect the effort to reorganize students' thinking in a holistic and flexible way.

At the heart of all the above-mentioned teaching approaches is an effort to help students to see a situation as a whole and better coordinate relationships between the quantities involved. Therefore, the success of these teaching methods can be strongly associated with the duality hypothesis and the development of holistic and flexible thinking.

A radically different approach was developed by Davydov (1982) and is currently being studied by several researchers (Iannece, Mellone, & Tortora, 2010; Schmittau & Morris, 2004). Davydov assigned a fundamental role to the *additive relationship* that he defined as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). He suggests that this concept development should start from the very beginning of schooling, in parallel with or even before learning addition and subtraction. He claims that the *additive relationship* is a pre-requisite for the mathematical concept of number. In his experiments, Davydov (1982) tried to develop in students a relational understanding of situations where continuous materials were used. For example, students discussed amounts of water in different containers and the composition and decomposition of geometrical figures as well as compared lengths of paper strips.

Some authors (Hatano, 1982; Verschaffel et al., 2009) stress the key role that conceptual knowledge plays in relational thinking and adaptive expertise development. They stress that the development of conceptual understanding should occur before the procedures have been trained. However, it is not clear which concepts we should refer to in the case of solving simple **addition and subtraction word problems**. The terms suggest that the concepts to develop are *addition* and *subtraction*. However, learners may understand them as processes of adding, putting together or taking away. Discussing the adaptive use of calculation strategies, researchers (Verschaffel et al., 2009) also argue in favour of the development of "personal framework of number relations" (p. 348). Is it more appropriate to consider the additive relationship (Davydov) as a mathematical concept by itself, separate from the concepts of addition and subtraction? If we consider the additive relationship to be an essential mathematical concept, should we explicitly teach it in the classroom providing bricks for all students from the beginning?

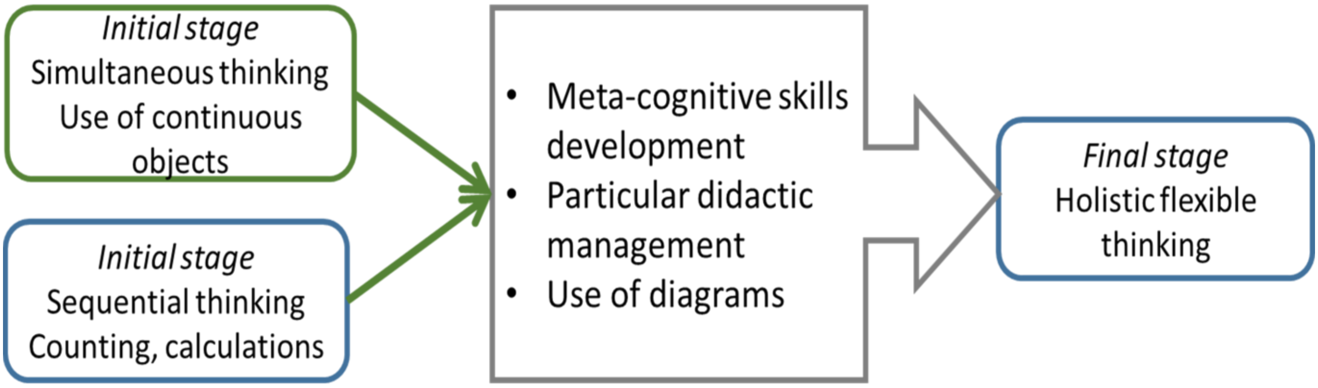

Figure 2. Equilibrated development approach

Keeping in mind all concrete teaching strategies, we still need to use or propose a general teaching approach in which they can all be incorporated. For example, in the scaffolding approach (Carpenter et al., 1999), students should first use their own thinking strategy and then be increasingly confronted with situations where this strategy does not work, thus leading to the creation of new strategies. In the theoretical thinking approach (Davydov, 2008; Schmittau & Morris, 2004), students should first develop the most abstract and general understanding of the additive relationship and then use this understanding in different concrete contexts. As we try to argue here, in any approach, the development of thinking duality should be the main educational goal. Thus, the sequential/simultaneous or relational/instrumental thinking coordination should be the focus of any word problem solving task, especially in early grades, at the beginning of word problem solving experience. Figure 2 presents the Equilibrated Development Approach we are currently developing (Polotskaia, 2014).

Working within this approach (Polotskaia, Freiman, & Savard, 2013; Polotskaia, 2014; Savard & Polotskaia, 2014), we ask students to analyze and model various situations involving strings of different lengths in **parallel** with the introduction of numbers and arithmetic operations. They must try to use a holistic approach to analyze word problems, often without any calculation being executed. Only when they are able to model the word problem and discuss its mathematical structure is the calculation part is integrated into the problem-solving process. According to our observations of Grade 1 and 2 students, this developmental process can take several weeks.

