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How can Students Generalize Examples? Focusing on the Generalizing Geometric Properties

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ABSTRACT
The purpose of this study is to determine the progression of exemplifying and example generalization by students. We investigated whether example generalization occurs by analyzing collected data by identifying whether students recognize, describe, and define general features of geometric examples. We also investigate how example generalization progresses by identifying students' use of abduction, induction, diagrams and deduction. As a result of this study, we revealed that the sub-mechanisms undertake the generalization of examples as supported by the coordination of abduction, induction, deduction, and the use of diagrams. We empirically confirmed generalization by students through exemplifying and found that their generalization of examples involved the coordination of abduction, induction, deduction, and the use of diagrams.
Keywords: exemplifying, generalization, geometric examples

INTRODUCTION

Exemplifying has been discussed as means by which to foster mathematical inquiries by students (Watson & Mason, 2005). An *example* is "a particular case of any larger class about which students generalize and reason (Watson & Chick, 2011, p, 284)." We can use the term example to represent "anything from which a learner might generalize (Watson & Mason, 2005, p. 3)." Watson & Mason (2005) emphasized that exemplifying can facilitate students to construct generalized rules and students may learn about generalities through induction from given examples—to "see the general through the particular (Watson & Mason, 2005, p. 129)."

Generalization has been regarded as the heartbeat of mathematics (Mason, 1996). Davydov (1990) argued that developing the ability in learners to generalize is one of the main purposes of mathematics education. Sriraman (2004) noted that "[generalization] begins with the construction of examples, within which plausible patterns are detected and lead to the formulation of theorems (p. 205)." Additionally, Zazkis, Liljedahl & Chernoff (2008) emphasized that generalization is afforded by considering particular examples. In other words, it is at the heart of mathematical inquiry to generate examples, discern the common

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State of the literature

- Studies claim the importance of generalization and the didactic potential of exemplification in mathematics teaching and learning.
- Although some authors have investigated generalization by students, elucidating and encouraging appropriate generalizations by students of their mathematical experiences remain as challenges for mathematics educators.
- The authors claim example generalization as an important process that can be used to support student inquiries using examples, but little is known about the mechanisms and characteristics of students' generalizations of geometric examples.

Contribution of this paper to the literature

- The paper presents a teaching experiment on integrating exemplification in classrooms for exemplifying and example generalization.
- The practical outcome is the design of an exemplifying task with the aim of facilitating student inquiries into generalization, with the task ready for use in classrooms. Additionally, deep empirical insight into example generalization by students is presented.
- This study makes contributions to local learning theories about example generalization and empirically confirms that the coordination of abduction, induction, deduction, and the use of diagrams is a key factor supporting the generalization of examples.

features of observed examples, and to verify and refine these provisional commonalities to generalize examples (Durand-Gurrier, 2003). Hence, a key issue when utilizing exemplification in the teaching and learning of mathematics is to comprehend students' generalizations of examples generated by themselves (Watson & Shipman, 2008).

Several studies have investigated student generalization, but elucidating and encouraging appropriate generalizations by students of their mathematical experiences remain challenging to mathematics educators (Cockburn, 2012). Although a few supporting and impeding types of examples of generalization have been investigated, identifying the mechanisms of example generalization is still a key issue in the mathematics education research community (Zazkis, Liljedahl & Chernoff, 2008). In this context, it is important to explore the connections between learner-generated examples and their generalizations by examining empirical studies of exemplifying activities aimed to enhance students' generalizations (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006; Zazkis, Liljedahl & Chernoff, 2008). Further, studies of students' generalizations have thus far mainly focused on generalizations of algebraic patterns (e.g., Radford, 2010; Watson & Shipman, 2008), despite the fact that geometry gives students opportunities to experience mathematical generalizations (Pytlak, 2015) and has different developmental paths compared those associated with algebra (Tall, 2013). There is also growing concern that in spite of this importance, example generalization remains a challenging task for learners (Becker & Rivera, 2005; Watson & Shipman, 2008; Watson & Chick, 2011). Therefore, researchers must fill in some of the missing gaps in the existing line of research regarding the relationship between the exemplification and generalization of learner-generated examples. Hence, we aim to identify whether exemplification by students can be generalized to geometric properties and how such generalizations of learner-generated examples progress.

THEORETICAL BACKGROUND

Exemplification and generalization

The connections between exemplification and generalization have been discussed as a central issue in the mathematics education research community. Watson & Mason (2005) introduced the term *example space*, and this term helps us to envision the potential roles of exemplification in generalization.

Examples are usually not isolated; rather, they are perceived as instances of a class of potential examples. As such they constitute what we call an example space (p. 51).

Generalization can be supported by enriching or explicating an individual's example space (Watson & Mason, 2005). We can generalize examples in various ways by diversely categorizing and characterizing examples in our own example space. Goldenberg & Mason (2008) noted that "generalization comes about through appreciating dimensions of possible variation and ranges of permissible change, by seeing generality through the particular" (p. 190).

Researchers have documented two major features of examples which are related to students' generalization. The first is related to *generic examples* which enable us to see the general in the particular (Mason & Pimm, 1984). To be more specific, Weber (2008) showed that mathematicians used generic examples in their validations of possibly incorrect proofs and that generic examples enabled mathematicians to deal with generalities through a particular example. The second feature is related to *example variation*. Watson & Mason (2005) emphasized that example variation draws attention to generalities because one aspect varies and others remain the same such that we can characterize these examples. To be more specific, students could revise their general conjectures by observing many examples (Alock & Inglis, 2008) and generalize algebraic patterns while experiencing numerical variation (Zazkis, Liljedahl & Chernoff, 2008; Watson & Shipman, 2008).

Although it is difficult to avoid the use of examples when we deal with generalities in geometry (Presmeg, 2005), example generalization remains a challenging task for students (Becker & Rivera, 2005; Watson & Shipman, 2008; Watson & Chick, 2011). Hence, we investigate whether exemplification by students can be generalized to geometric properties and how such generalizations of learner-generated examples progress. In next two sections, we clarify the notion of generalization and its constituent elements, as well as the stages of the generalization of geometric properties to establish theoretical lenses with which to analyze students' exemplifications and generalizations in this study.

Generalization in mathematics education

Ellis (2007a) defined generalization as engaging in at least one of three activities: (a) identifying commonalities across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases (p. 227). She then classified generalization into two major categories: *generalizing actions* which are students' activities as they generalize, and *reflection generalizations* which are students' final statements of generalization. Although the term generalization can be classified into two major categories, we focus on the former meaning, generalizing actions, to investigate how generalizations of learner-generated examples progress.

Researchers have investigated students' generalizing actions and found that they have several constituent elements of generalization. Rivera (2010) emphasized that pattern generalization by students involves the coordination of two independent actions: abductive and inductive actions. He also argued that "the abductive phase is when students begin to offer an explanatory hypothesis for a given pattern on the basis of the available instances, which is then used to extend the pattern and is repeatedly tested—that is, the inductive phase—leading to either confirming a rule or further developing another abduction (p. 301)." From a traditional perspective, induction refers to the generation of new ideas from observations of multiple cases (Prawat, 1999). However, Peirce introduced the notion of *abduction*, distinct from deduction and induction, to avoid the pitfalls of empiricism and rationalism (Prawat, 1999) and focused on abduction as a phase in the discovery process (Pedemonte & Reid, 2011). Abduction refers to the process of forming an explanatory hypothesis A about an observed result B (C.P. 5.171). It takes the form 'if A is true then B would be a matter of course' (Prawat, 1999). Peirce emphasized that "abduction, on the other hand, is merely preparatory. It is the first step of scientific reasoning, as induction is the concluding step (C.P. 7.218)."

The conclusion of an abduction is capable of verification or refutation by a comparison with facts. It is the first stage in scientific reasoning, followed by deduction (of further consequences) and induction (testing those consequences) (Pedemonte & Reid, 2011, p. 284)

Studies have also emphasized that diagrammatic reasoning is closely related to generalization. Peirce emphasized that diagrammatic reasoning is a means of constructing abductions (C.P. 2.65, 2.77). In other words, diagrammatic reasoning is the linking ring between the 'deductive nature of mathematics and those elements of observation that lead to discovery and development' (Arzarello & Sabena, 2008, p. 190). A *diagram* is "a representamen which is predominantly an icon of relations" (CP 4.418), and it is possible to construct an abduction by *diagrammatic reasoning*- constructing a diagram, experimenting on it, and observing the results of the experiment (Hoffmann, 2005). An icon is a sign which represents its object by relying on its likeness to that object, and this likeness is aided by conventional rules (Otte, 2006). Most figures used in Euclidean geometry are both an icon and a diagram, as conventional rules apply to represent relationships among constituent elements (Hoffmann, 2004). An experiment on a diagram is the transformation of representations based on conventional rules; hence, the results of an experiment are assumed to have some degree of rationality (Park & Lee, 2016; Hoffmann, 2004). Accordingly, Otte (2006) took the role of diagrammatic reasoning into account when supporting the identification of hidden relationships by students in order to form new general properties. Although diagrammatic reasoning supports mathematical inquiry in diverse ways, we especially focus on diagrammatic reasoning as a tool for investigating students' example generalization, as several studies have noted the above interrelationships between diagrammatic reasoning and generalization in geometry.

