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How do pre-service teachers view Galois theory? A questionnaire study

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Abstract

Abstract algebra is an important part of mathematics teacher education as it provides the rigorous mathematical foundations for many mathematics topics covered in school classrooms. Throughout their academic career, many secondary mathematics teachers even enroll in more advanced algebra courses, which often culminate in Galois theory. However, very little is known about educational aspects of Galois theory and respective mathematics education research is scarce. We contribute to closing this gap by reporting on the results of an exploratory questionnaire study with a sample of n=39 pre-service mathematics teachers, inquiring about the raison d'être of incorporating Galois theory into teacher education: Is Galois theory viewed as useful for their later profession and which connections are drawn to the secondary mathematics classroom? On the one hand, the results of our study indicate that a vast majority of pre-service teachers do not perceive studying Galois theory as meaningful and struggle to exemplify connections between Galois theory and secondary school mathematics. On the other hand, a small share of the participants experienced Galois theory as an important part of mathematics that elegantly connects a variety of algebraic and geometric notions.

Keywords: Galois theory, mathematics education, teacher education, higher education

INTRODUCTION

Abstract algebra is highly relevant for developing a profound understanding of algebra and of algebraic thinking (Sibgatullin et al., 2022). The previous scientific output regarding educational aspects of abstract algebra concepts such as groups, rings and fields, however, is marginal even though those mathematical objects can be traced through the entirety of the educational chain from primary to secondary to tertiary education (cf. Murray et al., 2017; Suominen, 2018; Veith & Bitzenbauer, 2022; Wasserman, 2014, 2018). In particular, mathematics teachers are often confronted with abstract algebra to some degree throughout their academic paths, indicating a consensus that those contents are relevant for their upcoming teaching profession. As such, a small but continuous effort has been put forward to improve mathematics teacher education in this regard, e.g., by developing

- teaching strategies such as TAAFU-materials (teaching abstract algebra for understanding) (Larsen, 2013; Larsen & Lockwood, 2013; Larsen et al., 2013) or Hildesheim teaching concept (Veith & Bitzenbauer, 2022) to promote algebraic thinking among learners.
- concept inventories such as GTCA (group theory concept assessment) (Melhuish & Fagan, 2018) or CI²GT (concept inventory for introductory group theory) (Veith et al., 2022a), which enable the tracking of students' conceptual understanding of aspects of abstract algebra prior, during or after interventions.

While the above mentioned instruments and teaching materials have been implemented to identify learning difficulties (Veith et al., 2022b; Larsen, 2010) and to explore cognitive processes (Veith et al., 2022c), the body of research runs dry quickly when it comes to more

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Contribution to the literature

- This study provides an assessment and description of pre-service mathematics teachers' retrospective views on Galois theory.
- This study presents an assessment of the perceived connections between school mathematics and Galois Theory by pre-service mathematics teachers.
- This study suggests implications for mathematics teacher training in the area of abstract algebra.

advanced topics, such as Galois theory. This can most likely be ascribed to the fact that Galois theory is not taught to teachers on every educational level and when taught often only constitutes the finale or an epilogue of an abstract algebra course (cf. Leuders, 2016). So far, only little research has been published that deals with educational aspects of Galois theory (Leuders, 2016). This absence of focus both in educational practice and educational research inevitably raises the question whether Galois theory should be taught at all or whether it is destined to be a discipline only worth exploring for pure mathematicians: How are mathematical contexts in school informed by Galois theory? How can knowledge of this abstract theory be helpful or relevant for teaching mathematics in classroom? And how can teacher educators excavate and emphasize those connections? In his article, which we will examine more thoroughly later, Leuders (2016) poses the following questions: "Which mathematics should teachers know? How should we teach them? And how do we know that we have succeeded?" (p. 2). While the first of the three questions will always be a normative, and–at least to some extent–a political one, it can be informed by answers to the third question: Namely, by evaluating the state of Galois theory education. To this day, however, no empirical research has been published on which such an evaluation could be based on. Thus, in this article we intend to contribute to the discussion of curriculum design regarding Galois theory by presenting the results of a questionnaire study, exploring pre-service teachers' retrospective views of their own Galois theory education. To this end, we will first briefly depict the fundamental ideas of Galois theory in order to connect them to various aspects of school mathematics. Subsequently, we will revisit Leuders' (2016) work to assess the degree to which school mathematics is touched on in current ways of teaching Galois theory. This will help contextualize the results presented and discussed later onward.

BUILDING BRIDGES: GALOIS THEORY & SECONDARY SCHOOL MATHEMATICS

To contextualize this work, we will first provide a brief mathematical overview of Galois theory and in a next step connect it to secondary school mathematics. It is noteworthy, however, that a detailed exploration of Galois theory is far beyond the scope of this research paper. Thus, only core concepts will be sketched and for the details we refer the reader to pertinent textbooks (cf. Dummit & Foote, 2003; Weintraub, 2008).

Galois Theory: A Brief Mathematical Perspective

Galois theory is a branch of abstract algebra that studies fields and their extensions. It was developed by the French mathematician Évariste Galois in the 19th century and has since become an important tool in many areas of mathematics, e.g., number theory and algebraic geometry (cf. Stewart, 2003).

