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Exploring mathematical connections in the context of proof and mathematical argumentation: A new proposal of networking of theories

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Abstract
Extended theory of mathematical connections (ETC) and theory of mathematical argumentation (TMA) based on Toulmin's (1984) model were articulated for the study of mathematical connections activated in the argumentation process. For this purpose, a "networking of theories" was made to obtain the complementarities between both theories. Then, a class episode was selected that dealt with the demonstration of the continuity theorem of functions of real variable "*if a function is derivable at a point then it is continuous at that point*", made by an in-service mathematics teacher of differential calculus, who participated in a non-participant observation, where his classes were videotaped. The arguments of this episode were analyzed through with Toulmin's (1984) model, after with thematic analysis method to identify mathematical connections, and, finally, the connections in the proof and mathematical argumentation were analyzed. The main result of the research reveals that the mathematical connections play a fundamental role in the argumentation process of the episode, given that, connection is important for the establishment and identification the argument and the warrant that supports it. In addition, complementarities were found between both theories, which makes this networking a useful tool for a better analysis of mathematical argumentation processes.

Keywords: networking of theories, extended theory of connections, theory of mathematical argumentation, proof, derivative

INTRODUCTION

One of the topics of great interest in research in mathematics education has been the development of theoretical approaches or theories specially to analyze mathematical activity. These types of research topics were classified into two groups (Ledezma et al., 2022):

1. Researchers who have proposed general approaches or theories to analyze mathematical activity, for example, those who develop theoretical constructs within the framework of theories of mathematics education (Brousseau, 2002; Chevallard, 1992; Kuzniak, 2011, Godino et al., 2007; Radford, 2013).
2. Researchers focused on specific activities such as the use of semiotic representations (Duval, 2017), problem solving (Liljedahl & Santos-Trigo, 2019; Pólya, 1989), visualization (Presmeg, 2006), mathematical modeling (Borromeo, 2018), mathematical and ethnomathematical connections (Rodríguez-Nieto et al., 2021a; Rodríguez-Nieto & Escobar-Ramírez, 2022) and argumentation (Cervantes-Barraza et al., 2019; 2022; Conner et al., 2014; Toulmin, 2003).

It is worth mentioning that some research has focused on creating networking of theories between a general or broad theoretical framework and a specific theoretical approach focused on analyzing a particular mathematical activity, among which the processes of

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Contribution to the literature

- This research reports that the mathematical connections of ETC complement the arguments that emerge in the development of the proof process made by the in-service teacher, since in the parts of the argument (data, conclusion, warrant and backing) there are mathematical connections that allow detail its operation. In addition, it is important to recognize that the mathematical connection is also enriched by the argument that supports it.
- The relevance of the networking of theories (ETC and TMA) is shown using theoretical network strategies considered important from understanding to local theoretical integration, referring to the fact that theories do not fully complement each other, but with some of their tools, the episode on the derivability of a function was effectively analyzed.
- In fact, this paper articulates two specific theoretical models dedicated specially to promoting two processes: Mathematical connections and mathematical argumentation.

semiotic representations stand out (Pino-Fan et al., 2017), connection processes to promote mathematical understanding (García-García & Dolores-Flores, 2018; Rodríguez-Nieto et al., 2021b). For their part, Ledezma et al. (2022) studied the modeling processes and Molina et al. (2019) were interested in the argumentation process and types of arguments. In this context, it is valid to integrate theories with specific theoretical models or approaches (with different analyzes of mathematical activity) and explore how they could complement or coordinate with each other, recognizing commonalities and differences.

Mathematical connections have been considered a fundamental process standard in the teaching and learning of mathematics in similar ways as problem-solving, reasoning and proof, communication, and representation (National Council of Teachers of Mathematics [NCTM], 2000). When a subject establishes connections, between concepts, representations, meanings and between mathematics and real life, he has a better chance of understanding a mathematical concept better (De la Fuente & Deulofeu, 2022; García-García, 2019; Hiebert & Carpenter, 1992; NCTM, 2000; Rodríguez-Nieto, 2021; Rodríguez-Nieto & Alsina, 2022). NCTM (2000)

> "recognizes and uses connections among mathematical ideas by emphasizing mathematical connections, teachers can help students build a disposition to use connections in solving mathematical problems, rather than see mathematics as a set of disconnected, isolated concepts and skills" (p. 64).

Some research carried out in the field of mathematics education have focused its attention on the mathematical connections because they contribute to understanding, allow students and teachers to solve problems consistently, allow integration between concepts from mathematics and daily life (García-García & Dolores-Flores, 2018, 2019; Moon et al., 2013). Likewise, it is recognized that mathematical connections are key in the contents of different curricular approaches (Departament d'Ensenyament [Education Department] [ED], 2017; Mwakapenda, 2008; NCTM, 2000), which are seen as competence to identify mathematics in concrete everyday situations. In fact, Association of Mathematics Teacher Educators (AMTE, 2017) argues that, in the proper teaching of mathematics, connections must be established so that students also make connections and deepen conceptual and procedural understanding.

Research on mathematical connections has focused on a particular concept, for example, Moon et al. (2013) paid attention to the connections between different representations of the conic curves. Mumcu (2018) studied the connections with the derivative made by future teachers. In other research carried out by Mhlolo (2012) and Mhlolo et al. (2012) proposed and used a tool to analyze the quality of mathematical connections made by teachers, emphasizing representations. Particularly in Mexico, García-García and Dolores-Flores (2018, 2019) worked with the derivative and the integral with high school students and García-García and Dolores-Flores (2020) explored the mathematical connections made by high school students when solving problems application on Calculus. Dolores-Flores et al. (2019) emphasized in the concept of the rate of change. Likewise, Rodríguez-Nieto et al. (2021a) studied the connections that pre-service mathematics teachers established on the derivative and presented some causes that cause difficulties in establishing mathematical connections of meaning and different representations. Rodríguez-Nieto et al. (2022a) made a theoretical reflection on the mathematical connections made by a teacher of differential calculus when he taught the derivative topic, where the metaphorical connection that is presented in extended theory of mathematical connections (ETC) emerged. Also, in Rodríguez-Nieto et al. (2021b) was developed a networking of theories, where they integrated ETC with the onto-semiotic approach (OSA) for a more detailed analysis of the mathematical connections on the derivative, considering a connection as the tip of iceberg made up of a conglomeration of practices, objects, processes and semiotic functions.

Now, just as mathematical connections are important, it is also essential to know how a mathematical connection is identified or established. In this context, García-García and Dolores-Flores (2018) affirmed that, "mathematical connections emerge when students solve specific tasks and can identify them in their written productions or in the oral or mimic *arguments* they develop" (p. 229). Also, it is important to recognize that argumentation involves students in making quality connections, since they are well support (Mhlolo, 2012). These research show how argumentation is directly related when a mathematical connection is established. The arguments are the means by which the mathematical connections that have a justification can be identified, this is equivalent to the mathematical content that justifies the relationship between the data and the conclusion.

Research studies recognized argumentation as a fundamental part in the process of teaching mathematics (Conner et al., 2014; Erkek & Isiksal-Bostan, 2018; Solar, 2018) and the learning process of mathematics (Cervantes-Barraza et al., 2019, 2022; Krummheuer, 2015). Curricular proposals encourage teachers to involve their students in the production of arguments, generating opportunities of discussion, validation of mathematical ideas, making connections and convincing others (Common Core State Standards Initiative [CCSSI], 2010; Ministerio de Educación Nacional [Ministry of National Education] [MEN], 2006; NCTM, 2000). This because, argumentative activities help teachers to construct pedagogical knowledge (Metaxas, 2015), and engaged them with understanding of their students' mathematical concept development and reasoning (Erkek & Isiksal-Bostan, 2018). Moreover, Molina et al. (2019) studied students' abductive and analogy arguments used to solve spatial geometry problems, to do so, authors integrated OSA theory with mathematical argumentation to provide evidence of different objects, process and mathematical statements implied in their arguments.

Studies on mathematical connections have focused mainly on identifying the connections made by high school students, undergraduate students in mathematics, pre-service mathematics teachers and in-service mathematics teachers (García-García & Dolores-Flores, 2018, 2019; 2020; Rodríguez-Nieto et al., 2021a, 2022a, 2021b, 2021c), worrying about answering questions aimed at the meaning of mathematical objects, what are the representations of mathematical objects, logical relationships of implication and part-whole, features of the concepts, but have not focused on the reasons, arguments, justifications that support a connection. It is important to mention that, in Rodríguez-Nieto et al. (2021d) the mathematical connections were detailed in terms of OSA, without deeply emphasizing the arguments that a subject construct. In addition, it is important to study the connections on the derivative because students and teachers have difficulties connecting multiple representations (verbal, symbolic, and geometric), meanings of this concept, which also indicates a lack of understanding of the concepts of function, tangent line, slope, among others (Borji et al., 2018; Galindo-Illanes et al., 2022; Pino-Fan et al., 2018).

