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Grade 9 Learners' Structural and Operational Conceptions of the Equal Sign: A Case Study of a Secondary School in Soshanguve

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ABSTRACT

This article gives an account of Grade 9 learners' understanding of the concept of the equal sign and how they move from an arithmetic to an algebraic equation. A case study, using a sequential mixed method research design, was conducted in a secondary school in Soshanguve, a township in Gauteng, South Africa. Out of the 49 learners who wrote a test on the concept of the equal sign, eight were selected for an interview. The study revealed that Grade 9 learners in this school interpreted the equal sign as a 'do something' and unidirectional (one-sided) sign, not as the concept that represents an equivalent (concept of keeping both sides of the equal sign equal) of two quantities. The researchers attributed misinterpretation of the equal sign to how learners had been taught the concept of number sentences in lower grades, where greater emphasis was placed on rules than on the meaning of a concept.

Keywords: arithmetic, equal sign, equation, operational understanding, structural understanding

INTRODUCTION

Given its important role in mathematics as well as its role as gatekeeper for future educational and employment opportunities, algebra, in particular the understanding of the concept of an equal sign, has become a focal point of both reform and research efforts in mathematics education. Much thought and research have gone into exploring learners' understanding of the equal sign in elementary and middle schools (Essien & Setati, 2006 Hattikudur & Alibali, 2010; Hohensee, 2017, Jones, Inglis, Gilmore, & Dowens, 2012; Stephens, Knuth, Blanton, Isler, Gardiner, & Marum, 2013). Linchevski and Herscovics (1996) argue that teachers and learners should be made aware of the demarcation between arithmetic and algebra. In other words, teachers should be able to understand the gap between arithmetic and algebra to enhance learners' readiness to learn secondary school mathematics. The gap between arithmetic and algebra may explain learners' lack of readiness and the poor results achieved in algebra (Linchevski & Herscovics, 1996).

The difficulties that secondary school learners experience in learning algebra may involve the misinterpretation of the equal sign in given expressions or equations. Unfortunately, most mathematics teachers and textbook authors are not aware that learners misinterpret the equal sign. *Common core for mathematical practice,* which was recently released in the United States by the National Governors Association (2010), asserts that learners who passed Grade 7 are expected to master the content and skills to be well prepared for Grade 8 algebra.

The Notion of the Equal Sign

Most research has documented that many elementary and middle school learners demonstrate inadequate understanding of the meaning of the equal sign, frequently viewing the symbol as an announcement of the result of an arithmetic operation rather than as a symbol of mathematical equivalence (Hohensee, 2017; Jones et al., 2012). Jones et al. (2012) argue that learners should learn the meaning of the equal sign as being an equivalence relationship between two objects, such as numbers or expressions. In this instance, the notion of the equal sign is correlated with an arithmetic competence, and assists learners to master algebra and other mathematical concepts

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Contribution of this paper to the literature

- The study intended to understand how learners move from arithmetic to algebra when using the equal sign in arithmetic (9+6=■+5) and in algebra (9+6=x+5). It was anticipated that the interpretation of the algebraic equation would reveal the learners' lack of understanding of the equal sign. No prior study, which focused on how learners move from arithmetic to an algebraic equation, has been conducted in South Africa, except those that focused on learners' understanding and interpretation of the equal sign.
- The teaching and learning of arithmetic and algebraic equations should not be dichotomised. Although the two mathematical concepts appear to be incompatible, they are in fact complementary.

(Matthews, Rittle-Johnson, Taylor, & McEldon, 2012). Jones et al. (2012); Li, Ding, Capraro, and Capraro (2008) and Matthews et al. (2012) postulate that learners should have a flexible understanding of the equal sign for arithmetic competence and algebra.

Essien and Setati (2006) concur that the equal sign is used to indicate the relationship between quantities or values in mathematics. In other words, learners should view the equal sign as a relational symbol to compare numbers or expressions in order to solve mathematical problems effectively. However, learners view the equal sign as an operation: they interpret it as the total of adding two quantities or getting the answer to a particular mathematical problem (Rittle-Johnson & Alibali, 1999). Essen and Setati (2006) explored Grade 8 and 9 learners' understanding of the equal sign in the South African context and found that the learners viewed the equal sign as a 'do-something' or a unidirectional sign. The researchers found that Grade 8 and 9 learners viewed the equal sign as a tool to compute the answer rather than as a relational symbol to compare the quantities. Essien (2009) further argues that the equal sign is introduced operationally, which mainly focuses on addition and subtraction of numbers. The National Council of Teachers of Mathematics (2000) suggests that learners should learn about the equal sign in the early grades as it is regarded as an algebraic concept. This means that learners should be able to interpret the equal sign with understanding, instead of having narrow knowledge that creates difficulties in understanding.