Let K and L be fields with $K \subset L$. Then one of the key ideas of Galois theory is that the structure of certain field extensions L/K (Galois extensions, cf. Weintraub, 2008) can be related to the structure of the automorphisms of L that fix K. These automorphisms form the so-called Galois group:

$$\mathrm{Gal}(L/K) \coloneqq \{\sigma \in \mathrm{Aut}(L) \colon \sigma(k) = k \text{ for all } k \in K\}$$

of the Galois extension L/K. This group plays a central role in Galois theory, as it provides a way to study the structure of the extension and to understand its algebraic properties by means of group theory. Consequently, one of the most important results of Galois theory is the Fundamental Theorem of Galois theory, which states that there is a one-to-one correspondence between the intermediate fields of a finite Galois extension and the subgroups of its Galois group:

$$\{\text{intermediate field of } L/K\} \leftrightarrow \{\text{subgroup of } \mathrm{Gal}(L/K)\}\,.$$

The theory emerging from this can be utilized, among other things, to study roots of polynomial equations. For example, if $f \in K[X]$ is a (non-constant) polynomial with distinct roots, one can show that its splitting field L yields a Galois extension L/K. Thus, one can define Galois group of such a polynomial via $\mathrm{Gal}(f) \coloneqq \mathrm{Gal}(L/K)$.

Since L is obtained from K by adjoining the roots of f, e.g., $\alpha_1, \dots, \alpha_n$, and all automorphisms $\sigma \in \mathrm{Gal}(L/K)$ fixate K by definition, every σ is already defined by how it permutes the set $\{\alpha_1, \dots, \alpha_n\}$.

Consistent with this algebraic interdependence, one can show that a polynomial equation $f(X) = 0$ for certain polynomials f is solvable (through radicals, cf. Weintraub, 2008) if $\mathrm{Gal}(f)$ is solvable. This vastly opens up the areas of application for Galois theory since many problems in arithmetic and algebraic geometry relate to finding solutions of polynomial equations. We can conclude that Galois theory reduces the complexity of a problem by transferring it into a group theory context, allowing mathematicians to investigate it with well-studied objects. In other words, Galois theory

(a) is a prime example of how mathematical contents of seemingly different disciplines are interlinked and

(b) provides powerful tools to tackle highly complex problems in an elegant manner.

For university mathematics, this is certainly an obvious truism. Regarding secondary school mathematics, however, some more work needs to be done to demonstrate how knowledge of Galois theory can be relevant in educational settings–and hence, for pre-service mathematics teachers and their profession.

Connecting Galois Theory to Secondary Education

As sketched above, Galois theory is a branch of abstract algebra that has wide-ranging applications in many areas of mathematics. With regards to secondary school mathematics, Galois theory is particularly relevant for understanding the algebraic and geometric concepts underlying polynomial equations, straightedge and compass constructions as well as arithmetic. Thus, in this section we will explore the connections between Galois theory and secondary school mathematics by providing some examples of how the theory can be used to solve equations, construct geometric figures, and understand the arithmetic properties of number fields.

Algebra

In the realm of school algebra, many problems require students to find solutions of linear and quadratic equations, for example when intercepting geometrical objects such as parabolas and lines. In this particular context, the general quadratic equation $a_2x^2 + a_1x + a_0 = 0$ for some $a_0, a_1, a_2 \in \mathbb{R}$ can easily be solved and it is recognized that its solutions can be expressed algebraically, as Eq. (1):

$$x_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}. \tag{1}$$

This raises the broader question of whether equations of higher order can be approached similarly. In other words, can solutions of Eq. (2):

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0, \tag{2}$$

where $a_0, \ldots, a_n \in \mathbb{R}$, be expressed in terms of the coefficients $\alpha_1, \ldots, \alpha_n$ using only the basic binary operations in fields (addition, subtraction, multiplication, and division) and root extractions? While solutions similar to Eq. (1) exist for $n = 3$ and $n = 4$–known respectively as the cubic and quartic formulas–the Abel-Ruffini theorem establishes that these results cannot be generalized for $n \geq 5$ (Stewart, 2003). The original proof of this theorem predates Galois theory, but this theory facilitates the construction of counterexamples and significantly simplifies the proof by showing that

- for every $n \geq 5$ one can construct a polynomial f with $\deg(f) = n$ such that $\mathrm{Gal}(f) \cong S_n$.
- the symmetric group S_n is not solvable for $n \geq 5$.

As mentioned before, this directly implies that the equation $f(X) = 0$ is not solvable through radicals in cases, where $\deg(f) \geq 5$.

Consequently, the question arises as to whether solutions for Eq. (2) exist at all, leading to the fundamental theorem of algebra: The field $\mathbb{C}$ is algebraically closed. Galois theory once again offers a considerably simplified proof (cf. Bosch, 2004) by

- assuming that there exists a non-trivial finite extension $L/\mathbb{C}$ and
- inferring that there exists a $K \subset L$ with $[K:\mathbb{C}] = 2$, which contradicts that all $z \in \mathbb{C}$ have a root in $\mathbb{C}$.

In conclusion, Galois theory allows to study polynomial equations in a very elegant manner and provides concise answers for questions stemming from school algebra. In addition, it also provides a new perspective on the involved number sets via applications in arithmetic and number theory.