We also focus on the roles of deduction in generalization. In general, deduction is a necessary inference driving a necessary result which must be implied on the premises (Meyer, 2010). Although deduction mainly supports mathematical coherence (Meyer, 2010), recent studies focusing on generalization have shown that deductive reasoning supports the students in their efforts to revise their generalization actions (Ellis, 2007b). Studies have also documented that justifications of generic examples enable mathematicians to deal with generalities through particular examples (Weber, 2008). Hence, we believe that the justification of examples can foster refutations and revisions by students of the provisional general features of examples with respect to a Lakatosian perspective on the role of justification when developing mathematical theories (Lee, 2011).

Thus, given the above theoretical considerations regarding generalization, we adopted abduction, induction, diagrammatic reasoning, and deduction tools for the present analysis of students' exemplifications and generalizations in this study. On the one hand, we expect that these four theoretical constructs will help us to analyze our issue in detail. On the other hand, abduction, induction, and deduction have been discussed as key means of reasoning to create generalities. Therefore, investigating their interactions with generalization may provide the research community with insight with which to detect key issues pertaining to students' knowledge construction in educational research (Prawat, 1999).

Generalization of geometric properties

Tall (2013) presented four stages for elaborating geometric properties. These stages are recognition, description, definition, and Euclidean proof.

Each of these stages involves new insights into the properties of figures. First is the categorization of the shape by its general appearance, then a focus on specific properties that can be sensed perceptually and described verbally, then as special generative properties that formulate a definition, and on to the deduction of relationships between various properties using Euclidean proof (Tall, 2013, p. 57).

The first stage involves recognizing the shape and visual appearance of geometric figures, and the second is to describe perceptually sensed properties verbally, after which various properties can be identified. The third stage is to define geometric properties using carefully selected geometric properties that enable them to be recognized and constructed, and the final stage is to prove by deduction. Radford (2003) noted that there are two interrelated elements of generalization: grasping local commonalities and expressing generalities. From Radford's perspective, the recognition stage involves grasping local commonalities, and the description and the definition stage are related to expressing generalities. Radford (2010) also sub-categorized the expression of algebraic generalities into contextual generalities and symbolic generalities. While the general objects are named through an embodied and situated description of them (e.g., "the next figure," "the top row," etc.) in contextual generality, the general objects and the operations created with them are expressed in the alphanumeric semiotic system of algebra in the form of symbolic generality. Compared to Radford's perspective, the description stage encompasses expressing geometric generalities through situated descriptions and the definition stage can be interpreted to mean the expression of generalities in the terms and properties of geometry. The fourth stage for elaborating geometric properties is the Euclidean proof, where 'deduction is utilized with *the principles of Euclidean proof* (Tall, 2013, p. 57).' Because the principles of proof should be employed during the Euclidean proof stage, the fourth stage involves students' constructions of formal proofs. We restrict our focus on the students' example generalizations and do not focus on their construction of formal proofs; we restrict the scope of this research to recognition, description, and definition and deemphasize Euclidean proof.

From this theoretical consideration of geometric generalization, we adopted Tall's stages of elaborating geometric properties as a tool for an analysis of students' exemplifying and generalization in geometry.

METHOD

Participants

The participants in this study were twenty students in the seventh grade with high levels of mathematical achievement. They were selected for a program held by a science-gifted education center of a university and were educated in the same class for more than six months. To be more specific, these students were selected for this gifted education center by mathematics examinations, and all had grades in the top ten percent in the province in which the university is located. The students participated in classes held by the gifted education center every two weeks. These classes are inquiry-oriented mathematics classes within the national curriculum for students. For the purpose of this study, the students participated in a regular class at the science-gifted education center which lasted 3h. The names of the students are coded as S1 to S20.

Data, instructional background, and research focus

The main data for this study were obtained from the students' written answers, video recordings, and lesson observations during the three-hour mathematics lesson in the spring of 2015. In order to facilitate the use of exemplifying for generalization in mathematics classrooms, one would expect experimentation to take place within a regular mathematics classroom. However, the aim of this study is to identify whether exemplifying by students can be generalized to geometric properties and how such generalizations of learner-generated examples progress. Other studies have noted that selecting talented and verbal participants is one option for achieving rich theoretical constructs with regard to the processes and characteristics of learners' mathematical inquiries in an explorative study such as the present research (Park, Park, Park, Cho & Lee, 2013; Dreyfus & Tsamir, 2004). In this respect, we selected high-achieving and gifted students as the participants here to identify (i) whether students' generalizations occur during exemplifying, and (ii) how generalizations of learner-generated examples progress. Given that our analysis is based on high-achieving students, we consider the resulting theoretical constructs as sensible proposals for components of a theory which explains the didactic potentials of exemplifying rather than as a complete and refined theory. However, we consider that such proposals are the tools of the trade of theory building.

Watson & Mason (2005) emphasized example generation by students can be effectively supported by dividing students into several small groups. In addition, Maaß (2006) pointed out that students' inquiries can be effective orchestrated by supporting each individual's inquiries in the context of small group interaction. Hence, the 20 participants worked together in five groups consisting of four students each and also worked on their own individual worksheets. To be more specific, we asked the students freely to exchange their ideas and solutions in their groups if necessary. We also asked the students to record their explorations on the given tasks in their individual worksheets to encourage the students actively to participate in exploration. As noted by Lee (2011), we also considered that these group tasks facilitate students' reflective thinking and that they shift the students' attention to support their mathematical inquiries. Students were asked to represent their exemplifying in various ways in order to foster their diagrammatic reasoning.

The collected data were first analyzed in chronological order and divided into distinct but related episodes, as suggested by Cobb and Whitenack (1996). We also analyzed the results while focusing on the following two points:

- Does example generalization emerge or not?
- How does example generalization progress?

We investigated whether example generalization emerges by analyzing collected data using the stages of geometric elaboration suggested by Tall (2013). That is, we analyzed whether students recognized, described, or defined the general features of geometric examples. We also investigated how example generalization progresses by identifying students' use of abduction, induction, diagrams and deduction.

The task

The aim of the task was to determine every line which divides a parallelogram into two parts with the same area. This task consists of four subtasks.

A parallelogram is a quadrilateral with two pairs of parallel sides.

1. Find a line which divides the parallelogram into two parts with the same area, and explain why this line divides the parallelogram into two parts with the same area.
2. Find four more lines which divide the parallelogram into two parts with the same area, and explain why these lines divide the parallelogram into two parts with the same area.
3. Explain the commonalities and differences of the lines you found in tasks 1 and 2.
4. Find a way to obtain every line which divides the parallelogram into two parts with the same area, and explain why your solution is reasonable.

In this exemplifying task, the students were asked to generate examples of lines which divide a parallelogram into two parts with the same area and then to characterize these examples. The design of this exemplifying task is thus considered to allow students to generalize examples which divide a parallelogram into two parts with the same area. Zazkis, Liljedahl & Chernoff (2008) noted that it is important to share with readers several strategies that worked for researchers, presuming that similar strategies could be helpful with other students in other settings. Hence, we briefly illustrate how the task was designed to share several key points of the task.

In subtask 1, the students were asked to generate one example which satisfies the given conditions. We assumed that the students could generate a simple example in task 1 and that they could clearly identify the given conditions by justifying their example. We initially asked students to generate a single example and justify it because an abduction can be constructed by observing a single example (Cañadas, Deulofeu, Figueiras, Reid, Yevdokimov, 2007; Prawat, 1999). We also believed that asking students to justify a single example may prevent their exemplifying from progressing to a trial-and-error strategy, as cautioned by Radford (2010).

Subtask 2 dealt with variations of examples. This example variation step sought to facilitate student explorations of non-trivial examples. Marton (2006) emphasized the necessity of variation in learning. That is, learners can notice the generality encompassing particulars by discerning differences among particulars (Marton, 2006). Similarly, Watson & Mason (2005) noted the importance of example variation when generalizing learner-generated examples. Thus, the key to example generalization is to discern differences as well as commonalities when we experience example variation. As in the variation theory of Marton (2006), we assumed that example variation would enable the students to discern the central features of their examples. On the other hand, one way to facilitate example variation by students is to assign the number of examples to be generated by the students. That is, asking students to generate more examples than would be trivial may help them construct non-trivial examples or more exemplary examples and then to discern the general features of the examples (Watson & Mason, 2005). Hence, we asked the students to generate four more examples in subtask 2 such that they would construct examples which were not trivial. We considered that trivial examples are lines extended by diagonals and segments which link the two midpoints of the corresponding sides.

