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Improving the learning of geometric proportionality using van Hiele's model, mathematical visualization, and GeoGebra

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Abstract

This qualitative, action research methodology study aims at the construction of a teaching unit for the development of learning processes, analyzing the learning trajectories of students around the study of geometric proportionality in topics related to similarity, homothecy, and Thales' theorem. The didactic unit was designed under the principles of the phases of the van Hiele's (1986) model, mathematical visualization, and the use of GeoGebra software. Among the results obtained, it can be evidenced that students went from not having a clear notion of similarity to constructing a definition for similarity, proportionality, and homothecy; from not identifying criteria for similarity to identifying and understanding the mathematical properties that remain invariant in similar figures; from not using visualization skills and not communicating their arguments using an appropriate mathematical language to using visualization skills and processes.

Keywords: geometric proportionality, van Hiele's model, mathematical visualization, GeoGebra, learning trajectories, teaching unit

INTRODUCTION

The teaching of mathematics is a constant challenge for the community of educators. Depending on how it is practiced, correct development of learning can be guaranteed or not. In the area of geometry, the teaching-learning process has been carried exclusively in the rote study of concepts such as areas, volumes, some geometric definitions, and the solution of completely decontextualized problems (Gualdrón, 2011; Gualdrón et al., 2020). This creates difficulties in learning geometry and concepts that relate to other areas of mathematics (Afonso, 2003).

On the other hand, Array et al. (2019) suggest that the lack of geometry teaching or the absence of teaching of some topics of school geometry affects comprehension in different subjects of the university cycle. Among the geometric concepts, whose lack of understanding hinders the learning of other fundamental mathematical concepts, are those related to proportionality, similarity, and homothecy (Gualdrón, 2011). These concepts are addressed, according to the Colombian curriculum, between the ages of 14-16 years, specifically in the ninth and tenth grades of high school, and are of interest due to their implications, especially in some of the first courses of science and engineering careers.

In Colombia, basic competency standards [or Estándares Básicos de Competencia-EBC] (MEN, 2006) propose the teaching of the similarity of plane figures from the first years of schooling. First, the similarity of figures should be worked on in the recognition of their enlargement or reduction. As they advance in their schooling, students should be able to identify some similarity relationships between figures and justify them, later, they should be able to use those justifications and properties for problem-solving through visualization skills and justification of theorems and similarity criteria (Chávez, 2012).

However, the reality in practice is very different. The subject of similarity is approached in most cases until the

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Contribution to the literature

- The teaching unit used for the development of this study is one of the most significant contributions that can be rescued, since it allowed, through the transit through the learning phases of van Hiele's (1986) model, adequate handling of the GeoGebra tool. By organizing the classroom intervention in five moments, the students were able to interact with the software in different ways. In phase 1, they identified prior knowledge in relation to the topic of study. In phase 2, they worked under the guidance of the teacher, through instructions; for example, to carry out some construction or the simple task of measuring angles. In phase 3, they conceptualized the topics studied with the teacher's help. In phase 4, they explored on their own, applying what they had learned in the previous phases and using the software to solve novel tasks. In phase 5, they shared their learning and conclusions with their peers, assimilating and accommodating the acquired knowledge.
- The characterization of the students' learning trajectories allowed verifying the evolution in the learning process, its analysis indicates that the development of the teaching unit favors the learning of proportionality, similarity, and homothecy and, on the other side, that this can be replicated in teaching units that address other mathematical objects.
- The results of this study confirm that the use of dynamic geometry, especially the GeoGebra software together with the development of well-structured teaching units, from van Hiele's (1986) model, contributed to the development of geometric reasoning.

ninth grade, in a decontextualized way. Students have not received sufficient teaching on the subject and when they reach ninth grade, they only recognize the enlargement and reduction of some figures. Furthermore, in EBC geometric proportionality is not explicit anywhere; it is only present in relation to numerical thinking and its arithmetical properties, and therefore the textbooks do not address it. Geometric proportionality is of vital importance for the understanding of the concept of similarity (Chávez, 2012).

Continuing with the traditional way of teaching mathematics does not contribute to the formation of competent individuals with twenty-first-century skills (ICFES, 2019), this limits the correct development of mathematical skills necessary for the understanding of concepts in other areas of knowledge. At the basic secondary level, it creates limitations reflected at the university level. It is the duty of teachers to contribute with strategies that allow better learning.

Teaching activities must be well thought out, involving the context and, in a certain way, motivating students to learn. Still, it is also necessary that these activities, tasks, or strategies are grounded to ensure that they actually fulfill their purpose. In the case of geometry, the use of dynamic software has revolutionized its teaching, generating a new line of research aimed at the construction of strategies that promote its use, but also verify its contribution to the construction of mathematical reasoning (Parada et al., 2023).

Based on the above, in this paper, we report the results of a study that addressed the research question: How do students reason geometrically when they study the concepts of similarity, proportionality, and homothecy using GeoGebra? To solve the question, we proposed the design and experimentation of a teaching unit using van Hiele's (1986) model, together with elements of mathematical visualization and the use of dynamic geometry, especially GeoGebra software, which aimed to promote the development of geometric reasoning in the study of the concepts of proportionality, similarity and homothecy in ninth grade students in a private institution in the city of Bucaramanga-Colombia.