Arithmetic & number theory

Abstract algebra generalizes the ideas of operations on sets by employing the notion of "algebraic structure" in different contexts. This is used in arithmetic to construct the number sets $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ as well as $\mathbb{Z}_n$ and equip them with basic operations, e.g., such that polynomial equations can be solved. While abstract algebra itself offers a deeper insight into the algebraic properties of said structures, Galois theory can foster a more rigorous understanding of the involved algebraic concepts by connecting the key notions of groups and fields, as elaborated before. For example, using Galois theory one can view the field extension from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ in terms of modular arithmetic via $\mathrm{Gal}(\mathbb{Q})(\sqrt{3}, \sqrt{5})/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or view the extension fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ as virtually identical. Further connections between polynomial equations and modular arithmetic are established by cyclotomic fields (cf. Weintraub, 2008): If $\zeta \in \overline{\mathbb{Q}}$ is a primitive n^{th} root of unity then $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong \mathbb{Z}_n^*$.

Thus, through Galois theory seemingly disjoint topics (such as modular arithmetic and the unit circle in $\mathbb{C}$) can be unified, perhaps facilitating a more delicate view of mathematics as a whole. After all, modular arithmetic appears in many different guises throughout school mathematics but is never addressed as such (cf. Veith & Bitzenbauer, 2022; Wasserman, 2014, for a more in-depth discussion). This potential of Galois theory to intertwine mathematical ideas and reveal connections is especially remarkable regarding geometry.

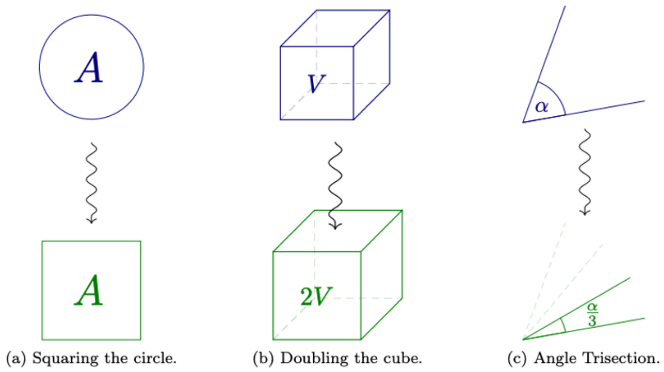

Figure 1. Illustration of three classical construction problems resolved by Galois theory (in general, it is not possible to construct figures shown in lower row from corresponding figures shown in upper row using only a compass & a straightedge) (Source: Authors' own elaboration, using TikZ in LaTeX)

Geometry

In school mathematics, students learn about geometrical figures by constructing them with tools on paper, e.g., by using a compass, a ruler or a set square (Freeman, 2010). On the one hand, this allows learners to explore geometrical properties of various figures. On the other hand, students can encounter algorithmic problems in a hands-on way, e.g., when tasked to construct the center of a line segment or an equilateral triangle. Historically, the constructibility of figures "is a key ingredient in Euclid geometry and [...] unconstructibility gave birth to famous open problems of the ancient Greece" (Schreck, 2019, p. 1). In the classical context, however, the construction of figures involved only a compass and a straightedge following a set of rules:

(1) Any two points may be adjoined through a line.

(2) For any two points one may draw a circle that is centered at one point and intersects the other.

(3) Any line segment may be transferred.

From this set of rules many problems originated that remained unsolved until Galois theory was developed, connecting the construction of lines with certain lengths to field extensions of $\mathbb{Q}$ (cf. **Figure 1**).

The examples shown in **Figure 1** illustrate how school mathematics (teachers) may profit from the perspectives offered by Galois theory.

Even apart from classical construction problems outlined above, geometrical thinking may be enriched by Galois theory, e.g., by mentally equating the symmetry of a polygons with the field extensions and thus drawing connections between number theory, abstract algebra and geometry. Moreover, Galois theory also offers applications outside of pure mathematics, for example in the field of cryptography (cf. Gueron & Kounavis, 2010; McGrew & Viega, 2004).

RESEARCH RATIONALE

As detailed above, there are numerous instances where Galois theory enriches educational contexts in school mathematics either by providing a more generalized and abstract view of the involved mathematics or by adding a sense of feasibility through real world implementations of mathematical objects. However, "although Galois theory is a core topic in most university algebra courses and often also part of teacher education programs there is only few literature, which discusses the role of this topic in university mathematics for future teachers" (Leuders, 2016, p. 2). To amend this, Leuders (2016) conducted a subject matter analysis in which he analyzed textbooks on Galois theory and the ways in which they cover this area of mathematics. The analysis concluded with three main approaches to Galois groups, namely

(1) "the Galois group as a group of automorphisms of a field extension,

(2) the Galois group as a group of permutations that leave the irreducible factors of the Galois resolvent polynomial invariant, and

(3) the Galois group as the stabilizer of the Galois ideal" (Leuders, 2016, p. 16).