Although an operational understanding of the equal sign can be sufficient for solving standard equations, it contributes to learners' ability to solve more complex problems (Hattikudur & Alibali, 2010). Byrd, McNeil, Chesney, and Matthews (2015) found that even late elementary learners still view the equal sign as operational instead of an indication of the relationship between quantities. Hattikudur and Alibali (2010) suggest that learners who hold an operational understanding encounter difficulties in the transition to algebra. Furthermore, learners who hold operational understandings perform poorly compared to those who have a relational understanding (Knuth, Stephens, McNeil, & Alibali, 2006). Grobman and Alibali (2007) argue that most learners who hold an operational understanding learn less from lessons involving linear equations than lesson involving quadratic equation. Studies conducted in Western countries also revealed that learners understand the equal sign not as relational, but rather as the operational meaning of expressions or numbers, to work out the answer (Knuth et al., 2006; Powell, 2012).

Sources of Learners' Operational Understanding

The equal sign has been defined and introduced differently in textbooks around the world (Jones et al., 2012). Studies conducted in the United States of America (USA) and China on teachers' guidebooks or textbooks were compared and it was found that, in the United States, the equal sign is rarely defined and is often used interchangeably with computations to obtain the answer (Li et al., 2008). The researchers revealed that the textbooks showed arithmetic equations as canonical, which is expression = answer. The textbooks contained little information for teachers to explain the equal sign to learners when teaching. Similarly, textbook explanations of the equal sign were studied in South Africa, where Essien (2009) found that textbooks introduced the equal sign operationally using addition and subtraction of numbers or expressions.

Textbook explanations of the equal sign in China differ from those in the United States and South Africa (Li et al., 2008). Textbooks in China define the equal sign in the first year of schooling, and use a variety of ways to do this, such as symbolic, verbal and pictorial contexts. In other words, the equal sign is introduced only after learners have grasped its meaning and other relational symbols, such as less than (<) and greater than (>). Learners' overall performance in China was found to be 98% when solving equations and they justified their answers correctly, while the performance of learners in the United States was only 28%. McNeil (2008); McNeil and Alibali (2005a) argue that the operational view of learners' understanding of the equal sign was reinforced by mathematics textbooks throughout the early years of schooling and learners may be resistant to change.

Moreover, the teaching approaches used during teaching and learning differ when teachers teach the equal sign as operation equals the answer (McNeil, 2008; McNeil & Alibali, 2005b). Hattikudur and Alibali (2010) found that the instructional approaches used in the United States could not help learners develop a relational understanding of the equal sign. These types of approaches hinder learners' understanding of the equal sign as they promote an operational understanding of the equal sign (Carpenter, Franke, & Levi, 2003; McNeil, 2007). Hattikudur and Alibali (2010) postulate that learners who are taught the equal sign as operation equals answer may foster the incorrect view of the equal sign as an operational symbol rather than a relational one. Li et al. (2008) argue that learners are rarely taught to interpret the meaning of the equal sign.

The purpose of this study was to replicate other studies on learners' interpretation of the equal sign, and was done in a Grade 9 mathematics class in a South African context in a selected school in Soshanguve. Furthermore, the study also intended to understand how learners transit from arithmetic to algebra when using the equal sign in arithmetic ($9 + 6 = \blacksquare + 5$) and in algebra ($9 + 6 = x + 5$). It was anticipated that the interpretation of the algebraic equation would reveal the learners' lack of understanding of the equal sign. No prior study, which focused on how learners move from arithmetic to an algebraic equation, has been conducted in South Africa, except those that focused on learners' understanding and interpretation of the equal sign (Essien & Setati, 2006). Hence, the presents study was based on previous international research to explore the interpretation of the equal sign and the transition of arithmetic to algebra (Hattikudur & Alibali, 2010; Jones et al., 2012) in a South African context.

RATIONALE OF THE STUDY

The study could play an important role in informing Grade 9 mathematics teachers about the importance of defining and describing the equal sign to learners when teaching mathematics. The teachers and researchers in mathematics education would obtain insight into how Grade 9 learners understand and interpret the equal sign, which could contribute to learners' understanding of other sections of mathematics. Grade 9 learners may realise the importance of understanding and correctly interpreting the equal sign in order to solve both arithmetic and algebraic problems. The study intended to find answers to the following research questions.

- How do Grade 9 learners understand and interpret the equal sign?
- What is the transition from arithmetic to algebra in relation to the equal sign?