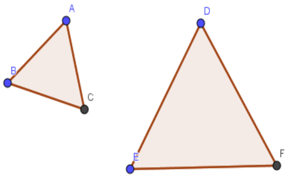
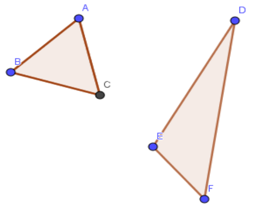
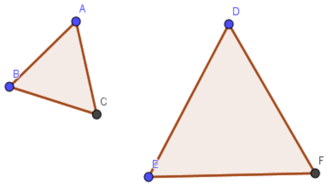
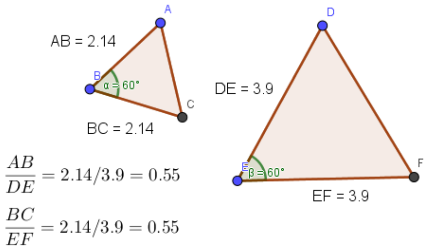
THEORETICAL FRAMEWORK

This section presents the theoretical elements that allowed the development of this research. As previously mentioned, we sought to analyze the learning process of proportionality, similarity, and homothecy in ninth-grade students, taking into account van Hiele's (1986) model, mathematical visualization, and GeoGebra; in this sense, the construction of the theoretical framework has three fundamental aspects, first, the characteristics of working with GeoGebra, followed by the presentation of van Hiele's (1986) model exposing the two parts of the model (descriptive and instructive) and, finally, the theoretical elements of mathematical visualization.

Dynamic Geometry Software

Dynamic geometry software supports the learning of mathematics, particularly geometric concepts, by proposing different ways of reasoning. According to MEN (2004), there is a substantial difference between working with dynamic geometry on the screen and geometry with pencil and paper, and it is precisely its dynamism "as the configurations are dynamic, the figures on the screen acquire a temporality: they are no longer static, but mobile, and therefore their properties must be present in all the possible positions they take on the screen" (p. 19).

Table 1. Principles of working with dynamic geometry, example with similarity

Doubting what is seen Is ΔABC similar to ΔDEF? 	This exercise is intended to verify the similarity of the triangles. At first glance, the student might say that "yes they are", but by dragging on one of the triangles, and changing its shape, the student modifies his perception of the similarity characteristic that the triangles shared.	Is ΔABC similar to ΔDE?  These no longer have the same shape.
Seeing beyond what is seen Is ΔABC similar to ΔDEF? 	If it is verified by dragging that at first sight, they seem similar, the student can use tools that allow verifying similarity, for example, measuring their angles and sides to verify the congruence of angles and the proportionality of the sides.	Is ΔABC similar to ΔDEF?  Here again, we must go to the first principle and, through dragging, verify that, no matter how the triangle is moved, its similarity characteristics do not change.

In addition, Colombian Ministry of National Education (MEN, 2004) states that working with dynamic geometry involves two fundamental principles to develop its potential: doubting what is seen, which is the use of dragging as a method to test the veracity of the relationships perceived in a static image, and seeing beyond what is seen, which implies enriching the static image through geometric construction to validate the perceived relationships (MEN, 2004). **Table 1** shows what the principles of working with dynamic geometry look like in relation to similarity.

On the other hand, Patsiomitou (2019) states some of the functions of dynamic geometry software, as follows:

1. As a construction tool: It provides an accurate constructor to create geometric configurations and has the ability to adjust and preserve variant and invariant properties of geometric configurations constructed under dragging in a visual, efficient and dynamic way.
2. As a visualization tool: It allows the use of visualization skills.
3. As a modeling tool: It allows the construction of mathematical models in ideal environments.
4. As a tool for experimentation, exploration, and discovery: When interacting with the software while studying a geometric concept, for example in GeoGebra to build an equilateral triangle you can go to the regular polygon tool and mark the number of sides, so a student could define an equilateral triangle as a regular polygon with three sides from their experience with the software.
5. As a tool for problem-solving and problem posing: The diagrams provided are important spatial representations that facilitate the understanding of the problem information as well as the conceptualization of the problem (p. 72).

van Hiele's Model

Dutch professors Pierre Marie and Dina van Hiele-Geldof consolidated this model under the main idea that in the process of learning geometry, students' reasoning increases gradually by passing through categorized, sequenced, and ordered levels without skipping any (van Hiele, 1986). The model has been recognized by several researchers as a powerful theoretical framework in the construction of teaching units, especially when addressing specific topics, including the concepts of similarity, proportionality, and homothecy (Gualdrón, 2011).