However, not once did the subject matter analysis touch on topics closely related to but outside of algebra, such as geometry or cryptography. In other words, Galois theory is often presented in a rather self-contained way, and its linking elements are viewed as side characters. This raises the question whether Galois theory as incorporated in current mathematics teacher education is too secluded and the links to school mathematics detailed before are disclosed in sufficient manner. To answer this question one first needs to assess the teachers' retrospective view of their own Galois theory education. As mentioned before, however, research into educational aspects of Galois theory is still in its infancy and there is no empirical data to draw conclusions from. Thus, it is still unclear whether prospective mathematics teachers become aware of this theory's potentials and implementations after being instructed in Galois theory as part of their university study program. The purpose of this study is to investigate this circumstance by asking (German) mathematics teachers about their

(a) perceived connections between Galois theory and school mathematics and

(b) retrospective view on Galois theory being part of teacher education.

Hence, with this research we address the following research questions:

Research question 1: How do prospective mathematics teachers retrospectively perceive their own education in Galois theory?

Research question 2: Which connections between Galois theory and school mathematics are perceived by mathematics teachers?

METHODS

Study Design & Sample

We conducted an exploratory survey study to address a clarification of our research questions. The sample comprised a total of $n = 39$ prospective German secondary school mathematics teachers. The study participants reported that they participated in algebra courses in the amount of 13.01 ECTS (short for European credit transfer system) credits on average during their university studies. We have only considered responses from participants who stated that they had been introduced to Galois theory during their university studies. Participation in this study was on a voluntary basis and was not compensated.

Instrument

The questionnaire used in this survey study consisted of three parts. The first part was used to collect information about the study participants' academic background regarding abstract algebra. The second part consisted of questions aimed to explore the teachers' retrospective views of their own Galois theory education. This part consisted of five two-tier items addressing participants' perceptions of Galois theory as an interesting and challenging content of their teaching degree program, the inclusion of applications in their studies, the impact of Galois theory on their perspective of mathematics, and the retrospective evaluation of the value of learning about Galois theory in their teaching degree program.

In the first tier, the participants were asked to rate given statements on a four-point rating scale. In the second tier, the participants were then asked for a detailed description and justification of their rating in tier one. The last part of the questionnaire included two open-ended questions designed to gain insights into the connections between Galois theory and school mathematics as perceived by the teachers participating in the study. To ensure validity of our instrument, we conducted an expert survey involving a panel of two esteemed researchers of different universities with extensive expertise in abstract algebra research as well as teacher training. The expert survey encompassed both content evaluation on the one hand and linguistic refinement on the other hand. An overview of the items of questionnaire is provided in **Table A1** in **Appendix A**. The questionnaire was administered in an online format using SociSurvey tool (https://www.soscisurvey.de).

Data Analysis

The response patterns of the rating scale items are described to present an overview of the participants' opinions. However, a more comprehensive exploration of our research questions is achieved through qualitative content analysis of

(a) respondents' explanations and

(b) their answers to open-ended questions.

The formation of categories in this analysis was based on inductive reasoning. Throughout the coding process, all categories were treated equally, resulting in the exclusion of repeated occurrences of a specific category within a participant's transcript, as they did not contribute novel insights into their views–a similar argument has already been used in prior research, e.g., by Bitzenbauer and Meyn (2021). Consequently, these redundant instances were not counted multiple times during the frequency analysis used to determine category occurrences. Detailed information on the categories, including coding rules and anchor examples is provided later. Each study participants' responses were coded by two independent raters and all cases of disagreement were discussed in face-to-face sessions to achieve full agreement in the end.

RESULTS

Retrospective View of Teachers´ Own Education in Galois Theory

Findings from rating scale items

Figure 2 presents an overview of the response distribution of the study participants to the rating scale items II.1a), II.2a), II.3a), II.4a) and II.5a) (see **Table A1** in **Appendix A**) and provides a first descriptive overview of the retrospective view of teachers' own education in Galois theory. In the subsequent subsection, we will expand upon these findings by conducting qualitative content analysis of participants' responses to open-ended questions, aimed at eliciting explanations for their selections on the rating scale items.

Findings from open-ended questions

Among the participants, a significant portion (33.3%) viewed Galois theory as not relevant to their teaching profession (see **Table 1**). However, a considerable number of respondents found the logical and practical aspects of Galois theory interesting (15.4%). Some participants expressed interest due to the applications of Galois theory (15.4%) and the connection between field theory and group theory (10.3%). Additionally, a subset of respondents found Galois theory intriguing because they were able to comprehend it better compared to other topics (10.3%).