Subtask 3 had students compare their examples to find the key characteristic of the examples, and subtask 4 required the students to characterize their examples. Subtasks 3 and 4 are designed to encourage students to express geometric generality by asking them to characterize their examples. Mason (2011) noted that "to characterise is to establish through mathematical reasoning that some other property classifies exactly the same objects (p. 44)." He also emphasized that mathematical definitions and theorems are global properties and that they can be characterized in terms of local properties. As Mason (2011) also emphasized, we take into account the role of example characterization in helping students express common features of their examples as characterizing mathematical examples.

In addition, every task asked students to justify whether their examples satisfy the given conditions. As reviewed previously, we assumed that a justification task may support the formation and revision of abductions based on a Lakatosian perspective (Lee, 2011).

RESULTS

In this study, we observed students' generalizations of their geometric examples. To be more specific, it was revealed that the students *recognized* examples and diagrams, *described* the commonalities of examples, and *defined* the general properties of the examples. This chapter is organized into three parts: (a) recognizing examples and diagrams, (b) describing the commonalities of the examples, and (c) defining the general properties of the examples.

Episode 1: Recognizing examples and diagrams

The students' inquiries into subtask 1 were mainly carried out individually, except for groups 1 and 3, as finding one example was not problematic for students. Thus, we mainly focus on students' inquiries with regard to this subtask at the individual level and discuss the group discussion held by groups 1 and 3. In this Episode, we initially address students' constructions of diagrams and their attempts to find more generic diagrams. We then illustrate example generation and construction of abductions by students with justifications of their examples.

Construction of diagrams and searching for a generic diagram

As the students read the exemplifying task, they initially constructed diagrams which signify the given problem situation. Representations constructed by students are diagrams because they organized the relationships between the constituents of the parallelograms (e.g., edges, angles, areas, and lines) (cf. Hoffmann, 2004).

The students' diagrams are categorized into two types (hereafter D1 and D2). Whereas the D1 diagrams are generic parallelograms which encompass every type of parallelogram, as shown in **Figure 1**, the D2 diagrams are specialized parallelograms or rectangles, as shown in **Figure 2**. The numbers of students who constructed each type of diagram are summarized in **Table 1**.

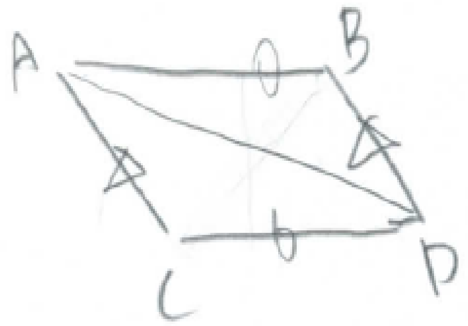


Figure 1. Diagram of S12 (D1 type)

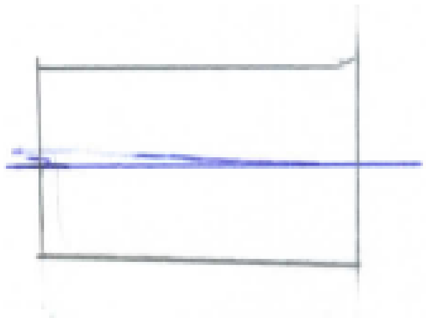


Figure 2. Diagram of S5 (D2 type)

Table 1. Types of diagrams constructed in subtask 1

Type	Diagram	Number of students
D1	Generic parallelogram	17
D2	Rectangle	3

The students who constructed the D2 diagrams drew lines extended by diagonals or segments which link two midpoints of the corresponding sides. These students changed their diagrams into generic parallelograms by communicating with other students or glancing at the worksheets of other students. The following dialogue among the students in group 1 shows how student S2 changed his rectangle to a generic parallelogram.

67	S2	Aren't rectangles ok? Don't parallelograms include rectangles?
68	S4	Yes, yes.
69	S1	That's right
70	S4	Ok
71	S3	There must be more conditions.
72	S2	Conditions are also the same.
73	S4	Rectangles are ok.
74	S3	But...
75	S3	But parallelograms are not rectangles. Rectangles are parallelograms.
76	S2	Rectangles are parallelograms; parallelograms are not rectangles, ok.
77	S3	Parallelograms have wider meanings. You shouldn't just pick a rectangle here.
78	S2	Hmm... ok, ok.

Student S2 initially drew a rectangle to generate an example and found that other students in the same group drew parallelograms which are not rectangles. He asked other students in his group about the relationship between rectangles and parallelograms. Though students S1 and S4 drew generic parallelograms, they immediately agreed with S2, as the statement of S2 was also partially correct (line 67, *parallelograms also include rectangles*). Student S3 participated in this discussion and explained the relationship between parallelograms and rectangles (line 75). She emphasized that rectangles have more conditions (line 71) and that parallelograms have wider meanings (line 77). In other words, she argued that they should draw parallelograms to resolve subtask 1 given that parallelograms occupy a larger category than rectangles. Student S2 then agreed with her idea and drew a parallelogram to find more examples.

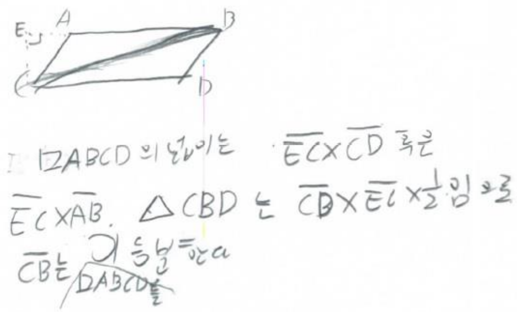
Example generation and construction of an abduction

Though students generated a single example in subtask 1, their examples are categorized into two types (hereafter E1 and E2). The numbers of students who generated each type are summarized in **Table 2**.

Table 2. Types of examples generated in subtask 1

Type	Example	Number of students
E1	Trivial examples	16
E2	Non-trivial examples	4

E1 examples are relatively trivial, consisting of lines extended by diagonals or segments which link two midpoints of the corresponding sides (**Figure 1** and **2,** respectively). The students who generated E1 examples drew lines which divide the parallelogram into two equiareal parts (**Figure 3**).



The area of ABCD is $\overline{EC}\times\overline{CD}$ or $\overline{EC}\times\overline{AB}$. [The area of] $\triangle$CBD is $\overline{CD}\times\overline{EC}\times\frac{1}{2}$, so $\overline{CB}$ equally divide ABCD.

Figure 3. Worksheet of S1

E2 examples are non-trivial examples which were generated during students' attempts to find lines which divide the parallelogram into two congruent quadrilaterals (**Figure 4**).

The students who generated E2 examples attempted to generate examples while relying on the construction of an abduction. To be more specific, they assumed the following: 'if two parts of a parallelogram divided by their example are *congruent*, then the areas of each part will be identical'. Therefore, they attempted to find a line which satisfies their abduction.

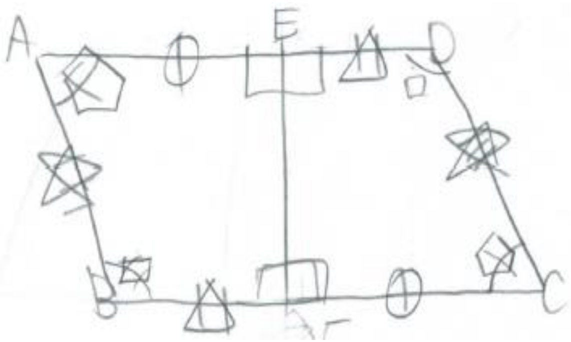


Figure 4. Example generated by S11 (E2 type)

The students who generated E2 examples constructed an abduction by attempting to justify their examples. The following dialogue took place between the students in group 3.

15	S10	I don't think it is possible clearly to explain [why my example divides a parallelogram into two equal areas]
16	S9	I can do…?
17	S10	How?
18	S9	These two [quadrilaterals] are congruent.
19	S10	How? We did not learn the congruence of a quadrilateral.
20	S9	No, make the length of this edge identical to this. Make the length of this edge identical to this edge too.
21	S10	So how can we know that these two are congruent?
23	S9	This is that…that…
24	S11	Right angle
25	S12	Perpendicular
26	S9	And this is also perpendicular…, perpendicular… and this [corresponding edges] is overlapped. So it's congruent. And this edge and this edge are parallel to each other.

Student S10 initially found a line extended by segments which links two midpoints of the corresponding sides, and he attempted to verify his example with student S9. S9 claimed that the two areas divided by S10's example are congruent. Justification of their first example was not easy work for them. S9 then rotated their examples to generate another line which was perpendicular to two edges of the parallelogram. She argued that the two areas divided by this new line are congruent because the corresponding edges and angles of these two quadrilaterals have identical lengths and sizes, respectively (Line 23~26). Although they already knew that two alternative angles are congruent when the two lines are parallel, they were not proficient in using this property. Accordingly, S9 found a line perpendicular to two edges of the parallelogram to justify her example easily based on the congruence of two quadrilaterals.