Among the characteristics presented by the model, from the work of van Hiele (1986), are recursivity, which allows the evolution of previous learning; Hierarchy and sequence of levels: it is not possible to access a higher level without having gone through the previous ones; language: as one advances through the levels, the language is transformed, it becomes more mathematical; locality: refers to the understanding of mathematical objects, the student may be at level 2 in a specific topic, while in another, he may be located at level 1; continuity: the passage from one level to another is done slowly, it may have passed one level and be slowly transiting through the next; instruction: the teacher must guide the learning process through the phases of the model.

van Hiele's (1986) model addresses two aspects, the descriptive one through which the "levels of geometric reasoning" are identified for the assessment of individuals' learning, and the instructive one, which suggests some guidelines on how sequences can be developed in the classroom to help students in the transition from one level to another, called "learning phases".

The following are the levels of geometric reasoning from van Hiele's (1986) model for topics related to geometric proportionality proposed by Gualdrón (2011), considering that we will only analyze the first two levels.

Level 1 (recognition): At this level students perceive the similarity of figures globally. The following aspects are considered:

1. They recognize similar figures based on their appearance, that is, using only visual strategies. There may be cases in which they do not recognize the similarity between two figures because one of them is rotated.
2. They see shapes as a whole and describe the differences and similarities between them using terms such as "larger", "smaller", "stretched", and "enlarged." For example, when a student is deciding on the similarity of two rectangles, they might say "this rectangle is not as long as this one." They may also include irrelevant attributes in the descriptions they make.
3. They begin to perceive some mathematical features of similarity but still do so in isolation. For example, some may take angle measurements and realize that in similar figures they are the same, but they do not see this as a necessary condition for similarity.
4. Can identify, using visual arguments, the similarity between figures when they belong to a Thales configuration (projection aspect or homothetic aspect) or are in homothetic arrangement.
5. Can identify and explain, using visual arguments, the similarity of figures in mosaics.
6. Construct or draw figures similar to a given figure without explicitly considering mathematical aspects such as angle measure or side lengths (p. 209).

Level 2 (analysis): At this level students are already aware that it is not enough to observe figures and decide them by their resemblance, but that there are also mathematical conditions of the similarity of figures that must be fulfilled through their elements and properties. Level 2 is characterized by the following aspects:

1. Construct or draw figures similar to a given figure explicitly considering mathematical aspects such as angle measure or side lengths.
2. Determine specific mathematical aspects of similar figures, such as the proportionality of segment lengths and the equality of angle measures, so they can induce the conditions necessary for the figures to be similar.
3. Discover that the position of similar figures is irrelevant, that is, it is not necessary for similar figures to have the same position.
4. To understand that the congruence of plane figures is a particular case of the similarity of plane figures.
5. To induce some properties related to similarity in right triangles.
6. Understand that the resulting figure when applying a homothecy is similar to the given figure.
7. Relate the similarity of triangles to Thales' theorem, understanding that triangles in Thales' position are similar.
8. Use configurations of triangles in Thales's position or in homothetic arrangement (with the center of homothety at a vertex) to demonstrate similarity relationships between them.
9. Make constructions or drawings of similar figures by giving them the similarity factor and also predict whether the resulting figure will be an enlargement, a reduction, or a figure identical to the given one. They can also make constructions or drawings of similar figures using homothecies and Thales' theorem.
10. Relate the ratio of similarity to scales. That is, understand that scale is the ratio of similarity between a reproduction (photo, map, plan, etc.) and the reality that the reproduction represents.
11. Identify similarity relationships in complex plane figures (two or more intersecting plane figures).
12. Demonstrate properties that have to do with the similarity of figures verifying that they are fulfilled in some cases. They deduce mathematical properties of similarity through experimentation and generalization. In addition, they generalize these properties to other types of figures.
13. Use the definition of similarity to solve mathematical situations, e.g., determining accessible or inaccessible lengths.
14. Identify the similarity of figures by relating it to the transformations of enlargement and reduction of one figure with respect to another.
15. Understand that rectangles coincident at a vertex and sharing a diagonal are similar (p. 210).

Another important aspect of the model is the learning phases, used in the development of the teaching unit.

1. **Phase 1:** Information. In this first stage, the teacher investigates the level of prior knowledge of the

students about the teaching concept. The exercises presented aim at highlighting the invariant characteristics shared by similar figures.

2. **Phase 2:** Guided orientation. In this stage, the teacher guides the students to develop knowledge about the topic in question. Using tools such as GeoGebra, students can construct figures and better understand their properties. These tasks are intended to prepare them to acquire a better understanding of the concept.
3. **Phase 3:** Explicitness. In this stage, the concepts being developed are reinforced, through conceptual formalization, using appropriate technical language and identifying the characteristics, properties, and relationships established. The assigned tasks focus on conceptualizing the topic of study together with the teacher, such as the formulation of definitions and a list of criteria to determine similarity, among others.
4. **Phase 4:** Free orientation. The student should have the opportunity to establish his or her own relationships with the objects of study, therefore, the teacher's participation should be limited to allow the student to develop his or her skills on his or her own. In this phase, the teacher should propose tasks that are creative, novel, and different from those previously used.
5. **Phase 5:** Integration. In this stage, the activity is concluded, where a summary of the learning achieved is evidenced. New topics are not developed, but rather those already acquired are compiled and organized. The tasks in this phase focus on students sharing and socializing the progress of their learning.