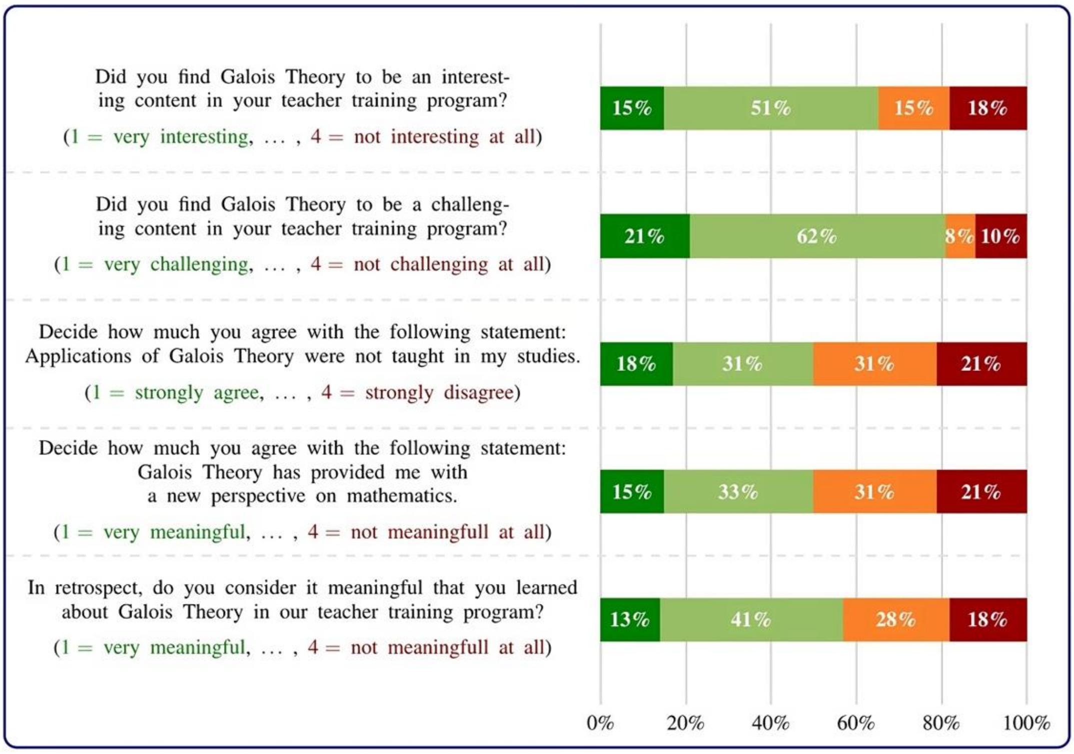

Figure 2. Overview of participants' response patterns in rating scale items as part of questionnaire to gain first insights into retrospective view of teachers' own education in Galois theory (Source: Authors' own elaboration, using TikZ in LaTeX)

Table 1. Categories (including explanation & anchor example) describing different justifications provided by participants regarding their interest in Galois theory alongside their (absolute & relative) frequencies

Category	Anchor example	Frequency
Not relevant for teaching profession.	"Little practical significance with regard to the future profession."	13 (33.3%)
Tasks related to Galois theory are logical & easy to practice.	"Galois theory is illustrative for students, as you can also ´calculate´ something in the corresponding tasks. It stands out somewhat from abstract algebra."	6 (15.4%)
Interest was sparked by applications of Galois theory.	"In the application, I got a good overview of which constructions are actually possible with ruler and compass. In mathematics lessons when I was at school, I rarely had to do such constructions myself."	6 (15.4%)
Interest was sparked by connection between field theory & group theory.	"In the course of the lecture on the theory of fields, I was already fascinated by the correspondence between intermediate fields of an extension of a field and the automorphism groups that allow the intermediate fields to be fixed."	4 (10.3%)
Galois theory was comprehensible, which sparked interest.	"One understood at least something, which is why it was more interesting than other topics."	4 (10.3%)
Missing explanation.	-	4 (10.3%)
Evariste Galois sparked interest.	"Evariste Galois has inspired me as a person."	2 (5.1%)

Furthermore, a majority (56.4%) perceived Galois theory as complex, particularly due to its theoretical nature (see **Table 2**).

Some participants (10.3%) highlighted the lack of allocated time for comprehension in modules as part of their study programs. The challenging aspect of understanding the connection between intermediate fields and subgroups was mentioned by a few participants (5.1%), as well as the lack of a standard procedure when solving tasks on Galois theory (5.1%).

Regarding Galois theory applications covered in the teachers' study programs, a significant portion (51.3%) reported that either no applications of Galois theory were covered during their studies or insufficient time was allocated for them (see **Table 3**). Additionally, some participants (20.5%) had limited or no memory of the applications that were taught. Interestingly, a subset of respondents (15.4%) even expressed difficulty in understanding the meaning of the term ``applications'' in the context of Galois theory. However, a small number of participants (12.8%) acknowledged that applications were indeed taught, although they felt the need for further review in some topics. We will come back to specific applications that are thought of by mathematics teachers explained later.

Table 2. Categories (including explanation & anchor example) describing different justifications provided by participants regarding how challenging they perceived Galois theory alongside their (absolute & relative) frequencies

Category	Anchor example	Frequency
Galois theory is complex.	"Galois theory is particularly complex, especially based on the theoretical tasks. If you don't fully understand and learn the theory, it becomes significantly more challenging compared to other topics in algebra."	22 (56.4%)
Accessible & practicable.	"In comparison to other content, it was easily accessible and practicable."	5 (12.8%)
Not enough time allocated in modules.	"Not particularly challenging, but squeezed in towards the end of the semester, leaving no time for comprehension (only in the exam preparation course)."	4 (10.3%)
Galois theory was understandable, especially compared to other areas.	"The beauty of the theory is that it solves many problems without being unnecessarily inflated."	4 (10.3%)
Understanding connection between intermediate fields & subgroups was challenging.	"The connections between intermediate fields and subgroups were not easy to see."	2 (5.1%)
No usual approach to solving tasks.	"There is a lack of a concrete approach to solving the tasks."	2 (5.1%)

Table 3. Categories (including explanation & anchor example) describing different responses provided by participants regarding applications of Galois theory introduced during university courses on abstract algebra alongside their total frequencies