The reconstruction of argumentation activity has increase in mathematics education field through theory of mathematical argumentation (TMA), several studies have making synthesis of related literature about teacher argumentation, we have identified these critical points:

1. More than a few studies have taken the Toulmin's (2003) model as a method to reconstruct teacher argumentation, Conner et al. (2014) adapted it to model middle school collective argumentation, including teacher participation as a part of the structure. Knipping and Reid (2015) provided a new way of reconstructing the structure of collective argumentation. In the same sense, Erkek and Isiksal-Bostan (2018) took up this adaptation to examine the nature of prospective mathematics teachers' argumentation in geometry setting.
2. Generally, participants involved in these studies are secondary or middle school mathematics prospective teacher.
3. Numerous studies have selected the mathematical object related with geometry such as proving properties of triangles, circles, rhombus, and other figures.
4. In fact, we recognize few research that take in account of high school and university (undergraduate) mathematical concepts, specially, derivative and a specific analytic tool to explore teacher's mathematical connections and arguments used while proving a theorem. For example, Giannakoulias et al. (2010) studied the argumentation of high school mathematics teachers when they try to convince their students about the invalidity of their statements and found that teachers use counterexamples and theory to refute, but few teachers use counterexamples in their argumentation and proceed to underestimate its importance as a special test method.

In this study we answer the following research questions:

1. What are the theoretical and methodological complementarities between ETC and TMA?
2. What is the role of mathematical connections in the arguments made by an in-service mathematics teacher in the context of the proof of the derivability theorem involving the continuity of a function?

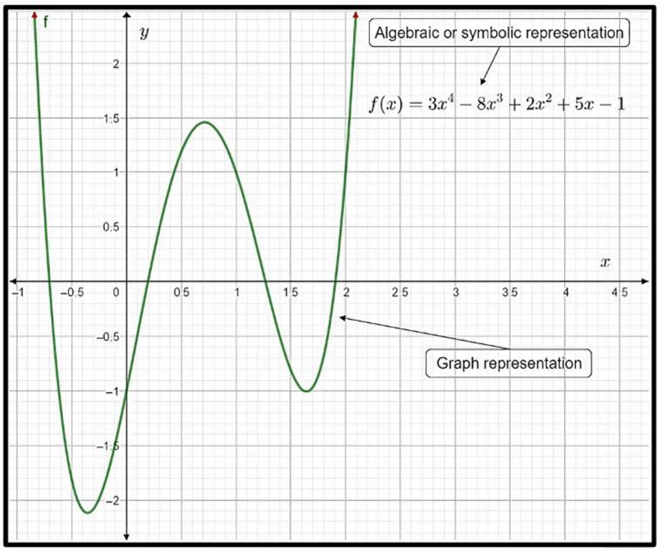

Figure 1. Example of connection between different alternate representations (Source: Authors' own elaboration, using GeoGebra software)

THEORETICAL FRAMEWORK

Extended Theory of Connections

In this research, a mathematical connection is understood as "a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations, and meanings with each other, with other disciplines or with real life" (García-García & Dolores-Flores, 2018, p. 229). In addition, it has been considered that there are models to characterize mathematical connections, for example, the one proposed by Businskas (2008), which has been used more frequently in mathematical connections research. Mathematical connections can be intra-mathematical "are established between concepts, procedures, theorems, arguments and mathematical representations of each other" (Dolores-Flores & García-García, 2017, p. 160), and extra-mathematical connections, which "establishes a relationship of a mathematical concept or model with a problem in context (not mathematical) or vice versa" (Dolores-Flores & García-García, 2017, p. 161). For the purposes of this work, we have retaken seven categories of intra-mathematical connections a priori: four (procedural, part-whole, implication, and different representations) of Businskas (2008), one (feature) of Eli et al. (2011), two (meaning and reversibility) of García-García and Dolores-Flores (2018, 2019, 2020) and metaphorical connection (Rodríguez-Nieto et al., 2022a). These mathematical connections are described below:

1. *Procedural*: This mathematical connection is evident when rules, algorithms or formulas are used to arrive at a result (García-García, 2019; García-García & Dolores-Flores, 2019). For example, if a line is not vertical and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points other than the line, then the slope of the line can be found using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, with $x_2 - x_1 \neq 0$.
2. *Part-whole*: This of connection type occurs when someone identifies that A is a generalization of B, where B is a particular case of A. For example, the function $P(x) = x^3 - x^2 - 9x - 9$ is a particular case of the general expression $f(x) = ax^3 + bx^2 + cx + d$ (Businskas, 2008). These relationships can be of inclusion when a mathematical concept is contained in another (García-García, 2019).
3. *Implication*: This type of connection is based in a logical relationship if-then (A→B) (Businskas, 2008; Mhlolo, 2012). For example, the logical relationship is presented if f is differentiable in $x = a$, then f is continuous in $x = a$.
4. *Different representations*: can be alternate or equivalent (Businskas, 2008). Is alternate if a student represents a mathematical concept in two or more different ways in different registers of representation: graph-algebraic, verbal-graph, etc. For example, an alternate representation is shown in **Figure 1**, where the polynomial $f(x) = 3x^4 - 8x^3 + 2x^2 + 5x - 1$ is graphed. While an equivalent representation is a transformation within the same register (algebraic-algebraic, graph-graph, symbolic-symbolic, etc.). For example, $P(x) = x^3 - x^2 - 9x - 9$ is equivalent to $P(x) = (x + 1)(x + 3)(x - 3)$ in the algebraic semiotic register.
5. *Feature*: It is identified when the subject manifests some characteristics of the concepts or describes its properties in terms of other concepts that makes them different or similar to others (Eli et al., 2011; García-García & Dolores-Flores, 2019). For example, García-García and Dolores-Flores (2019) affirm that, when the person mention some elements of a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ (derivative function or antiderivative function) are coefficients (all, a_i, with $i = 0, 1, 2, 3, \ldots, n$), literal or variables (in this case, the "x") and exponents of the variables ($n, n-1, n-2, \ldots, 1$).
6. *Meaning*: This mathematical connection is presented "when students attribute a meaning to a mathematical concept as long as what it is for them (which makes it different from another) and what it represents; it can include the definition that they have built for these concepts" (García-García, 2019, p. 131). In this sense, students express what the mathematical concept means to them, including their context of use or their definitions (García-García, 2019). In this research, we assume that this type can be more general, that is, we accept the existence of a mathematical connection between meanings. We consider that

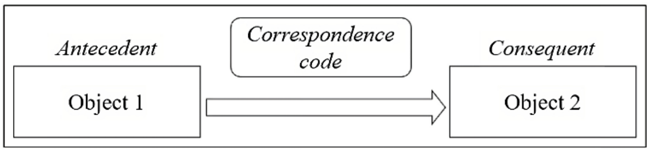

Figure 2. Basic scheme of mathematical connection (Rodríguez-Nieto et al., 2021b)

this type is manifested when the students relate different meanings attributed to a concept to solve a specific problem. For example, Stewart (1999) "the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$" (p. 153).

7. *Reversibility*: It is present when a subject starts from a concept A to get to a concept B and invert the process starting from B to return to A (García-García & Dolores-Flores, 2019). For example, this connection is established when the bidirectional relationship between derivative and integral, as operators, is recognized and when fundamental theorem of calculus is used as a way to link both concepts (García-García & Dolores-Flores, 2018).
8. *Metaphorical*: Metaphorical is understood as the projection of the properties, characteristics, etc., of a known domain to structure another less known domain. For example, when the teacher or the student uses verbal expressions such as "*travel through the graph without lifting the pencil from the paper*" that implicitly suggest the conceptual metaphor "the graph is a path" (Rodríguez-Nieto et al., 2022a).

On the other hand, García-García (2019) supports that research in mathematics education can validate and refine the categories of mathematical connections presented above, but it could also include other categories not yet identified. In addition, the mathematical connections can be schematized as presented in **Figure 2**, considering that there are two mathematical objects that are connected and, said connection is supported by a correspondence code (Rodríguez-Nieto et al., 2021b, 2022b). For example, connect the derivative at a point (object 1) with its meaning as the slope of the tangent line to the curve at a point (object 2).

Theory of Mathematical Argumentation and Proof

In mathematics education field the concepts of argumentation and proof have been studied broadly, several research studies recognize argumentation and proof as a main part in the process of teaching and learning of mathematics (Cervantes-Barraza & Cabañas-Sánchez, 2020, 2022; Conner et al., 2014; Erkek & Isiksal-Bostan, 2018; Krummheuer, 2015; Pedemonte & Balacheff, 2016).

In this paper, we refer to argumentation in the framework of informal logic, this related with the use of the language of everyday life (Toulmin, 2003; Van Eemeren & Grootendorst, 2015). Based on Toulmin (1984), argumentation denotes the activity of presenting conclusions supported by evidence, reasons with the objective of convincing an audience and implying the arguer respond to critics or refutations that emerge from the audience.