Visualization

For the development of this research, regarding the elements of visualization, the contributions made by Gutiérrez (1996), who exposes elements of visualization, were considered. According to this author, visualization in mathematics is a form of reasoning that uses visual or spatial elements, both mental and physical, in order to solve problems or verify properties.

1. **Mental image** is a way of representing a concept or mathematical property in the mind, using visual, graphic, or schematic elements to store information. Gutiérrez (1996) considers that "only a few types of mental images are necessary to solve a certain type of task" (p. 9). The following are the mental images considered for this study.
2. **Pattern images** that represent abstract mathematical relationships visually.
3. **Formula images** for some students may "see" in their mind a formula as it appeared written on the board or in the textbook.
4. **Kinesthetic images** that are created, transformed, or communicated with the help of physical movements.
5. **Dynamic images** with movement in the mind (p. 7).

External representation

According to Gutiérrez (1996), an external representation can be any type of verbal or graphic representation of concepts or properties. This includes images, drawings, diagrams, etc., which help to form or modify mental images and to perform visual reasoning.

Visualization processes

A mental or physical action involving mental imagery. There are two processes performed in visualization: "visual interpretation of information (VP)" to create mental images and "interpretation of mental images (IFI)" to generate information. Gutierrez (1996) established these visualization processes from the contributions of Bishop (1983). VP involves visualization, interpretation of relationships, and non-figurative data, as well as manipulation and transformation of visual representations and images. On the other hand, the IFI process involves the understanding of visual conventions, the reading of images to obtain useful information, and the use of spatial vocabulary for geometric works, charts, and diagrams. These processes were considered when designing the teaching unit since the tasks contain information for students to use each of the processes.

Visualization skills

Students need to develop and improve their visualization skills in order to be able to interpret and solve problems by forming mental images. Gualdrón (2011) specifies several visualization skills that were considered in the study.

1. **Visual identification skill (VI):** This is used, for example, to recognize the similarity of polygons in a figure composed of several overlapping or intermingled figures.
2. **The ability of mental rotation (MR):** This is the ability necessary, in the context of similarity, to mentally rotate a figure and imagine it superimposed on another and identify whether or not they are similar.
3. **The skill of conservation of perception (CP):** This is the skill necessary, for example, when recognizing that a figure, which has been given movement to verify similarity with another, maintains its shape and mathematical properties.

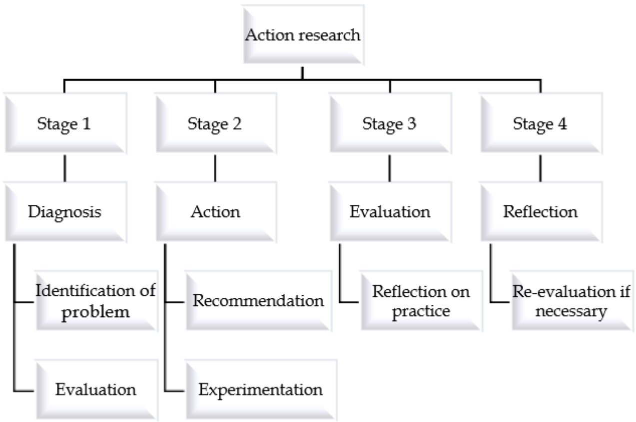


Figure 1. Stages for conducting action research (Source: Authors' own elaboration)

4. **The ability to recognize positions in space (RP):** This is the ability that allows, for example, to identify that a figure can "fit" into another in order to verify its similarity.
5. **The ability to recognize spatial relationships (RR):** This is the ability necessary to identify mathematical relationships between two figures, for example, to identify Thales' theorem in a graphical configuration.
6. **Visual memory skill (VM):** The skill needed, for example, to remember the visual and positional characteristics of triangles in Thales's position.
7. **Visual discrimination skill (DV):** It is the ability to identify visual relationships between two figures, for example, to compare several geometric figures or photographs by identifying their visual similarities and differences in order to identify their similarity (p. 88).

In conclusion, the theoretical framework gives solid foundations for the construction of the teaching unit and, consequently, for conducting the data analysis and generating valuable conclusions within the study.

Teaching Unit

Gutiérrez et al. (2021) define the teaching unit as an organized set of activities, not necessarily ordered sequentially, that has an overall learning objective and several specific learning objectives. These authors refer that the tasks are the diversity of assignments to students such as exercises, problems, evaluation questions, research, etc. And activities are tasks focused on direct, explicit, or implicit use of a mathematical content or procedure. Problems are activities, where students, after reading the statement, have difficulties solving them, because they either do not have the necessary prior knowledge or because they do not know a way to link knowledge with the statement to find a solution (p. 3).

In this sense, instructional planning is an essential part of the teaching-learning process. This implies organizing efficiently the mathematical content, as well as the activities, tasks, and problems necessary to achieve the established learning objectives. Such a process must consider the time and resources available for its execution.