In this article, I argue that the teaching and learning of arithmetic and algebraic equations should not be dichotomised. Although the two mathematical concepts appear to be incompatible, they are in fact complementary. To develop this argument, I drew from Sfard's (1991) theoretical framework on procedural and structural conceptions, which are regarded as relevant to understand learners' conception of the equal sign and the transition from arithmetic to an algebraic equation.

THEORETICAL FRAMEWORK

Sfard's (1991) theoretical framework of procedural and structural conceptions is useful in explaining how learners understand and interpret the notion of the equal sign, and how learners transit from an arithmetic identity to an algebraic equation. Sfard (1991, p. 1) indicates that any analysis of different mathematical definitions and representations brings us to the conclusion that abstract notions, such as the equal sign, a number or a function, can be conceived in two fundamentally different ways: structurally as an object and operationally as processes. She argues that the two approaches, although ostensibly incompatible, are in fact complementary. In some instances, especially in textbooks, a concept could be defined as if the mathematical notion referred to is an object. This is a structural definition. Seeing a mathematical concept as an object means being capable of referring to it as if it were a real thing – a static structure, existing somewhere in space and time, recognising an idea at a glance and manipulating it as whole, without going into detail (Sfard, 1991). However, the same concept can also be defined in terms of its computational process, algorithms and actions rather than objects. This reflects an operational conception of a notion. The operational conception is dynamic, sequential and detailed (Sfard, 1991).

In this article, I argue that the notion of the equal sign can be interpreted structurally (as a static relationship between two magnitudes) as well as operationally (as a concise description of some computation) (Kieran, 1981). It can be interpreted procedurally if it is viewed by learners as a symbol that requires them to provide an answer, and it can also be viewed by learners as a symbol that represents an equivalent of two quantities, that is, the right-hand side (RHS) = the left-hand side (LHS). In other words, it can be regarded as a symbol of identity or as a command for executing the operation appearing on the right-hand side. Hence, in this study, I was interested in investigating how learners interpret the notion of the equal sign. Sfard (1991) provides a summary of the operational and structural conceptions of any notion in mathematics as reflected in **Table 1**.

Table 1. Summary of operational and structural conceptions

	Operational conception	Structural conception
General characteristics	A mathematical entity is conceived as a product of a certain process or is identified with the process itself	A mathematical entity is conceived as a static structure – as if it were a real object
Internal representation	Is supported by verbal representation	Is supported by visual imagery
Its place in concept development	Develops in first stages of concept formation	Evolves from the operation conception
Its role in the cognitive process	Is necessary, but not sufficient, for effective problem solving and learning	Facilitates all cognitive processes (e.g. learning, problem solving)

Source: Adapted from Sfard (1991)

METHODOLOGY

This study was drawn from a bigger project, the Mathematics Teaching and Learning Intervention Programme (MTLIP), housed in the Department of Mathematics Education at the University of South Africa. The programme focuses on two provinces: Limpopo and Gauteng. Twenty secondary schools in each province were chosen based on their performance in the Annual National Assessment (Department of Basic Education, 2013). The programme is divided into two phases: the administration of a pre-diagnostic test in the schools to identify mathematical concepts in which learners did not perform well, and, after the results had been analysed, the training of teachers in the schools on the problematic mathematical concepts identified. This article reports on the first data collected during the administration of the pre-diagnostic test in one of the schools participating in the MTLIP in Soshanguve, Gauteng.

The present study followed an explanatory sequential mixed method design to gather both quantitative and qualitative data (Creswell, 2014). In the first phase of this study, the researchers collected and analysed quantitative data. In the second phase, qualitative data were collected as a follow-up to get an in-depth understanding of the quantitative results. The data that are the focus of this article consisted of learners' responses to items from a written assessment test that targeted their understandings of the equal sign and the transition from arithmetic to algebra. Firstly, a pre-diagnostic test comprising 13 question items was administered to 49 Grade 9 learners in the selected secondary school in Soshanguve. The test was aimed at assessing learners' understanding and interpretation of the equal sign and the relationship between arithmetic and algebraic equations. Thereafter, a purposive sample of eight Grade 9 learners participated in semi-structured interviews to explore their in-depth understanding of the equal sign and how they can move from arithmetic to algebra.

Data Analysis

The quantitative data collected from the diagnostic test and the qualitative data collected from learners' semi-structured interviews were analysed individually before being triangulated. The data collected from the diagnostic test answers were analysed using the following categories: correct and incorrect responses (Didis & Erbas, 2015). The incorrect responses were regarded as wrong answers that could have been influenced by the misinterpretation of the equal sign (see **Annexure A**). Themes and codes were derived from the qualitative data informed by the theoretical framework that underpinned the study and the literature review.