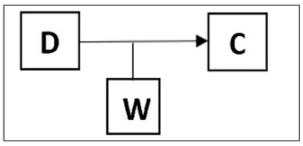

Figure 3. Basic structure of an argument (Toulmin, 2003)

Following, we need to define the argument content in terms of a set of reasons that support the conclusions provided by the arguer and indicate that its basic structure (**Figure 3**), is made of three key elements: The *data* (D) is the information or evidence that support the conclusion, the *conclusion* (C) refers to the arguer's conclusion and the relationship between the data and the conclusion is justified by the *warrant* (W), this element includes mathematical properties, patterns or general statements. There are three missing elements, the *backing* (B), the *modal qualifier* (Q) and the *refutations* (R) but they are not considered in this study.

Argumentation and proof are not the same thing, Duval (2000) affirmed that argumentation is related with common language with the purpose of convincing and a proof reveals the truth of a reasoning. In the context of school mathematics, NCTM (2000) defined proof as: "arguments consisting of logically rigorous deductions of conclusions from hypotheses" (p. 56), in the same sense, Stylianides (2007) pointed out that a proof is a mathematical argument that contains reasons that support a mathematical claim or conclusion, and has some characteristics:

1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification.
2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community.
3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

Considering argumentation from Toulmin's (1984) perspective allow us to highlight the function of the warrant in the context of mathematical proof, this element supports the conclusion based on mathematical axioms, definitions, or theorems (Pedemonte & Balacheff, 2016) and provides an idea about the type of reasoning adopted by the arguer (Cervantes-Barraza et al., 2020, 2022; Conner et al., 2014). In this paper, the

Table 1. Types of arguments in collective argumentation (Cervantes-Barraza, 2020, p. 40)

Argument	Definition
Argument from classification	Mathematical objects are classified based on invariant characteristics & mathematical properties.
Argument from mathematical properties	A conclusion is based on mathematical properties that geometrical objects satisfy.
Practical argument	Conclusion is based on comparison of characteristic & mathematical properties.
Argument from best explanation	Most reasonable conclusion in an argument is recognized when all possible cases are considered in its justification: *Case 1*: Mathematical objects that satisfy properties & invariant characteristics & *Case 2*: Mathematical objects that do not satisfy properties & invariant characteristics.
Argument from consequences	A conclusion is justified or refuted by showing a positive or negative consequence.

warrants play an important role because their content evidence the different mathematical connection that a student or teacher make in order to justify the linking of the initial data or hypothesis with the conclusion. In addition, Nardi et al. (2011) proposed a classification of warrants in the context of teacher argumentation, in fact, they pointed out that teachers can use warrants that involves these categories: a priori, empirical, institutional, and evaluative.

The first type of warrant foster teachers in using theorems or mathematical properties. The second type of warrant indicates how the teacher supports their conclusions using the mathematics' book content. The third type refers to what have been consolidated in the class, like consensus or a conclusion of the class. The last type of warrant is a justification of a pedagogical choice or the value or belief about an event. So what, teacher arguments cannot have analyzed for their mathematical accuracy only, can be reconsidered, arguably more productively, in the light of other teacher considerations and priorities: pedagogical, curricular, professional, and personal.

Types of mathematical arguments

Following the basic structure and elements of an arguments described in the last section, we need to talk about different mathematical arguments and how can have typified them. Godden and Walton (2007) pointed out that the type of arguments are stereotypical patterns of reasoning with the function of justify a conclusion. In the frame of formal logic, we can find mathematical structures that characterize the arguments based on mathematical syllogism (e.g., Modus Ponendo Ponens, Modus Ponendo Tollens, among others) (Walton et al., 2008). On the other hand, Toulmin (2003) in the frame of informal logic provided types of arguments based on reasoning: analogy, rule, effect, generalization, sign, cause, authority, classification (p. 213).

Recent studies about mathematical argumentation in classroom have analyzed students' arguments and provided typification's. Conner et al. (2014) typified arguments according to the integration of the key elements of mathematical reasoning (case, rule, and result) and the basic elements of arguments (data, warrant, and conclusion). Molina et al. (2019) characterized abductive and analogy arguments based on objects, process, and mathematical statements. In the context of collective argumentation, Cervantes-Barraza (2020) adapted five types of arguments (**Table 1**), these arguments are classified according to warrant content and provide a complete set of types.

Since this research focused on mathematical connections and mathematical argumentation, we consider important the perspective of García-García and Dolores-Flores (2018) who affirmed that, "mathematical connections emerge when students solve specific tasks and can identify them in their written productions or in the oral or mimic *arguments* they develop" (p. 229).

Networking of Theories in Mathematics Education

Research in mathematics education has been concerned with understanding of how theories can be successfully connected, highlighting that the conceptual and methodological elements underlying each theory must be respected. This process is called 'networking of theories' (Bikner-Ahsbahs & Prediger, 2010, 2014), which, various authors agree that it is useful for making more detailed analyzes on the understanding of the complexity of emerging phenomena in the learning and teaching processes of mathematics (Ledezma et al., 2022; Rodríguez-Nieto et al., 2021b, 2022a, 2022b).

In Kidron and Bikner-Ahsbahs (2015) reflect on the emergence of networking, but also on the key to doing so. For example, a phenomenon or research problem could be analyzed with the lenses of different theoretical frameworks, the results of which show innovative and creative mathematical knowledge structures. However, from the perspective of articulating theories, it is observed that the various analyzes with different theories of the same phenomenon often shed light on similarities, which is an adequate way to integrate theories. Arzarello and Olivero (2006), Bikner-Ahsbahs (2016), and Kidron and Bikner-Ahsbahs (2015) argue that some data are difficult to interpret with a theory, therefore, the integration of theories, even if they are different, can generate complementary analyzes and detailed with theoretical and methodological contributions.

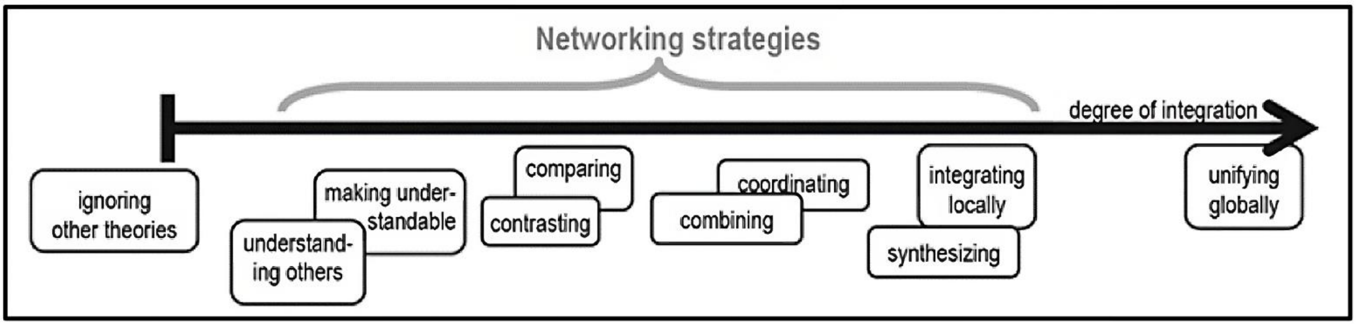

Figure 4. Networking strategies (Adapted from Prediger et al., 2008, p. 170)

Until now, it has been essential to refer to the creation and structuring of the networking of theories, but it is also important to delve into What is a theory? In particular, the literature recognizes various positions on the term 'theory', but for the purposes of this research we consider the view of Radford (2008), which is the definition of theory most used in research on networking of theories, in fact, this type of articulation depends on the particularities of the components of the theories [principles (P), methodology (M), and paradigmatic research questions (Q)], which must be connected. According to Radford (2008), a theory is made up of:

- A system, *P*, of basic principles, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
- A methodology, *M*, which includes techniques of data collection and data-interpretation as supported by *P*.
- A set, *Q*, of paradigmatic research questions (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified) (p. 320).

Artigue and Mariotti (2014) recognized that given the significant advancement of works focused on networking of theories, some researchers set out to structure methodologies with the aim of having access to theories and operability in research practices. To carry out these investigations, methodologically paths have been established or strategies have been created, for example, Bikner-Ahsbahs and Prediger (2010) and Prediger et al. (2008) reported that strategies and methods are connected by following four pairs of sub-strategies to articulate theories, ranging from completely ignoring another theoretical framework at one end, to globally unifying different approaches at the other. As intermediate strategies, the first two pairs of strategies that refer to the compression of both theories by the experts of each theory are presented in a hierarchical manner. The second pair of strategies invites the comparison and contrast of the theories to identify different points in common or differences between the theories. The third pair directs researchers to the combination and coordination of theories, leaving a framework of conceptual complementarities to generate a new theory or methodology. In the fourth pair of strategies, the complementarities that lead to the formation of a holistic theoretical framework are locally integrated and synthesized (**Figure 4**).