Learning Trajectories

The learning trajectory refers to the cognitive development that a learner experiences, as a construct when confronted with an educational activity. This learning trajectory is based on the work done by Clements and Sarama (2004), Patsiomitou (2019), and Simon (1995), which highlights the reasoning, procedures, and skills that the student uses when studying a concept. Since it is a mental construction that does not occur in a linear way, students can go through different trajectories for the study of the same concept. Some authors define these trajectories as "real learning trajectories" that serve to analyze the evolution of students' thinking when faced with significant teaching activities.

METHODOLOGY

This research is of a descriptive qualitative nature under the action research methodology. According to Hernández et al. (2014), "qualitative research focuses on understanding phenomena, exploring them from the perspective of the participants in a natural environment and in relation to their context" (p. 358). In this sense, in this particular case, we are going to intervene directly with ninth-grade students in the context of the educational institution (hereinafter EI), where they develop, so that their learning process can be evaluated and interpreted.

On the other hand, action research is characterized by studying local practices (of the group or community), involves individual or team inquiry, it focuses on the development and learning of the participants, it seeks to implement an action plan to solve the problem, introduce improvement or generate change; all while being guided by the researcher and one or more members of the group or community. Action research contemplates four basic phases or cycles: diagnosis, action, evaluation, and reflection.

Segal (2009) identified six stages for conducting action research: identification of the problem, evaluation, recommendation, experimentation, reflection on practice, and re-evaluation if necessary. These stages are aimed at defining the study problem, observing, studying, seeking background, creating a plan, implementing the recommendation, evaluating the recommended practice, and modifying the plan if needed. In this research, the previously described stages were associated with the phases of action research. These were established as moments of study. **Figure 1** shows the moments that took place during this study.

Stage 1: Diagnosis

In order to identify the problem, apart from the bibliographic review, it was necessary to characterize the initial state of learning in geometric proportionality, including similarity and homothecy, in the study population, considering their contextual reality. For this purpose, it was necessary to construct a diagnostic test that would allow evidencing the level of geometric reasoning and the use of visualization. Each question of the test allowed the students to express their reasoning, showing the different visualization skills they use when facing the different types of tasks presented to them. For the analysis of the study, some research elements of Gualdrón (2011) were considered.

Stage 2: Action

A teaching unit on the topic of study was designed, considering the use of the dynamic geometry software GeoGebra, the levels of geometric reasoning, the learning phases, and visualization elements. The teaching unit was constructed recognizing that the population under study has contextual and institutional curricular characteristics specific to ninth-grade students.

Stage 3: Evaluation

The analysis of the students' learning trajectories was carried out considering the theoretical framework as its support. In this sense, the data obtained by the different collection instruments were analyzed, making a contrast between the diagnostic test and the final test to show the evolution of the learning process, its transit between the levels of geometric reasoning, the processes, and visualization skills that took place.

Stage 4: Reflection

At this point, conclusions were included in response to the objectives proposed at the beginning of the research, bearing in mind the considerations or limitations that were presented in the study. Also, future research works and their scope in the development of mathematics didactics are projected.

Design of Teaching Unit

The activities that make up the teaching unit are presented below. The theoretical elements that support it are included: van Hiele's (1986) theoretical model, the elements of visualization, and the use of GeoGebra.

In this sense, the teaching of similarity can be introduced in different ways: "intuitive notion, basic definition, homothety, the composition of a homothety and a motion, etc." (Gualdrón, 2011, p. 109). Thus, the intuitive notion and the relationships between the basic definition of similarity with homothecy and proportionality were considered, as well as the theoretical elements previously exposed, van Hiele's (1986) model, the visualization elements, and the use of GeoGebra.

In relation to the use of GeoGebra, the different ways of working with the software and the principles proposed for its use were considered. Thus, activities were designed, where GeoGebra was a tool to explore geometric relationships and others, where it was used to model a situation in the real context, or as a construction tool.

The learning phases of van Hiele's (1986) model were used in the design of each section. Thus, each section was composed of two activities in which ten specific tasks are presented, two tasks for each of the phases. On the other hand, the contextual reality of the EI was considered; thus, the intervention activities were developed using the thematic axes proposed by the curriculum of the institution. This teaching unit has four intervention sections.

Section #1, aimed to introduce GeoGebra, and some preliminary notions for the study of similarity. The items to be developed were:

1. Identification of constituent and corresponding elements of geometric figures.
2. Angles formed by two parallels cut by secant lines and properties of some geometric figures such as triangles and quadrilaterals.

Section #2 intended the construction of the intuitive definition of similarity. The items to be considered were:

1. The recognition of physical characteristics "they are similar because they keep the same shape".
2. Acquiring a mathematical language regarding similarity.
3. To build criteria about when two polygons are similar.
4. Criteria for similarity of triangles.
5. Construction of similar figures.

In section #3, the objective was to demonstrate the relationships of similarity, proportionality, and homothecy. Items to be developed in this section were:

1. Relationship between similarity and homothecy.
2. Relationship between similarity and Thales' theorem.
3. Homothecy and Thales configurations.
4. Construction of a similar figure from a homothecy.