There were three alternate forms of questions, which varied only in specific numbers or the problem format used in some of the items (i.e. arithmetic format versus symbolic format):

- question 1 (arithmetic identity) required learners to interpret the equal sign and provide an answer;
- question 2 (algebraic equation) required learners to determine the solution to an algebraic equation; and
- question 3 (mathematical statements) required learners to interpret the equal sign by stating whether the number sentence was true or false (see **Figure 1**).

Question 1

1 Write the missing number in each square box below

1.1 $[\ \ ] = 8 + 4$

1.2 $4 + 5 = [\ \ ] - 1$

1.3 $[\ \ ] + 5 = 5 + 8$

1.4 $3 \times 7 = [\ \ ] \times 3$

Question 2

2. Solve for *n* in the following equations.

2.1 $7 + n = 6 + 9$

2.2 $126 - 37 = n - 40$

2.3 $4800 \div 25 = n \times 48$

2.4 $4n + 4 = 4n + 1$

2.5 $n + 15 = 4n$

Question 3

3 Are the mathematical statements below correct? If yes, say why. If no, say why you think it is not.

3.1 $7 = 5 + 2$

Reasons:

3.2 $4 + 5 = 9 + 1$

Reasons:

3. 3 $674 - 389 = 664 - 379$

Reasons:

3.4 $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$

Reasons:

3.5 $64 \div 14 = 32 \div 28$

Reasons:

Collection Instrument: Grade 9 test

Figure 1. A written test on the concept of Equal sign

For all three questions, learners' responses were coded as 'correct' or 'incorrect'. A response was coded as correct if a learner gave a correct response, and as incorrect if a learner provided an incorrect response. Given the fact that a learner could write a correct answer, but have an incorrect interpretation and understanding of an equal sign, both the incorrect and correct responses were followed up during the interviews. Furthermore, the incorrect responses were further analysed in terms of recurring patterns of those responses and the number of learners who gave a particular response. **Annexure A** further provides some reasons, which were corroborated with learners in the interviews regarding how they might have provided those incorrect responses (**Annexure A**).

ETHICAL ISSUES

Ethical clearance for the study was obtained from the University of South Africa. Furthermore, a permission letter to visit the school was obtained from the Department of Basic Education. A consent form was signed by the learners who participated in the study as well as by the researchers. The purpose of this study was explained to the Department of Basic Education officials, the principal, the mathematics teacher and the learners, and the participants' roles, rights and voluntary participation were explained to all concerned (Tashakkori & Teddlie, 2010). The participants were assured that the collected data would be treated confidentially and that no punitive measures would be taken against them should they decide to withdraw. Furthermore, the participants were assured that no personal information would be disclosed to anyone who was not part of this study. The collected data were kept in a safe place in a secure office on a password-locked computer. Below are the test items administered to Grade 9 learners. These test items were adapted from Van de Walle, Karp, and Bay-Williams (2014).

DATA COLLECTION INSTRUMENT: GRADE 9 TEST

FINDINGS AND DISCUSSION

The table below shows the learners' performance in each of the questions (Q) and the numbers and percentages of correct (C) and incorrect (IC) answers, and the analysis of the incorrect responses answers given by all 49 learners who wrote the test.

A brief analysis of how learners performed per question is provided below. However, further analysis on how the themes emerged from the data is captured in **Annexure A**.

Table 2. Analysis of the results

Questions(Q)	Correct (C)	%	Incorrect(IC)	%	Analysis of incorrect responses
Question 1 : Arithmetic identity / equation					
1.1	47	96%	2	4%	Answer 10 (1) Answer 1 (1)
1.2	28	57%	21	43%	Answer 9 (10) Answer 8 (9) Answer 5 (1) Answer 1 (1)
1.3	38	78%	11	22%	Answer 13 (6) Answer 0 (2) Answer 18 (2) Answer -5 (1)
1.4	35	71%	14	29%	Answer 21 (11) Answer 63 (2) Answer 28 (1)
Question 2 : Algebraic equation					
2.1	12	24%	37	76%	N= 22 or 22n (14) N= 15 or 15n (5) N= 16 or 16n (4) N= other (14)
2.2	8	16%	41	84%	N= 49 (20) N= 89 (5) N= other (24)
2.3	8	16%	41	84%	N= 192 x 48 (20) N= n x 48 (3) N= other (26)
2.4	0	0%	49	100%	N= 13 or 13n (16) N= 3 or 3n (7) N= 9 or 9n (4) N = other (22)
2.5	5	10%	44	90%	N= 19 (15) N= 4 or 4n (12) N= 15 (3) N= others (14)
Question 3 : True and False arithmetic identity / equation					
3.1	48	98%	1	2%	No, you cannot start with an answer, then follow with a question
3.2	39	80%	10	20%	Yes, because 4 + 5 is equal to 9
3.3	23	47%	26	53%	No, the answer must be 254
3.4	2	4%	47	96%	Yes, because I add 1 plus 1 is 2 and 2 plus 3 is 5