METHODOLOGY

This research is qualitative (Cohen et al., 2018), where two theories in mathematics education are articulated and then, with said articulation, the mathematical activity of a teacher is described and analyzed when he teaches his students to prove a theorem on derivability. To do this, the four pairs of strategies to develop networks of theories proposed by Prediger et al. (2008) were used, as presented in **Figure 3**. In this sense, the findings of this study contain the theoretical articulation and its use. therefore, in the first strategy the theories were understood, in the second the theories were compared and contrasted, in the third the theories were coordinated and combined, where the use of articulation was deepened to analyze in detail the teacher's proof. Finally, in the fourth strategy a local synthesis was found.

Particularly in the development of the third pair of strategies, the voluntary participation of an in-service mathematics teacher was required, who performs a proof on the derivability theorem that implies the continuity of a function. In the following sections, it will be shown in more detail that the data collection was done through participant observation and the data analysis was carried out using the thematic analysis method of the episodes to identify connections and arguments.

FINDINGS

Understanding and Comparing Theories

In the context of this research, the first two pairs of strategies:

(1) making under-standable-understanding others and

Table 2. Important aspects of comparison between ETC & TMA

Variable	Comparison
Principles	In theoretical foundation of ETC, establishment of mathematical connections is considered important, since various curricular organisms affirm that connections are an important indicator for subjects to understand mathematical concepts, relate meanings, properties of concepts, representations, etc. (AMTE, 2017; MEN, 2006; NCTM, 2000; SPE, 2011), which reveals a representational cognitive stance that emphasizes mainly search for meaning of mathematical concepts, its expansion & a teaching of mathematics on connections. Also, principles of ETC are supported by a set of investigations that have conceptualized term connection, for example, for Brown (1993) mathematical connections "are a causal or logical relationship or association, an interdependence" (p. 481). Metaphorically, Hiebert and Carpenter (1992) understand connections as part of a hierarchical network, like a cobweb, where an intersection or node can be seen as part of information represented, & threads between nodes can be understood as connections or relationships. In fact, connections are true relationships (Businskas, 2008), but they are really "a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations, & meanings with each other, with other disciplines or with real life" (García-García & Dolores-Flores, 2018, p. 229). Whereas that TMA is on viewing argumentation as a process & a basic ability develop in all education system, it allows students to develop conceptual understanding (Rumsey et al., 2019). Also, argumentation is a means to foster student's learning of mathematics through participation with arguments (Krummheuer, 1995, 2015) & promotes development of argumentative skills such us refuting & building counterarguments (Cervantes-Barraza et al., 2019; Rigotti & Greco Morasso, 2009). Also, several plans & programs of mathematics studies in basic education point out relevance of argumentation, they emphasize need to promote construction of arguments by students from first years of schooling, since it helps them gain sufficient confidence to justify conclusions & procedures with arguments oriented towards deductive-inductive reasoning & mathematical proof (CCSSI, 2010; NCTM, 2000; SPE, 2011).
Methods	In mathematics education research, ETC has been characterized by exploring connections in two moments: (1) application of semi-structured interviews or interviews on tasks included in questionnaires to collect data (Goldin, 2000) & (2) analysis of data from thematic analysis of content (Braun & Clarke, 2006; Rodríguez-Nieto et al., 2021b) to identify phrases or keywords boxed in codes & topics, where a type of mathematical connection is inferred. Unlike, TMA has adopted a particular method from philosophical field, Toulmin's (2003) model was designed under informal logic perspective allow researches in adapting this model & analyze mathematical argumentation in classroom (Boero et al., 2010; Conner et al., 2014), also is a methodological tool that allows to reconstruct the meaning of the classroom talk (Krummheuer, 1995, 2015), it has been modified in order to reconstruct complex argumentation occurred in mathematics class (Knipping & Reid, 2015; Cervantes-Barraza et al., 2019, 2020, 2022).
Research questions	In Rodríguez-Nieto et al. (2021b), it is evidenced that research carried out under ETC framework have answered research questions such as following: What connections are promoted when studying a particular mathematical object? What is level of quality of mathematical connections established by students or teachers? What factors must be present for a new typology of mathematical connections to be generated? What are connections teacher makes in classroom? What mathematical connections are presented in school mathematics textbooks, & which are promoted in curricula & curricula of different countries of the world? How could teaching interventions be developed that help promote connections & develop in students ability to use mathematical connections in different mathematical & extra-mathematical domains? What are beliefs that both students & teachers attribute to use & importance of mathematical connections & what is your perception of role they play in teaching-learning? (García-García, 2019; García-García & Dolores-Flores, 2019, 2020). Also, it is necessary to build & validate a frame of reference to study mathematical understanding from connections (García-García, 2019). In contrast to ETC, TMA research questions seek to understand nature of argumentation processes: (1) What are characteristics of complex argumentation structures emerging in a fifth-grade mathematics classroom, while students are refuting conclusions? (Cervantes-Barraza et al., 2019) & (2) Lin (2018) research about, how do young students develop argumentation when they are engaged in conjecturing tasks incorporated into mathematical contents through regular instruction in a primary classroom over two consecutive years? And has been stablished methodological research questions like Pedemonte and Balacheff (2016): how Toulmin's (2003) model enriched with cK¢ allows us both to make explicit knowledge bases of students during argumentation activity?

(2) comparing-contrasting, were developed under the vision of Radford (2008) in the detailed description of the theoretical framework and the literature review presented in this article' introduction, where the each theory' principles, data collection and analysis methods and some of the research questions that have been formulated in research with both theories were explained.

However, **Table 2** presents a summary of the research principles, methods, and questions.

On the other hand, once the authors of this article have understood and compared the theories, we proceed to the combination and coordination between them. In this process it is important to consider the perspective of Bikner-Ahsbahs and Prediger (2010) when they state that:

> Whereas the strategies of comparing and contrasting are mostly used for a better understanding of typical characteristics of theories and theoretical approaches in view of

> further developing theories, the strategies of coordinating and combining are mostly used for a networked understanding of an empirical phenomenon or a piece of data (p. 10).

It is for this reason that this research connects theories, but it also analyzes an empirical phenomenon that deals with the mathematical connections that a teacher established when performing the proof of the theorem of the continuity of a function at a point. Likewise, Kidron and Bikner-Ahsbahs (2015) affirmed that, networking not only emphasizes articulating work with two, three, four or more theories, it is also "a methodological approach for theoretical and empirical research that connects different theories to broaden and deepen insight into problems" (p. 221).

Coordination and Combination of Theories

In this section the complementarities between both theories are presented, for example, emphasis will be made on the notions of connection and argument, categories of mathematical connections and types of arguments and, finally, in the development of this strategy the points in common between the methods used in both theoretical approaches will be shown. This way of presenting coordination of theories suggested by Bikner-Ahsbahs and Prediger (2010) when they emphasize that coordination strategies for making theoretical networks should be followed carefully relating different elements of each theory and discovering possible points of compatibility to generate a new conceptual framework, because combination may not bring together all elements of theories but a coherent part made up of well-defined analytical tools to investigate a particular empirical phenomenon.

Family airs between the mathematical connection and the argument

This section reflects on the conformation of the mathematical connection, which refers to the relationship between meanings, representations, propositions, among themselves and with real life. These relationships are supported by arguments or justifications to give validity to mathematical connections. In this sense, we consider it important to deepen on which aspects of argumentation are fundamental in the conformation of mathematical connections or if connections are identified in the structure of argumentation. These family relationships between ETC and TMA allow consolidation of a theoretical and practical language for study of connections and argumentation in mathematics education research, which are two conceptualizations suggested by curricular bodies (e.g., NCTM, 2000) that should be considered in the approach to mathematics classes with different mathematical concepts.

Similarities in the types of mathematical connections and types of arguments

In the theoretical perspectives, the categories of mathematical connections of ETC and the types of arguments established and used in TMA were presented and described. The similarities between mathematical connections and arguments in a mathematics class involve recognizing a relationship of correspondence between the constituent elements of an argument and those corresponding to a connection. In this sense, an argument needs a datum, a warrant and a conclusion; on the other hand, a connection needs an antecedent, a consequent and a correspondence code. Under this comparison of elements, we identified that the warrant corresponds to the semiotic function, since they contain mathematical elements that support the relationship between the data and the conclusion for an argument and the relationship between the antecedent and the consequent in a connection.

However, relationships between connections and arguments were evidenced, for example, the feature connection is related to four types of arguments (e.g., argument of classification, argument of mathematical properties, practical argument and argument of the best explanation), which generally refers to the use that the teacher or student gives to the characteristics or properties that a mathematical object has in a given intra-mathematical or extra-mathematical situation.