Category	Anchor example	Frequency
No applications were covered, or not enough time was allocated.	"Applications were not mentioned."	20 (51.3%)
Participants have limited or no memory of applications.	"I can not remember well, but I think Sylow was an application we covered."	8 (20.5%)
Participants had difficulty understanding what is meant by "applications."	"It depends on what is meant by 'applications': Solving an algebraic problem (why unsolvable equations of degree five exist)? Or an application in another field?"	6 (15.4%)
Applications were taught, although some topics require further review.	"Applications were covered, although I would need to review all except angle construction."	5 (12.8%)

With our study, we further shed light on mathematics teachers' retrospective view of Galois theory education by asking as to whether Galois theory instruction offered a new perspective on mathematics as a whole: A majority (53.8%) perceived Galois theory as mandatory content in their study program, without providing a new perspective on mathematics (see **Table 4**).

However, a significant number of respondents (23.08%) reported an increased understanding of algebra as a whole, which led to a change in their perspective. Some participants (15.4%) highlighted the lasting application possibilities of Galois theory, particularly in terms of solvability of equations through closed formulas. A smaller subset of participants (7.7%) acknowledged some impact of Galois theory on their perspective, albeit to a limited extent.

Lastly, with regards to the perceived usefulness of participating in Galois theory courses, a considerable portion of study participants (41.0%) did not consider learning Galois theory useful (see **Table 5**) because they believed it was not needed in the teaching profession (see also the findings reported above). However, a significant number of respondents (20.5%) considered Galois theory valuable within the scope of field theory, emphasizing its application-oriented nature and its connections to group theory. Some participants (15.4%) mentioned the relevance of Galois theory for specific topics in school. A smaller subset of participants (7.7%) acknowledged the usefulness of Galois theory to some extent, particularly for understanding polynomials, their roots, and straightedge and compass constructions.

Perceived Connections Between Galois Theory & School Mathematics

In a first question, we asked the participating mathematics teachers to elaborate on inner-mathematical relevance of Galois theory (see **Table 6**). A significant proportion of the participants (35.9%) emphasized the applicability of Galois theory to specific problems, noting its ability to simplify certain concepts within mathematics. Unfortunately, the respondents did not elaborate further on what specific problems they had in mind. Another group of respondents (28.2%) highlighted the connection between Galois theory, field theory, and group theory, emphasizing its usefulness in solving equations. Some participants (20.5%) regarded Galois theory as the culmination of field theory, highlighting its significance within the broader mathematical framework. However, a smaller subset of participants (15.4%) expressed a perception that Galois

Table 4. Categories (including explanation & anchor example) describing different responses regarding question as to whether Galois theory education offered them a new perspective on mathematics alongside their total frequencies

Category	Anchor example	Frequency
No new perspective, seen as mandatory content of study program.	"I think for someone studying math because they want to be a mathematician, this is definitely an area that leads to new perspectives. As a teacher, I would (unfortunately) say that we took this module (field theory, which included Galois theory for us) because it is mandatory for the teaching degree program. But that applies to most modules in my math education studies."	21 (53.8%)
Increased understanding of algebra as a whole has changed perspective.	"Algebra became even more appealing to me, and I understood the connections between the relevant areas better."	9 (23.08%)
Remembrance of lasting application possibilities.	"It's not just about solving equations, but about solvability using closed formula."	6 (15.4%)
Some impact, to a limited extent.	"Yes, to a small extent. Whether it counts as one among many or as a significant perspective–probably somewhere in between."	3 (7.7%)

Table 5. Categories (including explanation & anchor example) describing different responses regarding question as to whether Galois theory education offered them a new perspective on mathematics alongside their total frequencies

Category	Anchor example	Frequency
Not useful as it is not needed in teaching profession.	"Unfortunately not. But that's not because of this theory. The study program is unfortunately very disconnected from the profession."	16 (41.0%)
Galois theory was interesting & therefore useful.	"Actually, I still find it quite valuable within scope of field theory, as Galois theory represents an area that is much more application-oriented than other areas. It also shows a connection to group theory & relationships between field extension & Galois group, for example. So, Galois theory certainly has its justification in study program."	8 (20.5%)
Interesting for specific topics in school.	"Relevant for some potential student questions."	6 (15.4%)
Solvable exam question & thus useful.	"Otherwise, I wouldn't have passed the algebra exam because it was covered."	6 (15.4%)
Yes, to some extent.	"Yeah, as I said, for understanding polynomials & their roots & constructions."	3 (7.7%)

Table 6. Categories (including explanation & anchor example) describing different justifications provided by participants regarding perceived relevance of Galois theory within mathematics alongside their (absolute & relative) frequencies

Category	Anchor example	Frequency
Applicability to specific problems simplifies certain concepts.	"It is an extension of field theory that provides tools that simplify certain problem statements."	14 (35.9%)
Connection between field theory & group theory makes it useful for solving equations.	"You can solve those annoying equations using group theory and therefore, the professors can ask less weird stuff in exams."	11 (28.2%)
Culmination of field theory.	"It is the culmination of field theory."	8 (20.5%)
No significance, at most important for passing exams.	"Is it not relevant? At most, it is important because it is required for the state examination."	6 (15.4%)

theory had limited significance, being primarily important for passing exams during the study program.