ANALYSIS OF LEARNERS' RESPONSES PER QUESTION

Question 1

Table 2 indicates that learners performed better in questions 1 and 3, which were about an arithmetic equation, than in question 2, which was about an algebraic equation. Furthermore, some Grade 9 learners still had difficulty in finding an unknown represented by a box in an arithmetic equation. For example, 34% out of 43% of learners who had the answer incorrect indicated that the answer to question 1.2 was 9 or 8. These learners did not interpret the equal sign correctly. Similarly, 12% out of 22% of learners who had the answer incorrect indicated that the answer to question 1.2 was 13. Again, 22% out of 29% of learners who had the answer incorrect indicated that the answer to question 1.2 was 21. They interpreted the equal sign as a 'provide, give an answer' verb instead of as a symbol that represents an equivalent of two expressions, that is, the RHS = LHS. This idea was confirmed by learners' interviews as indicated below. Please note that all responses are reproduced verbatim and unedited,

Question 2

As indicated in the preceding paragraph and Table 2, most learners did not perform well in the algebraic equation, as compared to the arithmetic equation. In question 2.1, 76% of the learners gave an incorrect answer. Learners provided a range of answers to the question, but the most prevailing answer was either 22 or 22n (14 learners out of 49). Five learners indicated that the answer to question 2.1 was 15, while four learners indicated that the answer was either 16 or 16n, and another 14 learners gave incorrect answers that could not be coded. Of the learners, 84% had question 2.2 incorrect. Of the 41 learners who had question 2.2 incorrect, 20 indicated that their answer was 49, while five indicated that their answer was 89. The responses of 24 of the learners on question 2.2 could not be coded. In question 2.3, 84% of the learners gave the incorrect answer. Twenty learners indicated that the answer of 192 x 48 = 9 216 was correct, three learners indicated that the answer was n x 48, and the answers of 26 learners were not coded because there were inconsistences compared to others. Surprisingly, no learner noticed that there was no solution to question 2.4. Instead, learners just added numbers that were in the question, irrespective of whether they were like terms or not. For example, through just adding numbers, most learners (16) indicated that the answer was either 13 or 13n. Similarly, 90% of the learners had question 2.5 incorrect, because they just added numbers in the question without considering whether they were like terms or not.

Question 3

Table 2 indicates that learners performed well in all questions except in questions 3.3 and 3.4. Most learners (53%) indicated that the answer was "No" because the answer must be 254. These learners just subtracted 389 from 674 to arrive at 254, ignoring the LHS numbers of the equation. One learner indicated in question 3.1 that the answer was "No", because one cannot start with an answer and follow with a question. Although question 3.4 was a fraction, it was intended to check whether learners could interpret the concept of the equal sign. Surprisingly, only two out of 49 learners had this question correct. Most learners (96%) indicated that the answer was "Yes, because, I add 1 plus 1 is 2 and 2 plus 3 is 5."

INTERVIEW FINDINGS

Table 2 shows how learners performed in the given tasks; however, it does not describe how learners were thinking responding to the questions. As a result, learners were interviewed to access their thinking based on the responses provided. Four themes emerged from the analysis of the interviews:

- articulation of the equal sign as 'equas to' instead of 'equal to';
- the cognitive gap between the arithmetic equation and the algebraic equation;
- the interpretation of the equal sign as 'give me an answer, do something' as opposed to equivalence; and
- just adding, subtracting, multiplying and dividing unlike terms.

The four themes are discussed below.

Articulation of the Equal Sign as 'Equas to' Instead of 'Equal to'

Participating learners did not know how to articulate the equal sign. They articulated it as 'equas to' instead of 'equal to'. If one cannot articulate a concept, it is unlikely that you would grasp its meaning. This is apparent in the excerpt below.

Researcher: Would you please explain to us how you got your answer for 1.2 as 9?

L1: The answer is 4 + 5 is equas to 9.

Researcher: How do you understand this sign = ? (The researcher referring to the equal sign)

L1: Equas to.

Researcher: What did you say? Equas to or equals to?

L: I said equas to.