Summary of Episode 1

In this Episode, we found how students constructed diagrams and searched for a generic diagram. We also identified how the students constructed an abduction and generated an additional example. Two key issues emerged from this section. First, the students constructed diagrams which signify the given problem situation, and we identified the students' attempts to search for a generic diagram by students in group 1. To be more specific, while exchanging opinions about the relationship between parallelograms and rectangles, the students in group 1 structured their example space on quadrilaterals and agreed over what was a generic example of a parallelogram. Though we identified the students' figures (parallelograms and rectangles) as diagrams, these are also examples of parallelograms. Given that one of the most notorious issues related to dealing with generality is the use of generic examples (Weber, 2008), these students' attempts to search for more generic examples deserved our attention.

As the statement of S2 was partially correct (line 67, *parallelograms also include rectangles*), students S1 and S4 agreed with his idea. Although student S2 did not draw a generic parallelogram, he may find lines which divide the parallelogram into two parts with the same area. However, his selection of an example of a parallelogram becomes problematic when he deals with generalities. As S3 noted, rectangles have more conditions than parallelograms. Therefore, the students may find more special commonalities among lines which divide the parallelogram into two parts with the same area if they deal with this task with rectangles. Thus, student S3 compared the conditions (line 71) and meanings (line 77) between parallelograms and rectangles and considered that dealing with a generic example of parallelogram is appropriate to resolve subtask 1. Though the students needed to investigate extreme examples to organize one's own example space and investigate the boundaries of the example space to consider possibilities beyond the obvious (Watson & Mason, 2005), it is also important to generate generic examples to deal with generalities (Weber, 2008). This flexibility is valuable for students to deal with generalities, though it also presented difficulties to the students.

Second, the students actively utilized diagrams during their exemplifying efforts, and we identified the construction of an abduction by the students in group 3. We found that most of the students generated trivial examples and that they recognized parallelograms and their examples (**Figure 3**). On the other hand, the students who generated non-trivial examples constructed an abduction while justifying their examples (**Figure 4**). Although these students carefully considered the properties of their examples, their abductions did not include the construction of every line which divides the parallelogram into two parts with the same area. Thus, they are describing features of their examples rather than defining. The construction of the abduction by the students in group 3 included hypothesizing that the two areas divided by their examples were congruent. These students generated additional examples which satisfy their abduction by conducting experiments on their diagrams. They could conduct an experiment on the diagrams by manipulating the relationships among the constituents of the parallelogram and other figures, as the diagrams signified their relationships. In the next Episode, we identify students' example variations and their discernment of the common features of their examples. The synergic relationships among abduction, induction, diagrammatic reasoning, and deduction are also identified in the next Episode.

Episode 2: Describing the commonalities of examples

The students were asked to generate four examples in subtask 2. However, they generated more examples because they understood that their final goal was to find a way to present every line which divides the parallelogram into two parts with the same area. The students in group 3 mainly focused on generating more than four examples which divide the parallelogram into two parts with the same area, as they had already generated non-trivial examples while resolving subtask 1. These students continued to find examples which divide the parallelogram into two congruent parts. On the other hand, the students in the other groups initially found it easy to generate trivial examples individually, but they encountered some difficulty when attempting to find examples different from trivial examples while resolving subtask 2. Accordingly, they actively exchanged their ideas about finding non-trivial examples. Thus, we address the students' inquiries at the group level in this Episode because our focus is on the students' generations of non-trivial examples and example generalization.

In this Episode, we mainly address how students begin to describe the commonalities of examples while experiencing example variation. We categorized students' example variation into two types and therefore, illustrate how the students describe the commonalities of the examples in accordance with their example variation approaches. Methods by which the students undertook example variation were categorized into two types (hereafter V1 and V2). The V1 type of example variation was mainly supported by the abductions formed by utilizing the ideas which emerged through deductive justification, as we found in Episode 1. Example variations by the students in groups 2, 3, and 4 were categorized as the V1 type. Other students in groups 1 and 5 experienced example variation which was mainly supported by diagrammatic reasoning. They discerned perceptual commonalities among examples by experimenting on their diagrams. To be more specific, these students conducted experiments on diagrams and generated various examples. They then constructed an abduction based on the results of their experiments on diagrams. Subsequently, they inductively verified and tested their abduction by generating further examples. The students who undertook V2 example variation generated many more examples then those who carried out V1 example variation.

In the following sections, we address how each type of example variation progressed and how abduction, induction, diagrammatic reasoning, and deduction supported example generalization.

Table 3. Example variation methods in task 2

Type	Example	Group
V1	construction of an abduction by deductive justification	2, 3, 4
V2	construction of an abduction by diagrammatic reasoning	1, 5

V1 type example variation

The students who used the V1 type undertook example variation by utilizing an abduction which emerged when they attempted deductively to justify their initial examples. The following is the dialogue of the students in group 4.

50	S15	This edge and that edge are the same in the parallelogram, and this and this are the same. That edge and this edge are parallel. These two angles are alternative angles and these are also alternative angles too. So, the angles are the same.
51	S14	Then we do not have to cut the parallelogram perpendicularly? We can just use this...
52	S15	Though the line is not perpendicular, this and this... the... the length of each edge needs to be the same.
53	S14	That is, these two areas are congruent even though the line is not perpendicular to the edges.

Student S14 initially generated an example similar to that shown in **Figure 1** for Episode 1. Student S15 attempted deductively to justify this line by verifying that the two areas divided by the line are congruent (line 50). By considering her statements (line 50) with the work on her worksheet, we considered that she attempted to utilize necessary inference to justify her example, despite the fact that her justification was not in the form of a formal proof. Thus, we interpreted her statements in the above dialogue as well as the explanations on her worksheets as evidence of her attempts at deductive justification. Based on the abduction of S15, S14 generated another example, as shown in **Figure 5** below, and other students in the same group (S13 and S16) agreed with their ideas.

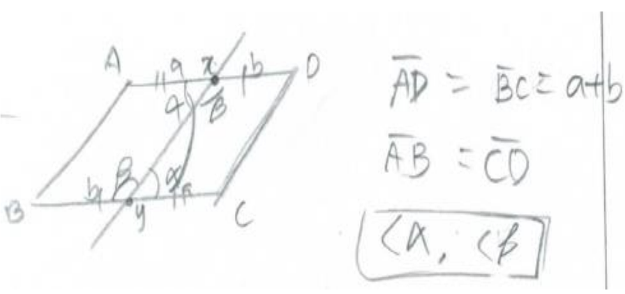


Figure 5. Example of S14 (group 4)

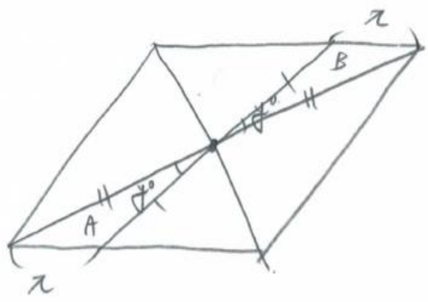


Figure 6. Example of S10 (group 3)

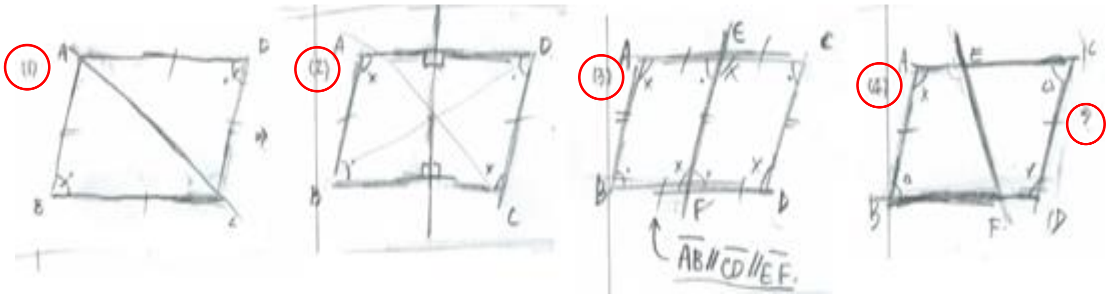


Figure 7. Examples of S7 (group 2, the red circle was added by the authors)

The students in groups 2 and 3 showed similar processes of example variation, as depicted in **Figures 6** and **7**. Example variation by group 3 was addressed in the first Episode. The students in group 2 also constructed abductions from deductive justification and utilized these abductions to find non-trivial examples. The four diagrams in **Figure 7** clearly show the V1 type of example variation. The students in group 2 initially generated trivial examples, as shown in the panel (1) in **Figure 7**, after which they deductively justified these trivial examples. They utilized the congruence between two areas deductively to justify their trivial examples and then generated other examples, as shown in the panel (2) in **Figure 7**, which satisfy their abduction while deductively justifying (1) in **Figure 7**: 'if two parts of a parallelogram divided by their example are congruent, then the area of each part will be identical.' The students in group 2 then deductively justified their example (2) in **Figure 7**, finding that lines which divide the parallelogram into two congruent parts do not have to be perpendicular to two sides of the parallelogram. These students utilized the properties of alternative angles with parallel lines and constructed a revised abduction: 'if one line bisects two edges of the parallelogram, then two parts of a parallelogram divided by this line will be congruent.' Therefore, the students in group 2 generated another example (3) in **Figure 7** which satisfies their revised abduction. These students attempted deductively to justify their example (3) in **Figure 7**. They then found that this line does not always have to bisect two sides of the parallelogram, as the two quadrilaterals ABFE and EFDC in the parallelogram (3) in **Figure 7** are congruent despite the fact that the lengths of segments AE and FD are not identical. Accordingly, they constructed example (4) in **Figure 7**. However, they did not characterize this example (4) in **Figure 7** while resolving subtask 2; therefore, we could not identify the abduction which they utilized to construct example (4) in **Figure 7** while resolving subtask 2.