The key findings from the questionnaire question regarding the connections between Galois theory and school mathematics, as perceived by mathematics teachers, can be summarized, as follows (see **Table 7**).

A considerable proportion (38.5%) of the participating teachers did not recall any specific problems from school mathematics that could be addressed using Galois theory. However, those who provided examples highlighted geometric constructions (30.8%) and the solvability of equations (23.1%) as potential areas, where Galois theory could be applied. A smaller subset of respondents mentioned other fundamental questions of arithmetic (7.7%) as relevant to this connection, however, without being specific about which fundamental questions they had in mind.

DISCUSSION

The findings presented before draw a very clear picture: The overwhelming majority of prospective mathematics teachers see Galois theory as a particularly complex field with little practical relevance or applications for their future profession. They view their education in Galois theory as a mandatory part that was enforced by study regulations and exams. On the

Table 7. Categories (including explanation & anchor example) describing different problems from school mathematics that can be solved, in principle, using Galois theory from a subject-matter perspective alongside their frequencies

Category	Anchor example	Frequency
No recollection of any problems.	"Unfortunately, I can not think of anything at all."	15 (38.5%)
Geometric constructions.	"Can one construct a regular heptagon with only a compass & a straightedge? Can one trisect an angle with only a compass & a straightedge?"	12 (30.8%)
Solvability of equations.	"The solvability of quadratic, cubic, etc., equations through a general 'formula'."	9 (23.1%)
Other.	"Fundamental questions of arithmetic."	3 (7.7%)

contrary, a small but stable fraction of participants voiced positive attitudes towards Galois theory throughout all questions. Those participants further elaborated that Galois theory sparked their interest in algebra even more by establishing "connections between the relevant areas" or emphasizing the beauty of the theory by solving many problems "without being unnecessarily inflated". Noticeably, the share of pre-service teachers articulating interest in Galois theory is comparable in size to the share describing it as "easily accessible" and "illustrative". Of course, we want to refrain from equipping this observation with a causal relationship, but our findings suggest that Galois theory is interesting in particular for those students who find it accessible. A similar evident observation was made regarding more basic aspects of group and field theory (cf. Christy & Sparks, 2017). The discrepancy and the lack of perceived connections between Galois theory and secondary mathematics raises the question whether the mediation of Galois theory contents should place an even more pronounced focus on applications: 38.5% of participants were not able to think of a single instance in which Galois theory enriches mathematical contexts in secondary education (cf. **Table 6**).

Especially, in abstract algebra, prior research has shown that "students seem to appreciate a new concept when it is launched as a useful apparatus, not as an ideal that exists only because of its definition" (Nardi, 2000, p. 187). In this regard, Nardi (2000) differentiates two different approaches to abstract algebra: On the one hand, a *pragmatic* approach, i.e., one that convinces learners of the significance of a theorem or notion by emphasizing its relevance for exam problems. On the other hand, an *epistemological* approach, which is described as providing an existential rationale, e.g., by justifying how one concept is important to build further theory upon. In his study, Nardi (2000) concludes that pragmatic approaches work as stronger motivators while epistemological approaches act more powerful on a cognitive level. Our findings regarding Galois theory tie in closely with these observations as the participating pre-service teachers used precisely those two ways of describing their interest in Galois theory–for some the learning of Galois theory was solely motivated by it being "required for the exam" while for others it "is the culmination of field theory". These varying modes of relevance for higher mathematics were studied before by Even (2011) who conducted a study in which 15 pre-service teachers, enrolled in a special master's program, expressed and explored the modes of relevance of higher mathematics education for classroom instruction. The study participants' views could be categorized into three distinct perspectives: Advanced mathematics courses

(a) as a resource for teaching secondary school mathematics,

(b) for advancing teachers' understanding about what mathematics is, and

(c) for reminding teachers what learning mathematics feels like.

While this categorization is a rather general one we argue that specifically the second view is fostered by Galois theory. After all, as elaborated on before, Galois theory constitutes a recurrent theme that links various elements of school mathematics along the entire chain of education–from the very first binary operations on $\mathbb{N}$ and $\mathbb{Z}$ that are studied in primary school to the first fields $\mathbb{Q}$ and $\mathbb{R}$ in lower secondary education, all the way to polynomial functions and symmetry in higher secondary education. Galois theory provides the rigorous mathematical foundation for these notions while also offering insights into adjacent topics such as geometry and cryptography and as such can be seen as a prime example of advancing mathematical understanding on an epistemological level. In this regard, the teachers in Even's (2011) study further emphasized that before teaching something, "a teacher needs to have a comprehensive background with a very wide-ranging basis. This is because a teacher needs to know where all the topics that he teaches "go", what one does with them later on and what mathematical theorems are implied from them" (Even, 2011, p. 945). This ascertainment is congruent with the assessed view of mathematicians in a study by Yan and Marmour (2022) according to which "the value of advanced mathematics courses for prospective secondary mathematics teachers lies in their potential to offer connections across mathematical domains, mathematical experience for the development of problem-solving abilities and increased epistemological awareness of the subject" (Yan & Marmour, 2022, p. 553). These aspects further manifested in a follow-up study by Shamash et al. (2018), where pre-service teachers were prompted to design a teaching unit dedicated to solving polynomial equations: The participating teachers chose a historical narrative for their unit in which they introduced the evolution of solving polynomial equations through radicals via Galois theory. Of course, the algebraic structures were never addressed in classroom, but played a crucial role in planning and designing the teaching unit. Consequently, it is both surprising and alarming that only 23.1% of our study participants connected Galois theory to secondary education by considering solvability of polynomial equations (cf. **Table 7**) while a higher share of 28.2% considered this connection only in terms of their own university exams (cf. **Table 6**). Overall, with 41.0% of pre-service teachers viewing their Galois theory education as essentially useless and disconnected from their later profession, it becomes clear that in university courses more time needs to be allocated towards exploring the applications of Galois theory to realize its epistemological potential. In other words, as long as pre-service teachers remain in a pragmatic stance (i.e., learning it only to pass exams) the value of Galois theory for secondary education will be suppressed by the feeling of enforced study regulations.