On the other hand, we find similarities between the argument of consequences and the implication connection given that both share the if-then relationship (P→Q) that makes sense in mathematics and in everyday life, especially because it is part of the propositional logic that underlies the mathematical connection that is made up of an antecedent (P) and a consequent (Q) related and supported by a correspondence code and/or argument. In turn, we can affirm that these cause-effect functions are fundamental to compose argumentative texts that help the subject to identify whether one situation depends on another or not. Key examples of this relationship are:

(1) if a polygon has three sides, then it is a triangle,

(2) if $f'(x) > 0$ in an interval I, then f is increasing in that same interval, and

(3) if $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I, among other examples.

In this section only two categories of connections were related to the types of arguments, but there are possibilities that there are more points in common, in particular, teachers and students when solving mathematical or application problems establish connections using arguments based on meanings, which can be understood when a person activates an expression-content relationship in a statement or using a meaning of a mathematical object in the resolution of a problem. Likewise, arguments based on different

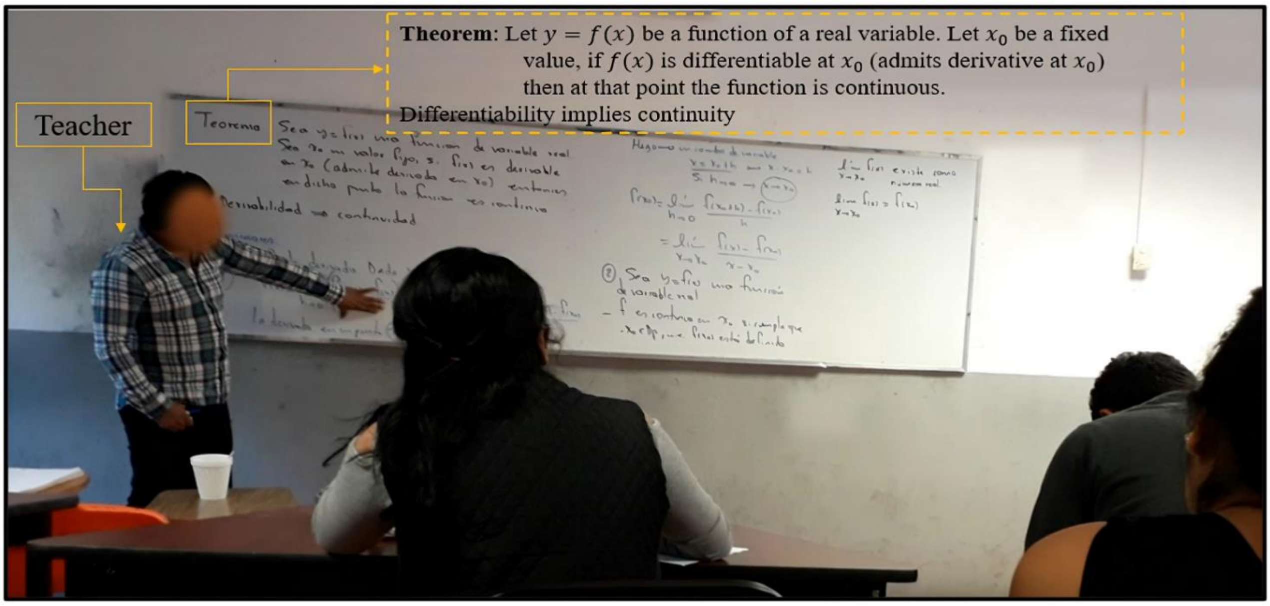

Figure 5. Evidence of the in-service teacher's observation in the class (Source: Authors' own elaboration)

representations and bidirectional implications, procedural, etc., could be further explored.

The path followed by ETC and TMA share similar destinations

For its part, ETC orients its methodological path towards the identification of mathematical connections through thematic analysis that consists of six phases such as the transcription of interviews, identification of codes, recognition of themes, review of themes, naming of themes and reporting of results, which can be inductive (theorizing from the data) and deductive (considering the theory a priori). Similarly, in TMA, the students' arguments are characterized with the help of Toulmin's (2003) argumentative model, a tool that facilitates the reconstruction of the constitutive elements of an argument and with this adapt, according to the theoretical framework to be used, elements that allow analyzing either the content, structure and argumentative functions of the arguments constructed by the students and/or the teacher. The product of conducting discourse analysis under TMA approach allows theorizing and recognizing patterns of argumentation, triggering elements of mathematical discourse on the part of the teacher or the student.

Subsequently, given that the complementarities between ETC and TMA have already been achieved, we proceed to the analysis of an empirical phenomenon that refers to a teacher's proof of the continuity theorem of a function.

Mathematical connections and arguments of a mathematics teacher

Participant and context: An in-service mathematics teacher with more than fifteen years of experience teaching differential calculus in a public university located in the capital of the state of Guerrero, Mexico, voluntarily participated in this research (**Figure 5**).

Data collection: The data were collected by the first author of the research, who requested permission from the participating teacher to observe the classes related to the derivative. The teacher considered it pertinent to voluntarily participate in the work. Subsequently, by means of the participant observation method (Cohen et al., 2018), eight classes on the derivative were observed starting from the formal definition of limit through the four-step rule, to the applications of said concept. For the purposes of this research, an episode on the proof of the continuity of a function was selected as a context of reflection to identify the connections and the argumentative process of the teacher.

Data analysis: This section presents the functionality of the articulation of theories, specifically the identification of mathematical connections and their relationship with the types of arguments.

The analysis of an in-service teacher's class is presented, pointing out that the connections and arguments are identified in the transcripts. For this purpose, a format was designed (**Table 3**) that includes

(1) transcript of the episode,

(2) the mathematical connections, and

(3) the arguments with their schemas.

The argument described is based on mathematical properties, according to what is proposed by TMA, while the analysis of mathematical connections based on ETC evidences the existence of mathematical connections of implication, meaning and instruction-oriented identified in the argument warrant (correspondence code in ETC), which are called micro connections that are in the argumentative core. It should be noted that, the macro connection of implication is the one that relates D1 and C1.

Table 3. Reconstruction of mathematical teacher's connections & argumentation

Transcription

Theorem: Let $y = f(x)$ be a function of real variable. Let x_0 be a fixed value, if $f(x)$ is derivable at x_0 (admits derivative at x_0) (D1) then at that point function is continuous C1 (**Figure 6**).

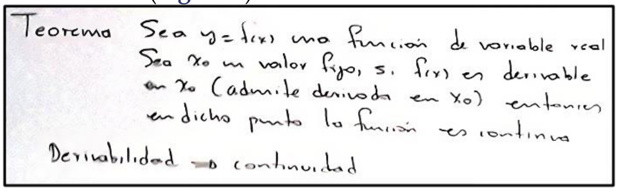

Figure 6. Evidence of the teacher's written production (Source: Authors' own elaboration)

Teacher: If we are then going to study this property, we already talked yesterday about relation of derivative & continuity, now we are going to prove that if a function has a derivative at a point then it is continuous. Thus, derivability implies continuity, but not necessarily other way around, there can be a family of continuous functions not necessarily derivable, but in case we prove that there is derivative then there is continuity. Before doing proof, we put some preliminaries: basis is definition of derivative that we saw last class. Given $y = f(x)$, then derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ (**W1**).

Teacher: We said yesterday that a function is derivable first if its incremental quotient exists & also if that incremental quotient has a limit when increment tends to zero, if that happens function is derivable (**W2**). Now here it means possibility of deriving at any point, of any value, this is going to be our first preliminary (**C1**).

Mathematical connections (MC)

MC1: **Implication connection:** If a function has a derivative at a point, then it is continuous there.

MC2: **Meaning connection** relates concept of derivative at a point (antecedent) to its meaning in terms of limit of average rates of variation of function.

MC3: **Different representations & instruction-oriented connection:** Derivative at a point (x_0) is represented as limit in a symbolic way & supported by instructions from teacher.

MC 4: **Meaning connection:** Definition of derivative in terms of existence of incremental quotient when increment tends to zero (**Figure 7**).