And, in section #4, the objective was to apply the concept of similarity in real context problems and to consolidate the definitions of similarity, proportionality and homothecy. The items to be developed were:

1. Ratio between areas and perimeters of similar figures.
2. Calculation of unknown distances and scales.

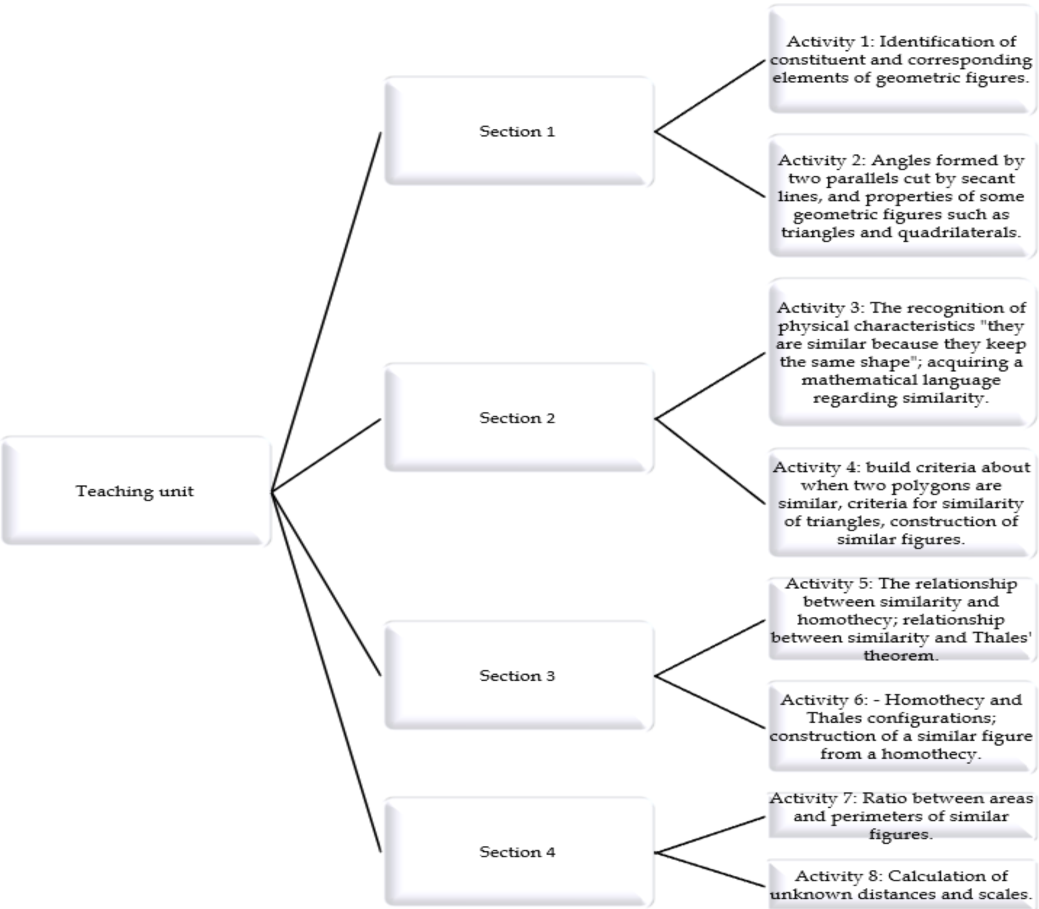


Figure 2. Concepts addressed in teaching unit (Source: Authors' own elaboration)

Table 2. Theoretical elements

Task	Learning phase	van Hiele's level	Visualization	Use of GeoGebra
Activity 1				
1-2 3-4 5-6 7-8 9-10	Stage 1 Stage 2 Stage 3 Stage 4 Stage 5	Level 1: Recognition	Images: Kinesthetic dynamic pattern skills: Visual identification (VI) & mental rotation (MR) Processes: Visual interpretation of information (VP) & interpretation of mental images (IFI)	As a tool for experimentation, exploration, & discovery (FILE 01)
Activity 2				
1-2 3-4 5-6 7-8 9-10	Stage 1 Stage 2 Stage 3 Stage 4 Stage 5	Level 1: Recognition	Images: Kinesthetic dynamic pattern skills: Visual identification (VI), perceptual conservation (PC), visual memory (VM), visual discrimination (VD), recognizing spatial relationships (RR). Processes: Visual interpretation of information (VP) & interpretation of mental images (IFI)	As a tool for experimentation, exploration, & discovery (FILE 02-FILE 02.1)

Figure 2 shows the mathematical concepts addressed in each section. The following is an example of relationship between the elements of the theoretical framework and the content of each activity (activities 1 and 2 of section #1).

Activity 1 fostered students to acquire the necessary language to approach the concept of similarity; for this reason, the process that was intended to be developed was the identification of attributes, both physical and mathematical. The tasks presented focused on identifying corresponding sides, corresponding angles, and elements that constitute a figure. The tasks presented also allowed the use of images and mental representations, in order for students to acquire visualization skills.