Limitations

There are several limitations that merit consideration in order to better contextualize our results. Firstly, it is essential to acknowledge the relatively low sample size of only $n = 39$ pre-service teachers. This is partly caused by Galois theory being located at the very end of the academic path with only a handful of secondary school teachers remaining to be questioned. Furthermore, it is crucial to recognize that questionnaire-based investigations inherently possess certain limitations: One noteworthy constraint is the inability to disambiguate vague terms such as "application" or "interesting" as these concepts can be interpreted differently by individuals. On the other hand, our research solely focused on capturing the teachers' perspectives at a surface level, thus limiting our ability to engage in detailed discussions or solicit further elaborations on their responses. A last but important limitation relates to the existing literature background: There is not a profound empirical research background regarding Galois theory education as far as we know. Consequently, our discussion had to rely on drawing connections between our findings and analogous domains within abstract algebra education. Although these connections provide valuable insights, they should be interpreted with caution due to the absence of direct empirical evidence within the specific context of Galois theory education.

CONCLUSIONS & OUTLOOK

While general difficulties of prospective teachers in excavating connections between school mathematics and abstract algebra concepts have been identified in prior research (Larsen, 2010; Veith et al., 2022b; Zaslavsky & Peled, 1996), similar observations appear to accompany Galois theory education. The overwhelming majority of pre-service teachers participating in our study experience this discipline as complex and struggle to equip this theory with a sense of meaningfulness for their later profession. The absence of provided exemplifications of such connections suggests that they were either not touched upon or not studied thoroughly enough in the respective Galois theory courses. This should be kept in mind by educators when designing courses specifically for prospective mathematics teachers.

However, as mentioned before, Galois theory education research is still in its infancy and the results of this first exploratory study can be substantiated significantly by further research. On the one hand, more in-depth empirical research is required, e.g., by conducting structured interviews or teaching experiments as has been done for other abstract algebra topics (Cook, 2018). On the other hand, the direct impact of Galois theory needs to be researched more thoroughly: How does introducing pre-service teachers to Galois theory inform and influence their instructional practices for teaching algebraic topics in secondary classroom? We will tackle this question in more detail in future research.

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APPENDIX A

Table A1. Overview of questionnaire items used in this study

Part	Questions	Answer format
I. Background	I.1) With this question, we aim to assess the extent of your algebra training in your mathematics teacher training program. Indicate the number of ECTS/CPs/credit points you acquired in algebra as part of your mathematics studies (excluding linear algebra).	Open-ended
	I.2) Was Galois theory an explicit component of your teacher training program (e.g., in the form of a lecture or seminar)?	Single-choice (yes/no)
II. Retrospection (see **RQ1**)	II.1 a) Did you find Galois theory to be an interesting content in your teacher training program?	Rating-scale (4=not interesting at all, ..., 1=very interesting)
	II.1 b) Please explain your decision in the above question, preferably with the help of examples.	Open-ended
	II.2 a) Did you find Galois theory to be a challenging content in your teacher training program?	Rating-scale (4=not challenging at all, ..., 1=very challenging)
	II.2 b) Please explain your decision in the above question, preferably with the help of examples.	Open-ended
	II.3 a) Decide how much you agree with the following statement: Applications of Galois theory were not taught in my studies.	Rating-scale (4=strongly disagree, ..., 1=strongly agree)
	II.3 b) Please explain your decision in the above question, preferably with the help of examples.	Open-ended
	II.4 a) Decide how much you agree with the following statement: Galois theory has provided me with a new perspective on mathematics.	Rating-scale (4=strongly disagree, ..., 1=strongly agree)
	II.4 b) Please explain your decision in the above question, preferably with the help of examples	Open-ended
	II.5 a) In retrospect, do you consider it meaningful that you learned about Galois theory in your teacher training program?	Rating-scale (4=not meaningful at all, ..., 1=very meaningful)
	II.5 b) Please explain your decision in the above question, preferably with the help of examples.	Open-ended
III. Connections (see **RQ2**)	III. 1) Please complete the following statement in detail. Feel free to use examples. “Within the subject of mathematics, Galois theory is important because ...”	Open-ended
	III. 2) Provide possible problems in school mathematics that can be addressed from a subject-mathematical perspective (not in the classroom) using Galois theory. The corresponding subject-mathematical arguments do not need to be provided.	Open-ended