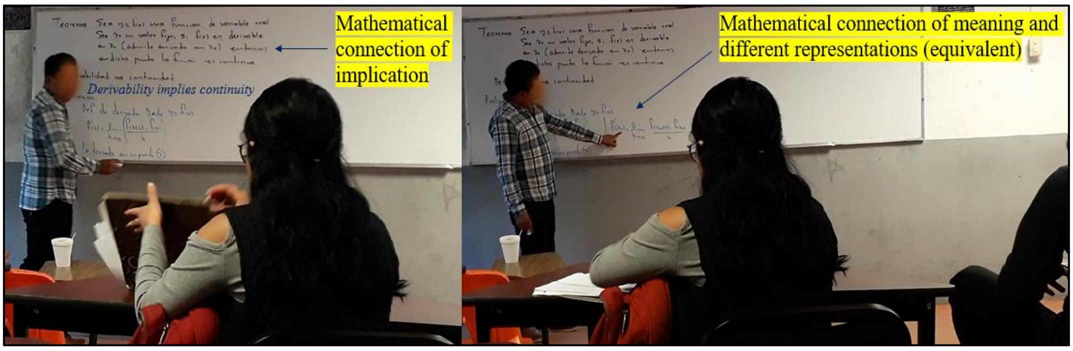

Figure 7. Evidence of mathematical connections in preliminaries (Source: Authors' own elaboration)

Type of argument

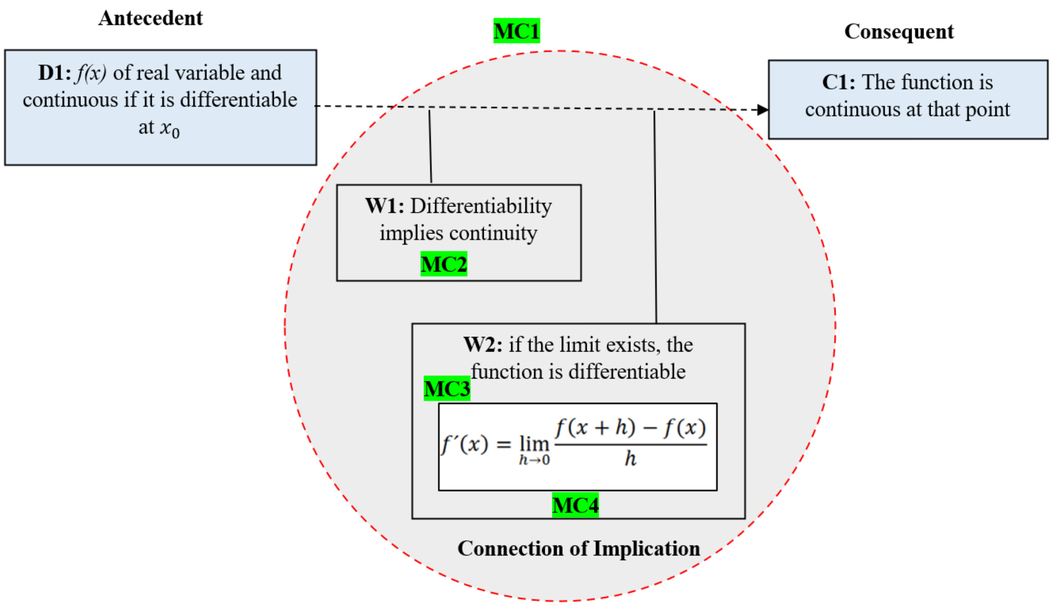

Figure 8. Argument based on mathematical properties (Source: Authors' own elaboration)

Table 4. Reconstruction of teacher's mathematical connections & argumentation

Transcription

In particular, if we are interested in exactly derivative at a point, for example, at x_0, then we will evaluate this in definition of derivative, as follows: $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ (**D2**). At any value, where derivative is assumed to exist, for example, at point 2, derivative of function at value 2 should be expressed in this way (see derivative at a point). Now, in this form, I will make a change of variable to find equivalence to this notation.

Teacher: Let's make a change of variable, let's name it: $x=x_0+h$, clearing $x-x_0=h$ now, h is tending to zero, if this h tends to zero then difference ($x-x_0=h$) what does it tend to? If it tends to zero, being an equivalence, it also tends to zero, but it tends to zero as x approaches this value (x_0) since x is changeable and can be any value and x_0 is a fixed value (**W3**). Then, if h tends to zero, it happens that x will tend to x_0, then I have this new variable & I have dependence of new variable. Thus, derivative at a point $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ to make change of variable I write it like this $\left(\frac{f(x)-f(x_0)}{x-x_0}\right)$. So, it is same, this limit & previous one are equivalent, derivative of a function I can write it like this $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ in classical notation or I can write it this way $\left(\frac{f(x)-f(x_0)}{x-x_0}\right)$ (**C2**). This is going to be our first preliminary before doing proof.

Mathematical connections (MC)

MC5: **Connection of different representations:** Derivative at a point (x_0) is represented as limit symbolically & definition of derivative as a limit is applied.
MC6: **Connection of different representations (equivalent):** $x = x_0 + h$ by clearing $x - x_0 = h$ (**Figure 9**).
MC7: **Procedural connection:** Applying definition of derivative as a limit.
MC8: **Connection of different representations (equivalent):** Presented again in C1.

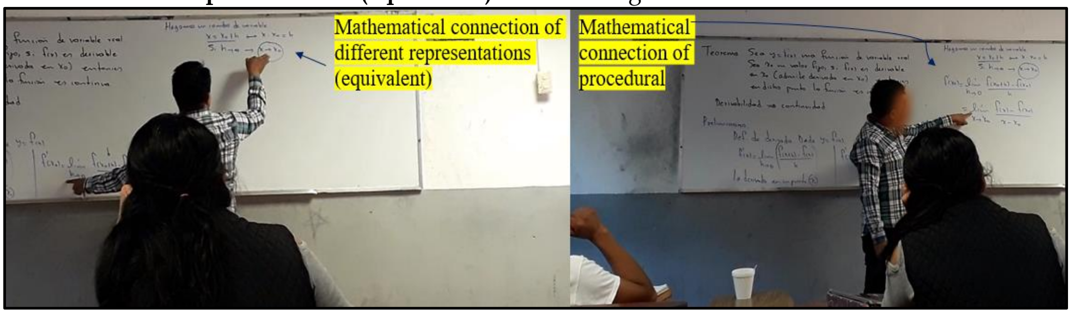

Figure 9. Evidence of connections of representations & procedures (Source: Authors' own elaboration)

Type of argument (**Figure 10**)

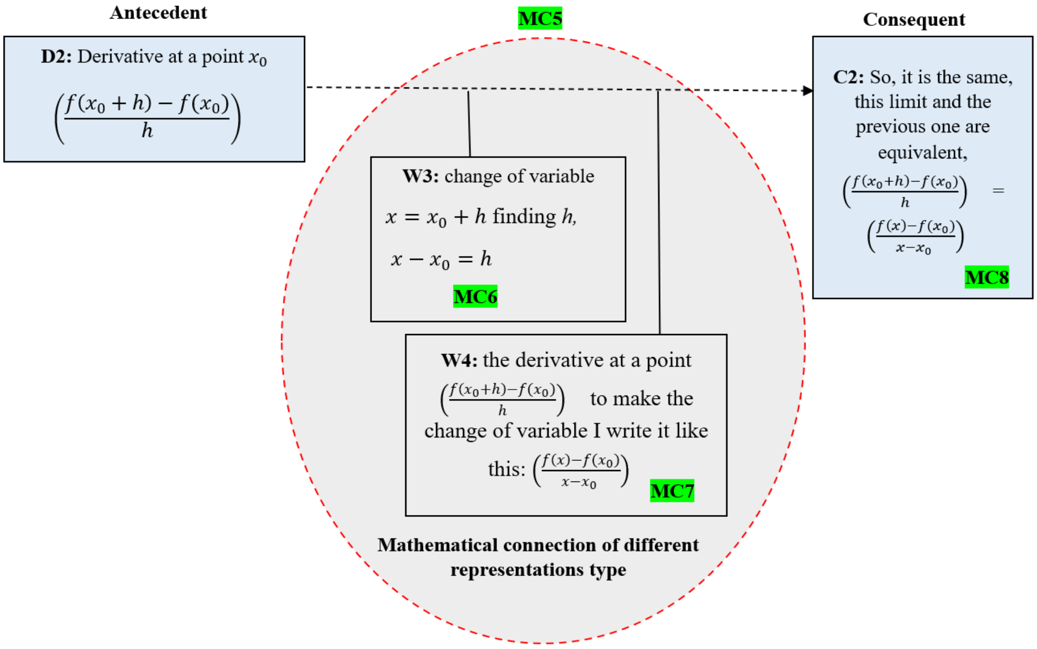

Figure 10. Practical argument (Source: Authors' own elaboration)

Next, the argument constituted by D2, W3, W4, and C2 is a practical type since the conclusion is based on the comparison of characteristic and mathematical properties of the limits (**Table 4**).