In the previous sections, we discussed two types of thinking in word problem solving: sequential and holistic. Hegarty et al. (1995) argue that one of the sources of difficulties in problem solving is the choice of thinking students make when approaching the problem. In the tale, nobody taught the pigs to choose their construction material. Can the appropriate coordination of different ways of thinking be developed as a stable mathematical habit in all students? We are not yet able to answer this question. However, we can try to reorient research efforts toward looking at the additive relationship as a concept contributing to the development of mathematical thinking. We should try to design specific tasks and teaching scenarios where holistic and flexible thinking in solving word problems can be strongly encouraged in students.[2] To highlight the relational perspective of the research in the area, we propose clarifying our terminology. We should use the terms *addition problems* and *subtraction problems* to refer to the calculation of sums and differences. The term *additive word problems* should be used to refer to word problems with an additive relationship as its mathematical structure, whether their solution involves addition or subtraction.

At Home

In different versions of "The Three Little Pigs," the destinies of the builder of the straw house and the builder of the stick house are different. According to some versions, the wolf just eats the little pigs—or, in other words, the bad problem solvers. Personally, we prefer the version in which they all survive and get to their brother's brick house (Mikhakov, 2010).

Once the three brothers were together in Naf-Naf's house, protected by brick walls, with the door and windows closed, the hungry wolf attacked them again. This time, the wolf tried to enter the house through the chimney. Nif-Nif and Nuf-Nuf were frightened and hid under the bed. Naf-Naf, who felt really at home in his own house, proposed a solution to this new challenging situation. He opened the lid of the pot heating in the fireplace. The wolf was scalded by the boiling water and ran away as fast as he could.

[2] In our research we have created and tested such tasks (Savard, Polotskaia, Freiman, & Gervais, 2013).

If we continue to explore this metaphor, we can note that the concept of *open* is completely opposite to that of *close*, which, until the end of the tale, was associated with the idea of "protection." We can suggest that the knowledge developed by Naf-Naf in building the brick house took on a new quality. The concept of protection, once constructed, became so natural and spontaneous that the student, Naf-Naf, was able to apply it to challenging situations, and to do so in a creative way. Why did Nif-Nif and Nuf-Nuf, who had just arrived at the brick house, fail to apply their knowledge in a new situation?

Based on Vygotsky's thinking (Vygotsky, Luria, & Knox, 1993; Vygotsky, 1996), Znamenskaya and colleagues (Znamenskaia, Ostroverh, Riabina, & Hassan, 2009, pp. 55-56) describe three levels of a student's relation to new knowledge:

1) Awareness (relating to it as familiar, not foreign)
2) Learning (relating to it as possible to reproduce)
3) Appropriation (relating to it as his or her own natural way) and mastery (relating to the knowledge as a special tool)

According to these authors, a competence cannot be considered as fully developed until students have appropriated their new knowledge and feel "at home" with the concept.

CONCLUSIONS: HYPOTHESES ABOUT TEACHING STRATEGIES AND RESEARCH PERSPECTIVES

This is the end of our didactic story. We imagine that our three little pigs ended up each building a brick house and lived happily ever after. Besides the sustainability of the brick house and Naf-Naf's insight, what can we propose based on this tale?

The story of the three little pigs helped us to reassess the situation of word problem solving learning and problem solving teaching. The theory of stages explains that holistic and flexible thinking about quantities and their relationships is knowledge that must be developed in students to ensure their success in problem solving. However, it does not explain why this is difficult for many students or suggest efficient ways for developing this thinking in students. Our hypothesis of cognitive duality in word problem solving can provide some more profound explanations. Above, we have shown that the difference in thinking development can be caused by initial difference in students' brain functions. We also explained how some students can stray from the knowledge development path, thereby creating obstacles for further learning. Therefore, the tendency of some students to preferentially use sequential thinking should be taken into consideration when designing and implementing problem solving tasks. *Problem solving tasks should be specially designed to really engage these students in a holistic analysis of the problem's structure and to help them develop the reversibility/flexibility of mathematical thinking.*

Sometimes, mathematically similar tasks, such as solving simple additive word problems, can imply quite different cognitive processes. Some of these tasks help to develop appropriate knowledge and others contribute to the construction of educational impasses. *We suggest that word problems with inconsistent language (our problem 3) should primarily be used as problem solving tasks in early grades.*

If the holistic analysis of a problem is not explicitly required, students risk approaching word problems instrumentally and can potentially shift their knowledge development toward the learning impasse. *We suggest that the knowledge of the additive relationship and the development of holistic and flexible thinking should be recognized as explicit teaching goals in elementary math education.*

We believe that the existing teaching theories do not fully embrace the relationship between the use of previous knowledge and the development of new knowledge. Research in neuro-education can be of valuable input in this area (Masson, Potvin, Riopel, Foisy, & Lafortune, 2012). *We suggest that thorough research based on brain imaging methods could potentially confirm our hypothesis about the fundamental difference between sequential and holistic thinking in word problem solving.*

When a student has a new knowledge just constructed, the teaching/learning process is not yet accomplished. More work is needed for the new knowledge/concept to be appropriated by student so that the student can feel "at home" with it and be able to apply it in a creative way. The relationship between the consolidation of knowledge and creativity is another interesting and far from being solved question in learning/teaching theory. Once again, neuro-education may be a good source for new ideas.

We hope that the points discussed in this article will bring attention to important problems in teaching/learning theories and nourish future research in this area.

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