Activity 2 sought to have students recognize the congruence of angles formed when two parallel lines are cut by secant lines. Also, they allowed the use of images and mental representations, in order for students to acquire visualization skills. **Table 2** shows the relationship between the tasks proposed in each activity and the theoretical elements.

Table 3. Diagnostic test content

General aspect	Task	Specific topic
Notion of similarity in plane figures	1	Notion of similarity
	6	Criteria of similarity of triangles
	9	Properties of triangles
Applications of similarity and proportionality	4	Proportionality factor
	5	Reduction of figures
	7	Reduction using a given proportionality factor
	10	Enlargement using a given proportionality factor
Congruent angles and proportional sides	2	Similar triangle from another
	3	Angles formed by a transversal and two parallels
	8	Proportional sides

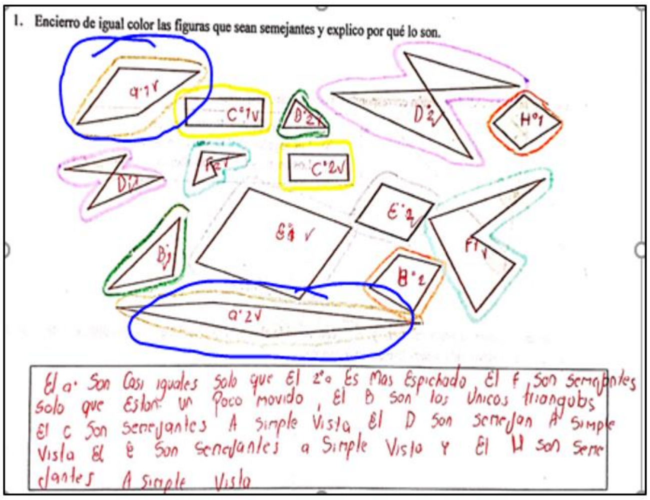


Figure 3. Herson's production in diagnostic test task 1 (Source: Authors' own elaboration)

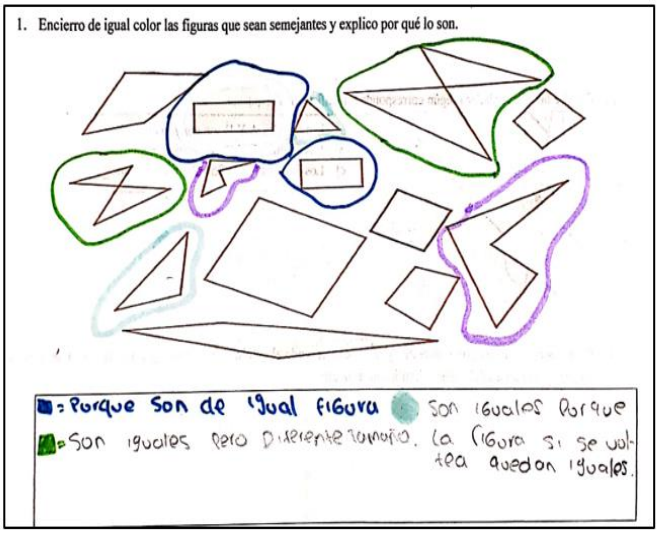


Figure 4. Andres' production in diagnostic test task 1 (Source: Authors' own elaboration)

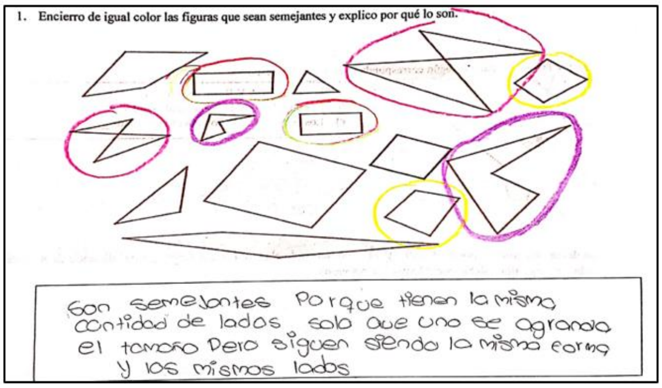


Figure 5. María's production in diagnostic test task 1 (Source: Authors' own elaboration)

ANALYSIS AND RESULTS

This section presents the analysis and results considering the different sources of information: the diagnostic test and the students' production when developing the teaching unit, particularly in activities one and two of section one, going through the tasks that make up each activity, identifying the students' learning trajectory, considering the categories proposed.

Analysis of Diagnostic Test

The diagnostic test consists of 10 tasks, which were aimed at identifying the students' reasoning. For the sake of synthesis, only the analysis regarding the general aspect of the notion of similarity in plane figures will be presented. **Table 3** shows the general aspect and the specific topic addressed by each diagnostic task.

General Aspect: Notion of Similarity in Plane Figures

In task 1, the student was asked to enclose in the same color the similar figures from a group of images and explain why they were similar. Similar responses were presented by several students in recognizing similarity by shape, although not the same size; in **Figure 3**, **Figure 4**, and **Figure 5**, the productions of Herson, Andres, and María (all fictitious names, the same as hereafter) are presented, indicating level 1 of geometric reasoning around the notion of similarity.