As an added value of analysis between TMA and ETC, it can be indicated that mathematical connections are identified in the content of the conclusions, warrants or data of an argument, in addition, each argument has

Table 5. Reconstruction of mathematical teacher's connections & argumentation

Transcription
Teacher: Our second preliminary, since what I am going to relate are concepts of derivability & continuity, I have to remember what it means for a function to be continuous at a point, if you remember? For second preliminary let $y = f(x)$ be a function with real variable, where f is continuous at a point x_0, f is continuous at point x_0 if what happens? When a function is continuous at a point x_0? If point belongs to domain of as a real number & third function ($x_0 \in Df$), that is, $f(x_0)$ is defined, second, that limit of function when x tends to x_0 exists that value of limit coincides with value of function at that point (**D3**). Then for a function to be derivable it has to fulfill these three conditions: (1) that point, where continuity is studied belongs to domain, (2) that limit of function when variable tends to point of interest exists as a real number, & (3) that value of limit coincides with value of function at that point (**W5**). And for there to be a derivative, function must have its incremental quotient & that this incremental quotient has a limit, thus, limit of this incremental quotient is going to be function, derivative at a specific point can be written in any of these two ways: $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ or $\left(\frac{f(x)-f(x_0)}{x-x_0}\right)$ (**C3**). Note that we are reviewing this because it is a strong part, it says hypothesis of theorem is that, if function is derivable at a point, it proves that it is continuous at that point, then this $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ we have to prove (...)
Mathematical connections (MC)
MC9: **Instructional-oriented connection:** It is evidenced when teacher comments to his students that, to study relationship between derivability & continuity, he must activate previous knowledge such as meaning of continuous function. MC10: **Meaning connection:** It was recognized when teacher enunciated meaning of a continuous function as presented in transcript excerpt: "if it fulfills that point belongs to domain of function ($x_0 \in Df$), that is, $f(x_0)$ is defined, second, that limit of function when x tends to (x_0) exists as a real number & third, that value of limit coincides with value of function at that point". MC 11: **Instruction-oriented connection:** It is identified when teacher reveals conditions for a function to be derivable, but in a general way that encompasses & supports connections MC9 & MC10. MC 12: **Connection of different representations (equivalent):** Identified when teacher states, where he/she has to go with his/her demonstration by highlighting that, $\left(\frac{f(x_0+h)-f(x_0)}{h}\right)$ & $\left(\frac{f(x)-f(x_0)}{x-x_0}\right)$ are equivalent symbolic representations.
Type of argument
Especially, this argument is based on meanings of derivability & continuity of a function, thus, its warrant is based on criteria used to affirm that function is continuous (**Figure 11**).

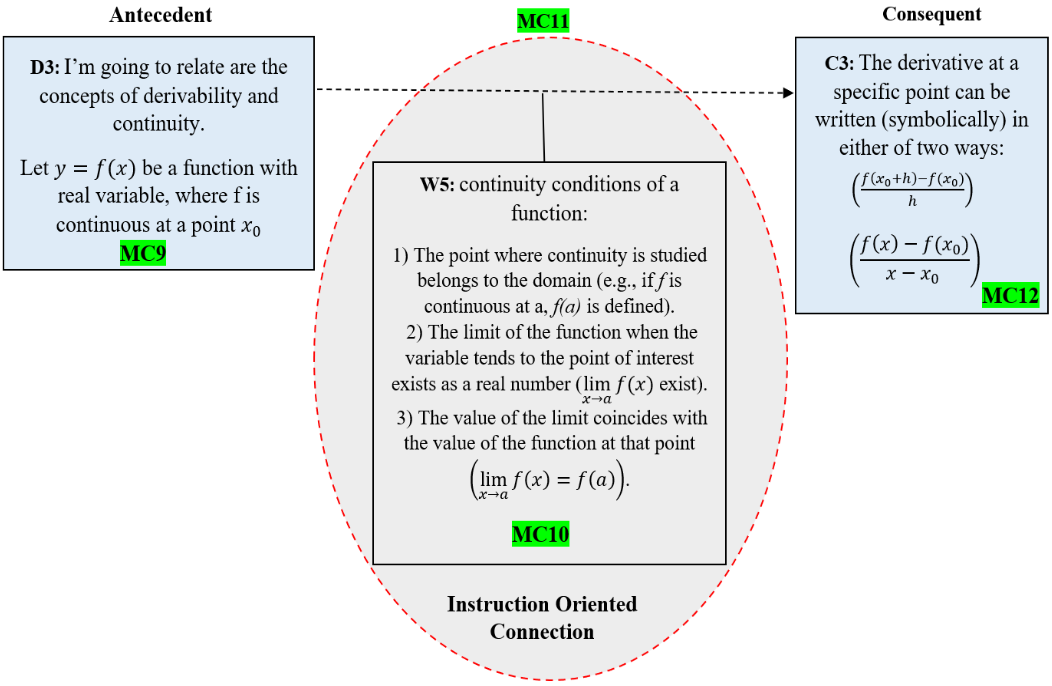

Figure 11. Argument based on meanings & mathematical properties (Source: Authors' own elaboration)

a mathematical connection (macro) that encompasses the other connections (micro) (**Table 5**).

Figure 11 presents an argument based on meanings and mathematical properties simultaneously, since the teacher, at the moment of stating the meaning of the continuous function, also emphasizes each of the properties and/or characteristics of the same concept. In fact, the teacher assumes that the function is continuous when its graph can be drawn without lifting the pencil from the paper, evoking the metaphorical connection that is important for the understanding of the intuitive idea of the continuous function.

Finally, the analysis of the final moment of the class, where she performs the demonstration is presented (**Table 6**) considering the aforementioned preliminaries.

Finally, in **Figure 13** the teacher argued that he proved the third condition of the continuity of the function at a point.

In addition, **Figure 14** shows the final part of the test, where the teacher concludes that f is continuous at x_0. Therefore, if the function is derivable at a point x_0 then it has been proved that continuity exists at that point. In this case, the teacher ended the proof using the mathematical connection of implication.

Local Integration and Theoretical Synthesis

In this research three pairs of theoretical networking strategies have been followed, in this case a local integration between ETC and TMA was achieved highlighting the fundamental role of connections for the activation of arguments and how argumentation is key to establish mathematical connections in the context of the proof and demonstration of a theorem. In this sense, in the analysis of the data, the coherent functionality of the tools of both theories in the analysis of the context of

Table 6. Reconstruction of mathematical teacher's connections & argumentation

Transcription

Teacher: $f(x) - f(x_0) = f(x) - f(x_0)$. We now see difference of something varying with something fixed, I will consider identity (**D4**). Now let's assume that these values of x are variable & not necessarily coincide with other variable are different, so on this side I'm going to multiply & divide by this subtraction, like this: $f(x) - f(x_0)\ (\frac{x-x_0}{x-x_0})$ if we assume that these are different quotient is not zero, it is one, so I'm going to have it like this: $f(x) - f(x_0)$, yes & here I'm going to make an arrangement, so that we are left with: $\frac{f(x)-f(x_0)}{x-x_0}\ (x - x_0) = f(x) - f(x_0)$, then on left side, I have a product that is equivalent to this one here: $f(x) - f(x_0)$ (**W6**). Now let's pass it to limit situation, I'm going to calculate limit on both members like this: $\lim_{x\to x_0} [(\frac{f(x)-(x_0)}{x-x_0})\ (x - x_0)] = \lim_{x\to x_0} (f(x) - f(x_0))$, by limit properties we must distribute the limit to each factor: $(\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0})\ (\lim_{x\to x_0} (x - x_0)) = \lim_{x\to x_0} f(x) - f(x_0)$, Now here this limit who is it? Who is going to be limit of first factor? Limit of first factor, according to preliminary that we study is going to be derivative of function at a point. For this one here by definition of derivative is going to be derivative of function at a point $f'(x_0)$, now here, this is varying & this other one is fixed, so this limit of x_0 when x tends to x_0, is same constant. Thus remaining: $(\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0})\ (\lim_{x\to x_0} (x - x_0)) = \lim_{x\to x_0} f(x) - f(x_0)$ (**W7**). This product is how much? It is equal to zero, I'm going to bring this little arrangement over here, so what I'm going to get is limit when x tends to x_0 of the function: $0 = \lim_{x\to x_0} f(x) - f(x_0)$. $f(x_0) = \lim_{x\to x_0} f(x)$. But this is just third condition to guarantee continuity of function at a point. Thus, f is continuous at a point (**C4**). Thus, f is continuous at (x_0), if function is derivable at a point then it is proved that at that point there is continuity (**C4**).

Mathematical connections (MC)

MC13: **Instructional oriented connection:** Teacher reminds students that they must prove that if a function of derivable at a point then it is continuous & require use of definition of derivative.

MC14: **Procedural connection** was recognized when teacher used identity property for multiplication (**Figure 12**), by teacher construction: $f(x) - f(x_0)\ (\frac{x-x_0}{x-x_0})$.

MC15: **Procedural connection:** It was evidenced when teacher used property of limits to distribute limits of each factor: $(\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0})\ (\lim_{x\to x_0} (x - x_0)) = \lim_{x\to x_0} f(x) - f(x_0)$ (**Figure 12**).