Although these students in groups 2, 3, and 4 used diagrams, we categorized these students' example variation efforts as the justification focused type given that their abductions were mainly constructed through their deductive justification efforts. Interestingly, the students in these three groups initially generated an example perpendicular to two edges of the parallelogram and then generated a more generalized example. As discussed in relation to the first Episode, this likely stems from the fact that the students were not proficient in using the properties of alternative angles with parallel lines. The activities of example variation and revision of the abductions by the students in groups 2, 3, and 4 are summarized in **Figure 8**.

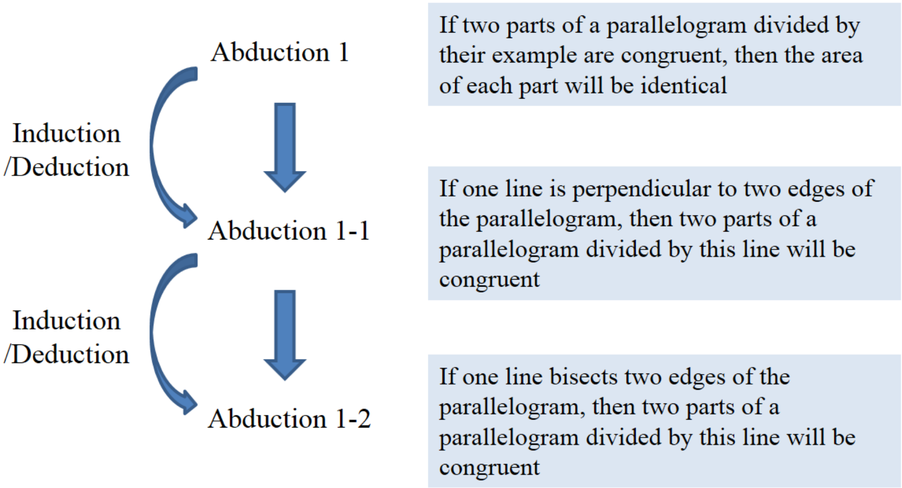


Figure 8. Revision of abductions by groups 2, 3, and 4

The students in groups 2 and 4 constructed and gradually revised their abductions along the path in the figure above. On the other hand, the students in group 3 constructed an abduction which was a mixed version of abduction 1-1 and abduction 1-2 because they generated examples which were both perpendicular to two edges and then divided these two edges of the parallelogram into two parts with the same ratio (**Figure 4** in the first Episode).

As noted above, the students in groups 2, 3 and 4 constructed abductions on commonalities among the examples, and these abductions consisted of carefully selected features of their examples. Although these students' abductions are neither perceptual nor contextual, their abductions nonetheless cannot include every line which divides the parallelogram into two parts with the same area. Thus, they are still in the description stage.

In the summary of the V1 type of example variation, the students initially generated trivial examples and then constructed an abduction during the deductive justification of their examples. They subsequently generated other examples which satisfied their abductions. This phase is similar to the inductive verification phase which was noted in Rivera (2008). These students then deductively justified these new examples and revised their abductions again (**Figure 7**). **Figure 9** summarizes the V1 type example generalization process.

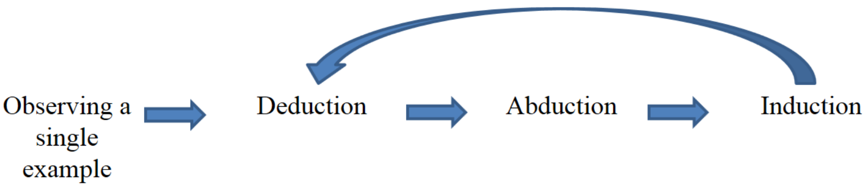


Figure 9. The V1 type of example generalization

V2 type example variation

The students who used the V2 type (groups 1 and 5) mainly experienced example variation by means of diagrammatic reasoning. These students constructed abductions by observing the results of experimentation on diagrams. They then generated more examples to test and revise their abductions by conducting further experimentation on diagrams with regard to their abductions.

The students in group 1 initially generated a few lines and found that the two parts of the parallelogram divided by their lines are located at a position of symmetry. At the outset, S4 initially argued that these two parts of a parallelogram are located at a position of line symmetry, but this abduction was reputed by other students' examples (Line 136). S3 found that their examples passed through one point and therefore modified S4's abduction such that it held that the two areas divided by a line are located at a position of point symmetry. S3 and S1 then claimed that 'every line passing through a point of symmetry divides a parallelogram into two parts with the same area.' The following is the dialogue of the students in group 1, and S2 agreed with this conversation.

135	S4	If line symmetry...if we fold, then do these two areas overlap?
136	S3	No, so this is point symmetry~
137	S1	point symmetry

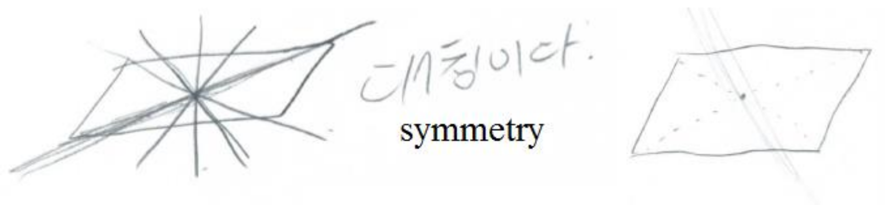


Figure 10. Diagrams of S1

Student S1 drew dotted lines to find the corresponding points to verify that the two areas divided by his example are located at a position of point symmetry, as shown in the diagram on the right in **Figure 8**. It is important to note that S1 marked and described the commonalities of lines as the point of symmetry, but he did not clearly characterize this point. He attempted to verify whether these dotted lines are equally divided by the point marked inside the parallelogram, but he did not complete his deductive justification. **Figure 11** below summarizes the activities of example variation and revision of the abductions by the students in group 1.

The students in group 5 also generated many examples in their diagrams, finding their perceptual commonalities. S17 initially argued that he can find every line which divides the parallelogram into two parts with the same area by rotating lines around the center point inside the parallelogram (**Figure 12**).

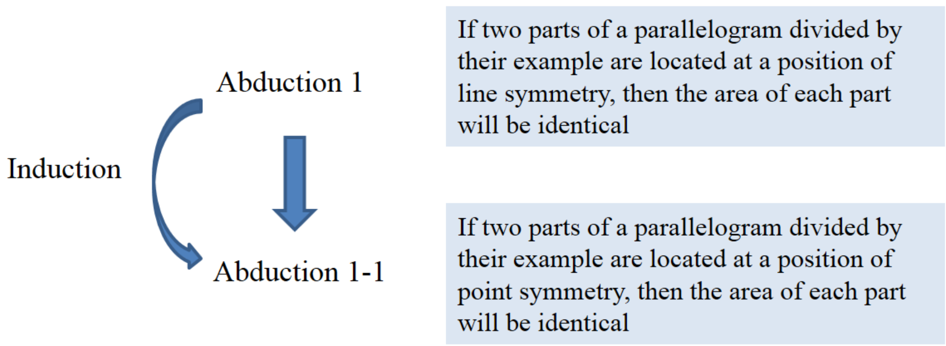
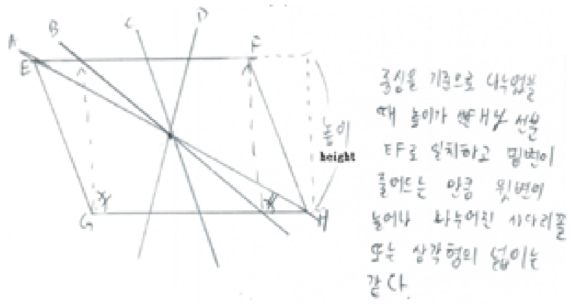


Figure 11. Revision of the abductions by group 1



Divide based on the center, heights are segment FH or EF and so are the same. The decrement of the length of the base-line is identical to the increment of the length of the topside edge, so the areas of the two trapezoids or triangles are the same.