Although the interviewer could not trace the origin of this incorrect articulation of the 'equas' sign, it appeared that this erroneous articulation had been sustained in class without correction, because it was not evident in only learner 1, but also in other learners. Learners 7 and 8 as indicated below also indicated the incorrect articulation of the equal sign.

L7: I did the same because 3 x 7 is equas to 21.

L4: I said 4n + 4 is equas 8n and added to 4n and 1 and gave 13n.

L 8: The question said 4 + 5 and a box and -1, so 4 + 5 is equas to 9 and 9 minus one is equas to 8.

When asked about the meaning of the equal sign, L1 said, "No I don't know what this sign means."

L4: Yes, sir, it says I must give an answer of 126 – 37, which is 89 and 89 – 40 is 49.

These learners did not know the meaning of the equal sign. Some have interpreted the equal sign as a 'do-something' sign rather than as a symbol indicating equivalence (Essien & Setati, 2006; Hattikudur & Alibali, 2010; Stephens et al., 2013). The operational conception of the equal sign can be attributed to incorrect articulation.

The Cognitive Gap between an Arithmetic and Algebraic Equation

Table 2 indicates that learners performed better in questions 1 and 3, which were arithmetic equations, than in question 2 which was an algebraic equation. When learners were asked about the relationship between the two questions, most did not articulate any relationship.

Researcher: What is the relationship between questions 1 and 2? Do you see any similarities or commonalities?

L3: No, sir, these questions are not the same. Question 1 has a box and question 2 has an n.

Researcher: Question 1 has a box and question 2 has an n? What do these symbols represent?

L6: A box means I must give an answer and n means I must solve for n.

Researcher: So, there is no relationship between these two symbols?

L3: No, sir, one is a box and the other one is an alphabet.

L6: I can't see any similarities between questions 1 and 2.

It was clear from the above extract that L3 and L6 did not see any similarities between questions 1 and 2. In fact, L3 saw differences rather than similarities: question 1 with a box and question 2 with the alphabet letter n. When asked about the representation of these symbols, L6 indicated, "[a] box means I must give an answer and n means I must solve for n". These learners do not see the box and the n as representing a number. This was further evidenced by their responses in question 2. They both responded in questions 2.1 and 2.3 as n= 22n and n= 13n, respectively. This was further supported by the learners' performance as indicated in **Table 2**. Most learners performed better in questions 1 and 3, which were about the arithmetic equation, than in question 2, which was about the algebraic equation. The difference in performance between questions 1 and 2 suggests a cognitive gap between arithmetic and algebra. Learners could not recognise algebra as a generalised arithmetic (Braithwaite, Goldstone, Van der Maas, & Landy, 2016; Linchevski & Herscovics, 1996)

The Interpretation of the Equal Sign

In questions 1.2, 1.3 and 1.4, most learners indicated that the answers to these questions were 9, 13 and 21, respectively. The interview showed that some learners interpreted the equal sign as 'give an answer or do something', but not as a symbol that represents an equivalent of two expressions.

Researcher: Okay, what does the equal sign mean? What does it tell you?

L5: It tells me to give an answer for 4 plus 5, which is 9.

Researcher: Okay, but there is -1 on the right-hand side of this equation, where did you take it to?

L5: No, I did not use it because it is not part of an answer.

The remark above by L5 indicates that the equal sign means 'give an answer', for example '4 + 5 is 9'. When L5 was asked about the negative 1 on the right-hand side of an equation, he said, "I did not use it because it is not part of an answer." This implies that this learner only considered numbers that were behind the equal sign and the rest were ignored. A similar pattern was also noticed in L7's responses below.

Researcher: In 1.3, you gave your answer as 13. Can you explain how you got it?

L7: It is because I added 5 and 8 and it gave me 13.

Researcher: Okay, what about this other 5? (referring to 5 on the left-hand side of the equation)

L7: I did not use it because I want an answer of 5 plus 8.

Researcher: What about 1.4?

L7: I did the same because 3 x 7 equals to 21.

When asked how he did question 1.4, L 4 said, "I multiplied 7 and 3 and it gave me 21." This is also an indication of learners interpreting the equal sign as a 'do something' sign irrespective of the operation with which they are working. This was confirmed by L2 who said, "I did not use it because the question says 3 x 7, so 3 x 7 is 21." Learners in the above excerpt understood the equal sign as a 'give the answer' signal. They had an operational conception of the equal sign rather than a structural conception. Herscovics and Kieran (1980) argue that students view the equal sign as a 'do something' signal rather than as a tool that means 'same as', that is, RHS = LHS. Essien and Setati (2006) confirm that Grade 8 and 9 learners see the equal sign as a tool for writing the answer rather than as a relational symbol to compare quantities. They regard the equal sign as a 'command' for executing the operation appearing on the right-hand side (Sfard, 1991). Many researchers (Bush & Karp, 2013; Essien, 2009; Essien & Setati, 2006; Hattikudur & Alibali, 2010; Jones et al., 2012; Stephens et al., 2013) have explored the interpretation of the equal sign as an invitation to an answer.