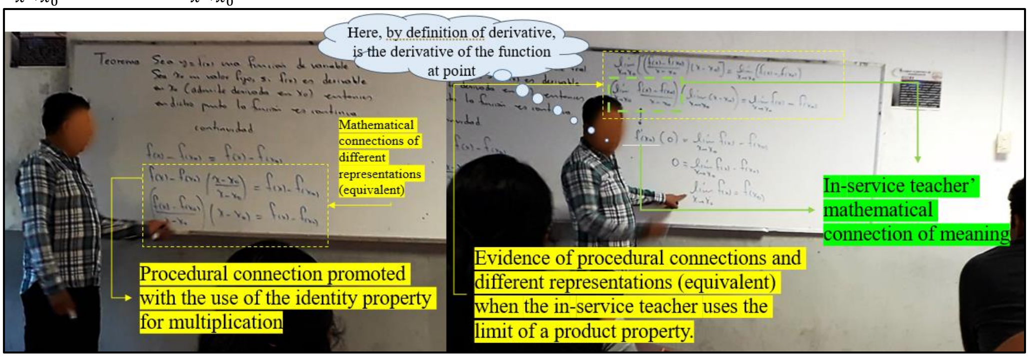

Figure 12. Mathematical connections activated with use of properties & definitions (Source: Authors' own elaboration)

MC16: **Instruction-oriented connection:** In-service teacher manifests to students that they will use preliminary referring to meaning of derivative.

MC17: **Meaning connection**: In-service teacher enunciates meaning of derivative as a limit.

MC18: **Procedural connection:** Teacher applies meaning of derivative to obtain $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$ & that $0 = \lim_{x\to x_0} f(x) - f(x_0)$.

CM19: **Connection of implication:** It was evidenced when teacher proves theorem and states that "if function is derivable at a point then it is proved that at that point there is continuity" (**Figure 12**).

MC20: **Connection of different representations (equivalent):** These types of connections were recognized in procedure used by teacher, for example, when he used identity property (MC14) & at end of demonstration: $0 = \lim_{x\to x_0} f(x) - f(x_0)$, $f(x_0) = \lim_{x\to x_0} f(x)$.

Table 6 (Continued). Reconstruction of mathematical teacher's connections & argumentation

Type of argument (**Figure 13**)

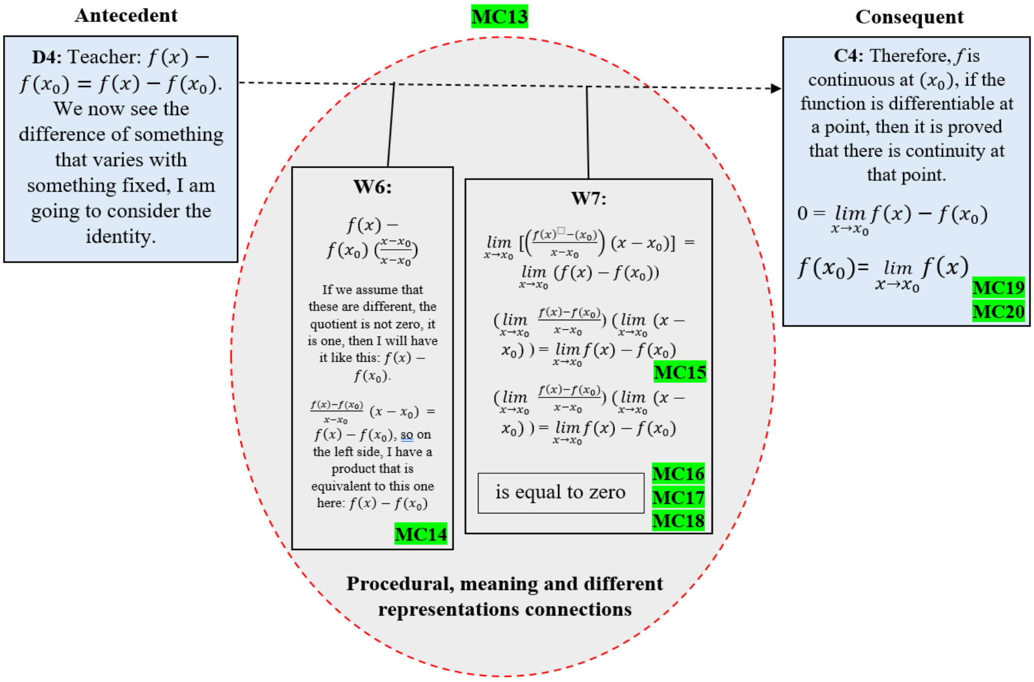

Figure 13. Argument based on different mathematical properties, meanings, & representations (Source: Authors' own elaboration)

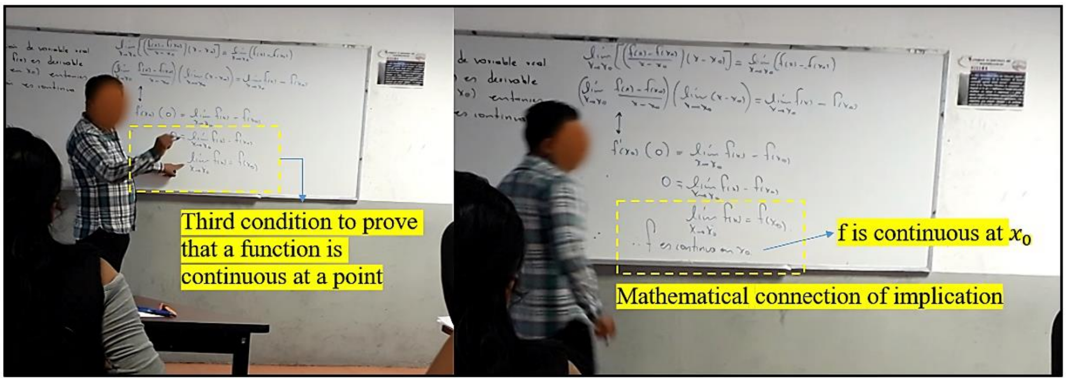

Figure 14. End of the proof on differentiability implies continuity (Source: Authors' own elaboration)

reflection on the derivative was evidenced, but it cannot be generalized with this particular case to affirm that the theories are articulated in their totality but a part of them, for example, the connections detail the functionality of an argument, and the connections are supported from a correspondence code or an argument of TMA. Now, in this articulation, a theoretical synthesis is not considered because there are tools of ETC such as the quality of the connections that were not contemplated in the articulation, as well as the refutation that is a means to promote argumentation in TMA.

FINAL REFLECTION

In relation to the first question, the networking between ETC and TMA was achieved, highlighting the fundamental role that argumentation plays in the constitution of mathematical connections and how the connections are implicit in the structure of an argument. For example, it is recognized that there are connections in the data, warrant, conclusion and backing.

Another important factor achieved with the articulation of ETC with TMA was that the categories of mathematical connections complemented the types of arguments based on mathematical meanings and properties of the concepts (analyzed in the previous sections), which had not been contemplated in the argument categorization proposed in Cervantes-Barraza et al. (2019) and Cervantes-Barraza et al. (2020). In fact, the types of arguments proposed in TMA emphasize the characteristics of mathematical objects, but they could be detailed with the connections of feature type and those of different representations (equivalent and alternate).

This way of analyzing the argumentation and the mathematical connection in an integrated way deepens the analysis of the mathematical activity of a subject, in this case that of the in-service mathematics teacher. However, this analysis can be deepened with the use of other theories such as OSA, in order to assess such mathematical activity, understood in terms of practices, processes/objects and semiotic functions that relate them (Rodríguez-Nieto et al., 2021b). On the other hand, the work of Molina et al. (2019) and Tabach et al. (2020) could be deepened by investigating the relationships between mathematical connections and the types of abductive, deductive and analogical arguments that could emerge from a person's mathematical practices when solving or explaining a problem.

Regarding the answer to the second research question, the combined use of networking between ETC and TMA allowed us to reconstruct the argumentation in terms of mathematical connections and show how these contribute to improving the functioning of the arguments that are structured from the Toulmin's (1984) method. In particular, the structure of the argument is based on a connection between data and conclusion supported by other connections with the warrant and the possible backings.

It should be noted that, with this new networking proposal, not only phenomena related to the proof of the theorem can be analyzed (e.g., if a function is derivable then it is continuous) but also other types of calculus, geometry, statistics, arithmetic, algebra, among others, where the teacher or the student has the possibility of organizing, thinking and proceeding step by step in their mathematical reasoning that describes the procedure to solve problems in detailed terms and mathematical connections supported or justified by an argument. However, in the explanation of the in-service teacher, a specific case of the most relevant and essential connections is shown, starting with the use of the definition of derivative as the limit of the average rates of variation of the function, identifying the connections of implication, meaning and different representations.

Finally, we suggest that this networking of theories can be used to analyze the mathematical practice of students and teachers when explaining and mathematical problems-solving involving other concepts or to further detail the idea of connection and argumentation in technological environments, other subjects, among others. In addition, we found that both theories complement and coordinate each other to make a detailed analysis of a demonstration in the context of argumentation and connections.

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