Figure 12. Worksheet of S17 (group 5)

He claimed that the areas of the two parts of the parallelogram divided by these lines are identical because the lengths of the two parts of the topside edge and the base-line divided by his examples change by an identical amount. It is important to note that S17 marked and *described* the commonalities of lines as the center in **Figure 11**. However, he did not clearly characterize '*the center*' of the parallelogram. In group 5 (S17~S20), S17 was the first to suggest this idea, and S18 and S19 agreed with this. Thus, S18 and S19 found examples in similar ways to S17. On the other hand, S20 attempted to justify his examples using congruence, but he could not complete this justification while resolving subtask 2.

As noted above, the students in group 1 described the commonalities of their examples as 'passing the symmetric point.' In addition, the students in group 5 marked and described the commonalities of their examples as 'passing the center of the parallelogram.' Hence, we confirmed that the students in groups 1 and 5 described commonalities among examples rather than defining, as they verbally described how the perceptual commonalities of examples cannot construct every line which divides the parallelogram into two parts with the same area.

In the summary of the V2 types of example variation, the students initially generated a few trivial examples in their diagrams and then constructed abductions (line symmetry/equiareal transformation) by experimentation on and observations of their diagrams. They then generated more examples in their diagrams and verified their abductions (group 5) or revised their abductions (group 1: point symmetry). However, none of the students in either group clearly characterized the points marked inside their parallelograms. They simply referred to this point as the center, or the point of symmetry. **Figure 13** summarizes the V2 type of example generalization.

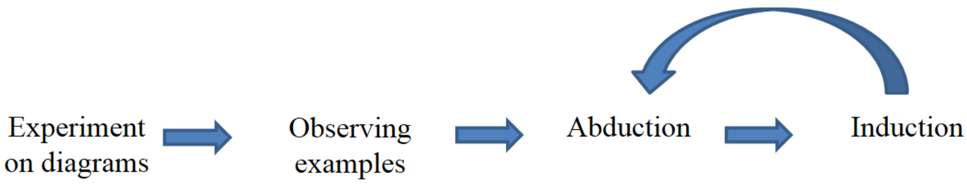


Figure 13. The V2 type of example generalization

Summary of Episode 2

In this Episode, we identified how students construct abductions and describe commonalities among their examples and then generate more generalized examples. The students constructed provisional abductions and verified or revised these abductions (**Figures 8** and **11**). Two key issues emerged from this Episode. First, we identified two types of example variation and generalization by the students while also identifying the synergic relationships among abduction, induction, diagrammatic reasoning, and deduction for each type of example generalization. The students in groups 2, 3, and 4 constructed abductions while attempting deductively to justify their examples (V1 type). Pedemonte & Reid (2011) reported that students construct abductions while undertaking deductive justification because deductive reasoning requires the construction of hypothetical premises to utilize necessary inferences. Similarly, the students in this study constructed abductions of their examples while deductively justifying these examples. They also utilized an inductive phase to verify and test their abductions, as noted by Rivera (2010), and revised their abductions while deductively justifying their newly generated examples. Although prior studies argued the importance of deductive reasoning when dealing with generalization, the major role of deductive justification during the generalization was to refute or modify provisional generalities (Lee, 2010; Ellis, 2007b). On the other hand, the students in this study utilized deductive reasoning to construct abductions and described commonalities among examples. In other words, the students' efforts at deductive justification directly supported the construction of provisional generalities. We also consider that the students' inductive phases were supported by their use of diagrams. Because these diagrams signified relationships between their examples and other constituents of their diagrams, the students' use of diagrams supported their efforts to generate modified examples which satisfied their revised abductions.

The students in groups 1 and 5 constructed abductions by diagrammatic reasoning (V2 type) and revised their abductions by generating further examples. As Rivera (2010) noted, we also identified abductive and inductive phases in the students' type V2 generalizations. We further empirically confirmed that the students in this study constructed abductions by diagrammatic reasoning, as claimed by other researchers theoretically (Hoffmann, 2005; Otte, 2006). We consider that the students' constructions of abductions were supported by their use of diagrams, as these diagrams signified relationships among their examples and other properties of the constituents of their diagrams. It is important to note that the above categorization of student exemplification is not intended to be hierarchical. We found that the students who focused on diagrammatic reasoning represented their examples by a single diagram, whereas the students who focused on deductive justification drew each of their examples in the form of different diagrams. Therefore, the students who focused on deductive justification did not find a perceptual commonality among their examples as identified by the students who focused on diagrammatic reasoning.

Second, we empirically confirmed that example variation supported the generalization of learner-generated examples. Studies have theoretically indicated that example variation may support the generalization of examples (Watson & Mason, 2005), and we empirically confirmed this. To be more specific, generalization by students progressed from the recognition stage to the description stage during the example variation processes. On the one hand, example variation supported students' constructions of abductions (V2 type). The students could construct abductions on their examples while experiencing example variation. On the other hand, the construction of abductions also supported example variation (V1/V2 type). To be more specific, the students could generate further examples by considering lines which satisfied their abductions. While the former type of example variation played a role in generating abductions, the latter type of example variation supported the verification as well as the revision of abductions.

Example variation by the students progressed dynamically, coming and going between the generation of particular examples and the construction and reconstruction of abductions. However, the students' generalization efforts remained in the description stage with regard to Tall's stages of generalization. Although the students constructed, utilized, and revised abductions while generating further examples, they could construct abductions only by describing examples which they had already generated. That is, their abductions could not encompass every example and only described partial examples generated by them, as the students mainly focused on the generation of further examples rather than considering every possible example while experiencing example variation.

Episode 3: Defining the general properties of examples

The students had already actively exchanged their ideas when they attempted to resolve the previous subtasks, and their inquiries during subtasks 3 and 4 initially began with group discussions. Thus, we examine the students' inquiries during subtasks 3 and 4 at the group level in this Episode. In this Episode, we initially address how the students named the commonalities among their examples, after which we illustrate how the students defined the general properties of their examples.

Naming perceptual commonalities

As noted for Episode 2, the students who focused on diagrammatic reasoning (groups 1 and 5) found this commonality among their examples and verbally described the commonality as the center or the symmetric point when they resolved subtask 2. The students in groups 2, 3 and 4 also found that lines dividing a parallelogram into two equal areas pass through a single point inside the parallelogram while resolving subtask 3. They were asked to find commonalities among examples in subtask 3. Although these students focused on congruence as they resolved subtask 2, they already represented some of their examples in a single diagram (**Figures 6** and **7,** in Episode 2). With these diagrams, the students in groups 2, 3 and 4 drew their examples in a single diagram and then found that their examples pass through a single point. To be more specific, S7 initially found this perceptual commonality in group 2, and the other students in group 2 verified whether their examples pass through this point. In group 3, S9 and S10 found this common point and shared this idea with other students in the same group. In group 4, S15 initially found this common point, and other students in the same group verified his claim with regard to their examples and agreed with his idea.

During this process, these students also verbalized this perceptual commonality as a single term which they heard from the discussions of groups 1 and 5, though they did not directly communicate with the students in the other groups. To be more specific, the students in groups 2, 3, and 4 initially used the term ‘center’ mixed with the term ‘symmetric point’ to describe the point passed through in their examples. However, these students began to disregard the term symmetric point and instead used the term ‘center’ in the end, as they did not utilize symmetric transformation to generate and justify their examples.

In subtask 4, the students were asked to find a way to obtain every line which divides the parallelogram into two parts with the same area. When the students started resolving subtask 4, their abductions were slightly revised, and they shared these abductions in each group. To be more specific, the students in groups 2, 3, 4 and 5 shared the following abduction in their groups: ‘every line passing through the center of a parallelogram divides the parallelogram into two parts with the same area.’ The students in group 1 shared the following abduction in their group: ‘every line passing through a point of symmetry divides a parallelogram into two parts with the same area.’ The students in groups 2, 3 and 4 started to focus on the center and attempted to clarify this point, as their abductions constructed during subtask 2 could not encompass every example. The students in groups 1 and 5 had already constructed the above abductions while resolving subtask 2. Thus, the students mainly began focusing on this point (the center, the symmetric point) to find a way to obtain every line which divides the parallelogram into two parts with the same area.

Defining geometric properties

Subtask 3 only asked the students to explain commonalities and did not ask them to justify their ideas. Hence, the students initially attempted to verify whether every line passing through the center of a parallelogram or a point of symmetry in fact divides the parallelogram into two parts with the same area when resolving subtask 4. During this process, the students could define the common point through which their lines passed, termed the center of a parallelogram (groups 2, 3, 4 and 5) or a point of symmetry (group 1). The key to their justification was to justify their abductions with an arbitrary line passing through the point inside the parallelogram. Because the students had already generated non-trivial examples while resolving the previous subtasks, they could attempt to justify their abductions with non-trivial examples, similar to the line inside the red circle shown in **Figure 14**.