In question 1.3, learners 5 and 8 just added and subtracted numbers without any understanding of the equal sign. They used the numbers and the operation sign at their disposal. This is confirmed in the following remarks below by L5 and L8.

Researcher: In question 2.2, you gave your answer as 8. Can you explain why?

L5: I said 4 + 5 is 9 and 9 – 1 is 8.

L 8: The question said 4 + 5 and a box and -1, so 4 + 5 is equal to 9 and 9 minus one is equals to 8.

Adding, Subtracting, Multiplying and Dividing Unlike Terms

Table 2 indicates that learners did not perform well in question 2, which was about an algebraic equation as opposed to question 1, which was about an arithmetic equation. Learners could not make a connection between the arithmetic and the algebraic equation. They could not see that an unknown in an arithmetic equation, which is represented by a box, and an unknown in an algebraic equation, which is represented by a letter, mean the same thing. Thus, learners could not make sense of a variable. When asked to solve n in question 2, some learners just added the numbers that are in the equation. When asked about how she did question 2.1, L3 said, "I have added 6 + 9 and it gave me 15, and added 7 to get 22n." L4 did the same by subtracting numbers that were in the equation, because there was a minus operation sign in the equation. When asked about how she worked out question 2.2, L4 said, "126 – 37 is equal to 89, and 89 – 40 gave me 49."

The excerpt below is evidence that the usage of variables and numbers was not limited to addition and subtraction signs, but also extended to multiplication and division operation signs. This finding is consistent with findings by other researchers (e.g. Osana, Cooperman, Adrien, Rayner, Bisanz, Watchorn, & Sherman LeVos, 2012; Prediger, 2010; Capraro, Capraro, Yetkiner, Özel, Kim, & Küçük 2010) namely that learners' conception of a variable is inadequate. Most learners' misunderstandings included viewing variables as abbreviations or labels rather than as letters that stand for quantities, assigning values to letters based on their positions in the alphabet, and being unable to operate with algebraic letters as varying quantities rather than specific values (Asquith, Stephens, Knuth, & Alibali, 2007)

Researcher: You provided your answer in 2.3 as 91216. Can you tell us how you went about this problem?

L5: I said how many times does 25 get into 4 800 and it gave me 192, and I times 192 by 48 and it gave me 91 216.

L6: I divided 4 800 by 25 to get 192 and multiplied 192 by 48 to get 91 216.

Researcher: What about question 2.4?

L4: I said 4n + 4 is equals 8n and added to 4n and 1 and gave 13n.

L6: I added 4n + 4n to get 8n, and 4 + 1 gave me 5, so 8n plus 5 gave me 13.

Researcher: What about 2.5?

L1: I have added 15 + 4 and it gave me 19n.

Researcher: Why 19n?

L1: Because an answer must have an n.

The above excerpt is an indication that the participating learners did not realise that a variable or a letter represents any number that can make an equation true. Learners who provided a solution without a letter or a variable, for example n = 22, did not even bother to test the equation by substituting the claimed value to the equation to see if it made sense. Although the researcher did not ask learners what the equation meant, it was clear that learners did not know. Sfard (1991) argues that the equal sign is usually interpreted as requiring some action rather than signifying the equivalence between two expressions, leading to the technical blunder that$x + 8 = 8x$, as some of the learners proposed in the above excerpt. Sfard (1991) further argues that individuals must move from an operational or computational orientation, that is, for example, seeing $x + 8$ as the process whereby 8 is added to the number x, to a structural orientation, as seeing $x + 8$ as an object, a 'whole', a 'thing', an 'answer'.

CONCLUSION

This article dealt with Grade 9 learners' understanding and interpretation of the concept of the equal sign and how they move from an arithmetic equation to an algebraic equation. I argue that the teaching and learning of arithmetic and algebraic equations should not be dichotomised. Although the two mathematical concepts appear to be incompatible, they are in fact complementary. Four themes emerged from this study. Firstly, the study revealed that the participating Grade 9 learners interpreted the concept of the equal sign as a 'do-something' and as a unidirectional (one-sided) sign, and not as a concept that represents an equivalent of two quantities (concept of keeping both sides of the equal sign equal). The researchers attributed an incorrect interpretation of the equal sign to how learners have been taught the concept of number sentences in lower grades, putting emphasis on rules rather than on the meaning of a concept. Most learners did not see the equal sign as a symbol of identity, but as a 'command' for executing the operation appearing on its right-hand side. Their interpretation of the equal sign was an operational–computational process, with algorithms and actions rather than interpreted it structurally as a static relationship between two magnitudes.