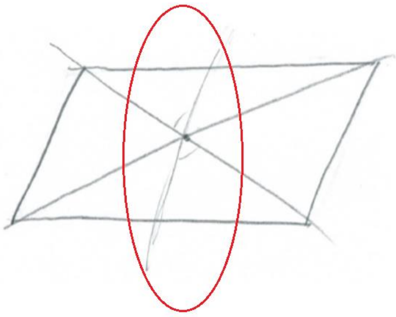
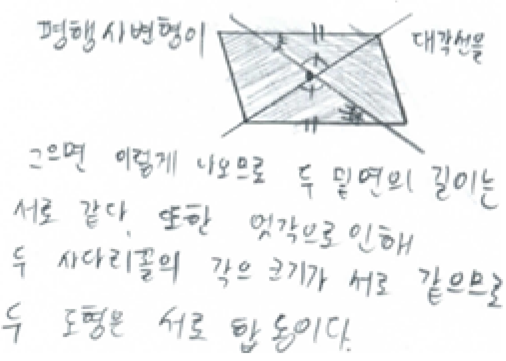


Figure 14. Diagram of S6 (the red circle was added by the authors)

The students in groups 2, 3, and 4 attempted to justify their abductions by utilizing the congruence of two areas. Their justifications were mainly led by one or two students in each group, and the other students verified the justification processes or presented additional opinions. Because every line dividing a parallelogram into two parts with the same area passes through a single point, it was clear for the students that diagonals also pass through such a point. Therefore, they naturally drew diagonals to use them for justification. They then divided the parallelograms into triangles to compare the areas (**Figure 14**) or attempted to argue for the congruence of trapezoids by comparing the lengths and sizes of every corresponding edge and angle (**Figure 15**), respectively.



If we draw diagonals, then the lengths of two base lines are the same. Also, the sizes of the alternative angles are the same and every angle of the two trapezoids is same. So, the two trapezoids are congruent.

Figure 15. Justification written on the worksheet of S17

During this justification step, the students could define the point inside the parallelogram as '*an intersecting point of two diagonals of the parallelogram*.' This can be considered as a *definition* of the examples, as this is a generative property which enables the construction of every line which divides the parallelogram into two equiareal parts. As noted above, the fact that every line passes through the center of the parallelogram is a special generative property of their examples, and their previous abductions did not have such a generative aspect. Thus, the students defined this point and suggested this generative property as a way to obtain every line which divides the parallelogram into two parts with the same area.

The students in group 5 also similarly advanced their inquiries, as S20 continually attempted to find a way to justify congruence between the two areas divided by their examples. Although S17 initially claimed to justify their abductions of equiareal transformation, but he could not clearly complete his justification. Therefore, S17 and the students in group 5 changed their justification method to focus on congruence.

The students who described the commonalities of lines as a point of symmetry claimed that there is a single point inside the parallelogram and that two parts of a parallelogram divided by every line passing this point are located at a position of point symmetry.

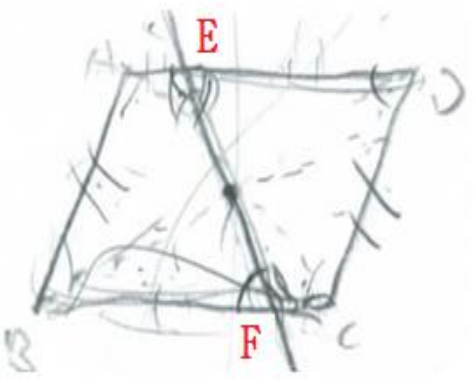


Figure 16. Diagram of S3 (the letters E and F were added by the authors)

These students drew dotted lines to find corresponding points to verify that the two areas divided by the example are located at a position of point symmetry. There are two dotted lines BD and AC in **Figure 12**, and two other dotted lines overlap the line dividing the parallelogram. The students attempted to verify whether these dotted lines pass through a point and are equally divided by the point. During this process, dotted lines linking two pairs of opposite points of the parallelogram were drawn for every line such that the students could discern the diagonals. Hence, these students also defined the point of symmetry as an intersecting point of two diagonals of the parallelogram.

However, these students could not complete their justification. They needed to show that two parts of the parallelogram divided by lines passing through the intersecting point of two diagonals are located at a position of point symmetry, but they had difficulty verifying that segment EF in **Figure 12** is equally divided by such a point. To verify this property, the students had to divide the diagram into several triangles and verify their congruence; hence, their style of justification was more complicated than those of the students who used congruence.

Summary of Episode 3

There are three issues that emerge from this section. First, the students termed the perceptual commonalities among their examples, and this was supported by representing their examples in a single diagram. As we noted in the Episode 2, the students in groups 1 and 5 already termed commonalities among their examples as 'passing *the center*' or 'passing *the point of symmetry*.' In this Episode, the students in groups 2, 3, and 4 also termed commonalities among their examples while representing their examples in a single diagram. Hoffmann (2004, 2005) emphasized that diagrams synthesize relationships among mathematical objects. In this sense, the students could relate their examples in a single diagram and could focus on and term commonalities among their examples.

Second, the students' ways of inquiry changed from the moment they termed commonalities among examples. As we noted, the students' inquiries were mainly focused on abduction, deduction, and induction of their examples while they undertook example variation. On the other hand, they begin to characterize the named point (*the center, the point of symmetry*), as they named the point inside the parallelogram which was passed by their lines. As Sfard (1991) emphasized, condensation of students' actions is very important when students conceptualize and manipulate their actions. From the moment students termed the commonalities of their examples, they no longer needed to examine many examples. The students only had to examine the center or the point of symmetry to deal with numerous examples or generalities. Thus, we consider that the students' naming of commonalities among examples enabled them to deal with generalities in a tightly condensed manner rather than dealing diversely with numerous possible examples.

Third, the students could define the commonalities of their examples while justifying their abductions. To be more specific, the students attempted to justify their abduction using a general example from among their examples (**Figures 14, 15,** and **16**) as well as trivial examples which were diagonals of the parallelogram. The students could focus on their abductions rather than on single examples when attempting to characterize and justify every example. That is, the students' example generation and justification actions mainly focused on additional examples generated by them while undertaking example variation. The students described the commonalities of their examples, but they could not define them when resolving subtask 2. On the other hand, the students utilized a general example among their examples to characterize every example and this enabled the students to justify their abductions rather than single examples while resolving subtask 4. As a result, the students could define the commonalities among examples. The students' use of generic example also played a key role in the creation of their definitions, as they had to select a representative example from among their examples to define and justify their abductions.

DISCUSSION AND CONCLUSION

In this study, we aimed to investigate student inquiries into exemplifying and example generalization. We especially focused on students' use of abduction, induction, diagrams, and deduction as the students generalized their examples. As a result, we determined that the students in this study used abduction based on both deduction and diagrammatic reasoning and generalized their examples by gradually revising their abductions. Though we could not show the overall mechanism used for example generalization, we revealed that the sub-mechanisms undertake the generalization of examples as supported by the coordination of abduction, induction, deduction, and the use of diagrams.

First, we identified the synergic relationships among students' use of abduction, induction, diagrams, and deduction during the generalization of learner-generated examples. Studies emphasized identifying interactions among the constituent elements of their generalization actions, as abduction, induction, and deduction have been discussed as key elements of knowledge creation as well as generalization (Prawat, 1999). The relationship between the abductive phase and the inductive phase during pattern generalization was investigated (Rivera, 2010), and we empirically identified the relationships among abduction, induction, the use of diagrams, and deduction as it pertains to students' generalizations of geometric examples.

The students utilized abduction, induction, and deduction in two ways when generalizing their examples. The first set of students focused on deduction, initially attempting deductively to justify their examples and then constructed abductions and revised them through induction (**Figure 9**). The second set of students focused on diagrammatic reasoning (**Figure 13**). They first constructed an abduction while experimenting on their diagrams, after which they gradually revised their abductions through induction. Whereas the first set of students described the commonalities of examples with regard to the properties of their examples, the second set of students described the perceptual commonalities of examples. We also identified the key roles of diagrams for each type of example generalization by students. First, we found that the students who experienced V1 type example variation utilized their diagrams to generate additional examples which satisfied their abductions. These students used diagrams to modify their examples to generate further examples which satisfied their abductions (Episode 2). Second, we identified that the students' use of diagrams supported their construction of abductions for the V2 type example variation. Studies have theoretically emphasized that diagrammatic reasoning plays a key role in the construction of abductions through experimentation on and observations of diagrams (Hoffmann, 2004). Because diagrams signify relationships among particulars (Otte, 2006), the students could identify and manipulate the relationships and commonalities among their examples. To be more specific, the participants in this study identified relationships among the constituents of the parallelograms in diagrams. They then conducted experiments (e.g., rotated lines, found correspondences, combined their examples into one diagram) on diagrams and synthesized their example generation and variation tasks to construct abductions on the commonalities found in their examples (Episodes 2 and 3).

The types of student generalizations were not hierarchically divided in this study. Although we assumed that this difference is related to how the diagrams were used, further studies of the differences and similarities with regard to these two ways of coordinating three types of reasoning in other learning contexts are encouraged, as diagrammatic reasoning (Otte, 2006) and the Lakatosian approach (Lee, 2011) are both key methods of geometric inquiry, and we identified different ways by which they progress.