Secondly, there was a cognitive gap between the arithmetic and algebraic equation. Therefore, it is recommended that, in order to bridge the gap, the concept of a variable should be introduced in the early grades.

Thirdly, most learners articulated the equal sign as 'equas to'. This encouraged them to develop a procedural understanding rather than a conceptual understanding of the concept.

Fourthly, the misunderstanding of the equal sign resulted in learners adding, subtracting, multiplying and dividing like and unlike terms. I therefore recommend that, for learners to develop a relational rather than an instrumental understanding, they should be encouraged to solve equations through inspections and trial and improvement, before solving them procedurally.

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ANNEXURE A

Table 3. Analysis of the incorrect responses

Questions		Most frequent incorrect responses	Reasons
Question 1 1.1	$[\ \ ] = 8 + 4$	Answer = 10 (1) Answer= 1 (1)	Most learners answered this question correctly; only two learners had it wrong.
1.2	$4 + 5 = [\ \ ] - 1$	Answer= 9 (10) Answer= 8 (9) Answer= 5 (1) Answer =1 (1)	Ten learners, who gave the answer as 9 interpreted the equal sign as "do something" or "give an answer" symbol. Nine learners who gave the answer as 8 indicated 4 plus 9 is 9 minus 1 is 8. They did not interpret the equal sign as "same as" symbol.
1.3	$[\ \ ] + 5 = 5 + 8$	Answer= 13 (6) Answer= 0 (2) Answer= 18 (2) Answer= -5 (1)	Six learners who gave an answer as 13 have interpreted the equal sign as "give me an answer" signal. Two learners who provided an answer as 18 just added numbers in the number sentences without interpreting the equal sign that shows equivalence, that is, the RHS quantity being equal to the LHS quantity.
1.4	$3 \times 7 = [\ \ ] \times 3$	Answer= 21 (11) Answer= 63 (2) Answer = 28 (1)	Eleven learners responded that an answer is 21. These learners seemed to have interpreted the equal sign as a "do something" signal. Two learners who have provided 63 as answer have just multiplied numbers in the number sentences.
Question 2: Solve for n			
2.1	$7 + n = 6 + 9$	N = 22 or 22*n* (14) N= 15 or 15*n* (5) N= 16 or 16*n* (4) N= other (14)	Fourteen learners who gave the answer as 22 or 22*n* have just added numbers in the equation without taking the meaning of the equal sign as showing equivalence of the RHS quantity being equal to the LHS quantity. Five learners who gave the answer as 15 or 15*n* have just added the right-hand side of the equation and ignored the left-hand side
2.2	$126 - 37 = n - 40$	N= 49 (20) N= 89 (5) N= other (24)	Twenty learners who provided an answer as 49 have subtracted 37 from 126 and subtracted 40 from the results to get 49. These learners seemed to have used numbers and operation appearing in the equation without understanding the meaning of the equal sign as representing two equivalent quantities. Five learners who gave the answer as 89 worked out the left-hand side of the equation and ignored the right-hand side.
2.3	$4800 \div 25 = n \times 48$	N= 192 x 48 (20) N= n x 48 (3) N= other (26)	Twenty learners have just divided 4800 by 25 and multiplied by 48 to get 9216 without considering the meaning of the equal sign as the RHS = LHS, while three learners just wrote their answer as $n \times 48$.
2.4	4*n* + 4 = 4*n* +1	N= 13 or 13*n* (16) N= 3 or 3*n* (7) N= 9 or 9*n* (4) N = other (22)	Sixteen learners who wrote their answer as 13 or 13*n* have just added numbers as appearing in the equation, without noticing that there is no solution in this equation.
Question 3 Is the mathematical statement below correct? If yes, say why. If no, say why you think it is not.			
3.1	$7 = 5 + 2$	2% of learners indicated No	No, you cannot start with the answer, then follow with a question
3.2	$4 + 5 = 9 + 1$	20% of learners indicated Yes	Yes, because 4 + 5 is equal to 9
3.3	$674 - 389 = 664 - 379$	53% of learners indicated No	No, the answer must be 254
3.4	$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$	96% of learners indicated yes	Yes, because I added 1 plus 1 is 2 and 2 plus 3 is 5