Compared to algebraic generalization, deductive reasoning played a key role in the generalization of geometric examples by supporting the construction of abductions and the defining of commonalities. To be more specific, the students' deductive reasoning supported the construction of abductions (Episodes 1 and 2) and encouraged the students to define commonalities among their examples (Episode 3). Researchers have reported students' construction of abductions when they attempt to justify a mathematical hypothesis (Pedemonte & Reid, 2011). The deductive process requires students to construct and utilize abductions from their observed results (Pedemonte & Reid, 2011), and this explanatory hypothesis about their examples also supports the students' efforts to describe the common properties of their examples in Episode 2. The students could also define the commonalities among examples while deductively justifying their abductions in Episode 3.

Second, naming commonalities among examples by the students helped them to pay attention to the generalities. When the students experienced example variation, they mainly focused on additional examples generated by them and attempted slightly to modify their abductions via these additional examples (Episode 2). As noted by Radford (2010), it is necessary when engaging in generalization to notice local commonalities and extend them to all cases. However, the students could not focus on general features as opposed to local commonalities during Episode 2. In Episode 3, the students named the commonalities of their examples and justified their characterizations, and this encouraged them to deal with generalities to handle the commonalities of every example, and they focused on abduction itself rather than on particular examples.

Third, the students' use of generic examples was an important factor in how they dealt with generalities. Although researchers have emphasized seeing generalities through examples and noted the role of exemplary examples (Watson & Mason, 2005), we especially identified the role of generic examples during the example generation process (Episode 1) and in defining the commonalities of examples (Episode 3). The students generated several abductions while undertaking example variation but did not clearly define their commonalities, as they did not focus on the most generic examples from among their examples (Episode 2). In Episode 3, the students could define the common features of their examples while utilizing a generic example to generate non-trivial and exemplary examples from among their examples. The students' attempts to organize their example space enabled them to focus on generic examples (Episode 1) as Watson & Mason (2005) theoretically noted. The diagrams also supported students in their efforts to link a generic example, trivial examples, and the properties of the parallelogram. To be more specific, the students drew generic examples and trivial examples in their diagrams, and they could define the commonalities of their examples by utilizing trivial examples (diagonals) and the properties of parallelograms (Episode 3).

In this study, we empirically confirmed the students' example generalization and the sub-mechanisms of their example generalization. Although we partially found a synergic relationship among the students' use of abduction, induction, diagram, and deduction, this effort remains incomplete. Moreover, a limitation of this research resides with the participants. The student participants constitute high-achieving or more advanced learners. The study groups did not contain achievers of differing levels. Another limitation relates to the limited use of exemplifying task types. Watson & Mason (2005) discussed various types of exemplifying tasks. Further studies involving exemplifying tasks of different types as well as task sequencing are encouraged in order to verify the possibility of including exemplifying in the teaching and learning of mathematics.

REFERENCES

Alock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures, *Educational Studies in Mathematics, 69*, 111-129.

Arzarello, F. & Sabena, C. (2008) Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics, 70*, 97–109.

Becker, J. R., & Rivera, F. (2005). Generalization strategies of beginning high school algebra students. In H. L. Chick, & J. L. Vincent (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 121–128). Melbourne: International Group for the Psychology of Mathematics Education.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 126-154). Prague: International Group for the Psychology of Mathematics Education.

Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., Yevdokimov, O. (2007). The conjecturing process: Perspectives in theory and implications in practice, *Journal of Teaching and Learning, 5*(1), 55-72.

Cobb, P., & Whitenack, J. W. (1996). A method for conducting longitudinal analysis of classroom video recordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213–228.

Cockburn, A. D. (2012). To Generalise, or not to generalise, that is the question, In B. Maj-Tatsis and K. Tatsis (Eds.), *Generalization in mathematics at all educational levels*, (pp. 11–21), University of Rzeszow, Rzeszow.

Davydov, V. V. (1990). Type of generalization in instruction: Logical and psychological problems in the structuring of school curricula. In J. Kilpatrick (Ed.), *Soviet studies in mathematics education* (Vol. 2). Reston, VA: National Council of Teachers of Mathematics.

Dreyfus, T. & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets, *Journal of Mathematical Behavior, 23*, 271-300

Durand-Gurrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective, *Educational Studies in Mathematics, 53*, 5-34.

Ellis, A. B. (2007a). A Taxonomy for Categorizing Generalizations: Generalizing actions and Reflection Generalizations, *The Journal of the Learning Sciences, 16*(2), 221-262.

Ellis, A. B. (2007b). Connections between Generalizing and Justifying: Students' Reasoning with Linear Relationships, *Journal for Research in Mathematics Education, 38*(3), 194-229.

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces, *Educational Studies in Mathematics, 69*, 183-194.

Hoffmann, M. H. G. (2004). How to get it. Diagrammatic reasoning as a tool of knowledge development and its pragmatic dimension, *Foundation of Science, 9*, 285-305.

Hoffmann, M. H. G. (2005). Signs as means for discoveries: Peirce and his concepts of "diagrammatic reasoning,' "theorematic deduction," "hypostatic abstraction," and "theoric transformation", In M. H. G. Hoffinann, J. Lenhard & F. Seeger (Eds.) *Activity and sign - Grounding mathematics education* (pp. 45-56). New York: Springer.

Lee, K.-H. (2011). Modelling of and conjecturing on a soccer ball in a Korean eighth grade mathematics classroom, *International Journal of Science and Mathematics Education, 9*, 751-769.

Maaß, K. (2006). What are modelling competencies? *ZDM-International Journal on Mathematics Education, 38*(2), 113-142.

Marton, F. (2006). Sameness and difference in transfer, *Journal of the Learning Sciences, 15*(4), 499-535.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.) *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht, Netherlands: Kluwer.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*, 277–289

Mason, J. (2011). Classifying and characterising: Provoking awareness of the use of a natural power in mathematics and in mathematical pedagogy, In O. Zaslavsky & P. Sullivan (Eds.) *Constructing knowledge for teaching secondary mathematics* (pp. 39-56), New York: Springer.

Meyer, M. (2010). Abduction—A logical view for investigating and initiating, processes of discovering mathematical coherences, *Educational Studies in Mathematics, 74*, 185-205.

Otte, M. (2006). Mathematical epistemology from a Peircean semiotic point of view, *Educational Studies in Mathematics, 61*, 11-38.

Park, J. & Lee, K.-H. (2016). How can students generalize the chain rule? The roles of abduction in mathematical modeling, *Eurasia Journal of Mathematics, Science, and Technology Education, 12*(9), 2331-2352.

Park, J., Park, M.-S., Park, M., Cho, J. & Lee, K.-H. (2013). Mathematical modelling as a facilitator to conceptualization of the derivative and the integral in a spreadsheet environment, *Teaching Mathematics and Its Applications, 32*, 123-139.

Pedemonte, B. & Reid, D. (2011). The role of abduction in proving processes, *Educational Studies in Mathematics, 76*, 281-303.

Peirce, C. S. (C.P.) (1931–1935, 1958) *Collected papers of Charles Sanders Peirce*. Cambridge, MA: Harvard University Press.

Prawat, R. S. (1999). Dewey, Peirce, and the learning paradox, *American Educational Research Journal, 36*(1), 47-76.

Presmeg, N. (1998). Metaphoric and metonymic signification in mathematics, *Journal of Mathematical Behavior, 17*(1), 25-32.

Presmeg, N. (2005) Metaphor and metonymy in processes of semiosis in mathematics education. In M. H. G. Hoffinann, J. Lenhard & F. Seeger (Eds.) *Activity and sign - Grounding mathematics education* (pp. 105-116). New York: Springer.

Pytlak, M. (2015). Learning geometry through paper-based experiences, In K. Krainer & N. Vondrová (Eds). *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 571-577). Prague, Czech Republic.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning, 5*(1), 37-70.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA-Pensamiento Numérico Avanzado, 4*(2), 37-62.

Rivera, F. D. (2010). Visual templates in pattern generalization activity, *Educational Studies in Mathematics, 73*, 297-328.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics, 22*, 1-36.

Sriraman, B. (2004). Reflective abstraction, uniframes and the formulation of generalizations, *Journal of Mathematical Behavior, 23*, 205-222.

Tall, D. O. (2013). *How humans learn to think mathematically*. New York: Cambridge University Press.

Watson, A. & Chick, H. (2011). Qualities of examples in learning and teaching, *ZDM-International Journal on Mathematics Education, 43*, 283-294

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates.

Watson, A. & Shipman, S. (2008). Using learner generated examples to introduce new concepts, *Educational Studies in Mathematics, 69*, 97-109.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*, 431–459.

Zazkis, R., Liljedahl, P. & Chernoff, E. J. (2008). The role of examples in forming and refuting generalizations, *ZDM-International Journal on Mathematics Education, 40*(1), 131-141.

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