Herson justifies the similarity of the two polygons enclosed in blue by using expressions about their shape such as "it is spikier", "it is moved", etc. As for the visualization elements, he uses perceptual conservation skill (PC). It is evidenced by saying "D is similar to the naked eye" when the figure has been moved and its size is different.

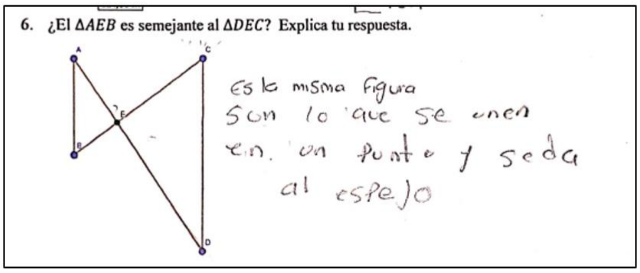


Figure 6. Camilo's production in diagnostic test task 6 (Source: Authors' own elaboration)

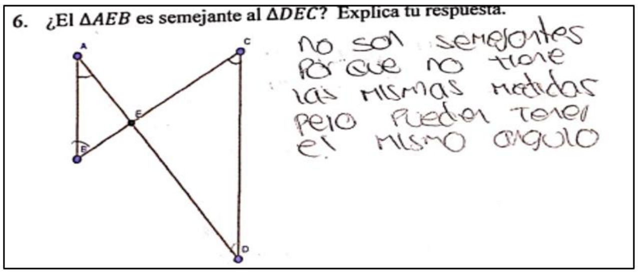


Figure 7. María's production in the diagnostic test task 6 (Here, she answers question "is ΔAEB similar to ΔDEC? Justify your answer". She answers, "they are not similar because it does not have the same measures, but they can have the same angle") (Source: Authors' own elaboration)

Andrés recognizes the similarity of the figures enclosed in green, using the definition of equal shape, and different size, although he does not do so with other figures that seem similar. He also uses the ability of mental rotation (MR), by saying that the two triangles are similar. "If they are flipped, they are the same", on the other hand, descriptor N1.1 [recognize similar figures based on their appearance, that is, using only visual strategies] was assigned.

María identifies the similarity of plane figures based on the number of sides of the figures, a condition that is important for similarity, but not sufficient; it is evident that she uses the ability of visual identification (VI).

In task 6, they were asked to verify the similarity of two triangles joined by a vertex; most of the students' answers tend to affirm that the triangles are similar because their angles were congruent, however, they did not show a justification that allowed them to reach that conclusion, even so, in Figure 6 and Figure 7 the answers of Camilo and María are highlighted, which shows the lack of basic knowledge for the development of the task.

Camilo does not recognize the notion of similarity, does not specify the difference in size, but sees one triangle as the reflection of another; the ability to conserve perception (CP) is evident.

María can identify the congruence of the angles without specifying the reasoning that led her to that conclusion, but she does not see it as necessary to define similarity; it is evident that María uses the ability to recognize spatial relationships, she can see angle B and C as internal alternates.

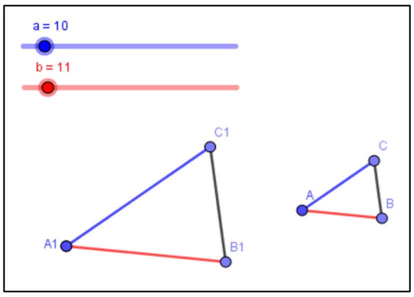


Figure 8. File 01: Activity 1 (Source: Authors' own elaboration)

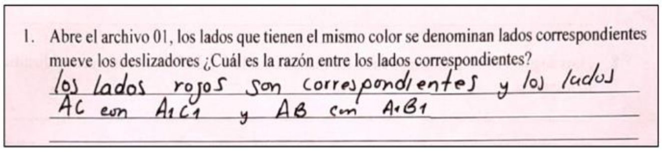


Figure 9. Andrea's production: Activity 1-task 1 (Source: Authors' own elaboration)

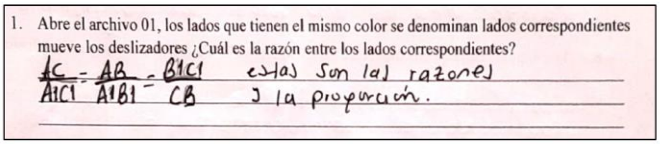
The purpose of the diagnostic test was to know the students' pre-knowledge. The following conclusions were highlighted:

1. Concerning congruent angles and proportional sides, students did not recognize the necessary characteristics of side correspondence, they did not know what a proportion is, and they did not recognize internal and external alternate angles.
2. Regarding the notion of similarity of plane figures, students did not present a clear notion of similarity, they did not know similarity criteria, they did not use visualization skills correctly, did not communicate their arguments, and did not use appropriate mathematical language.
3. Regarding the application of similarity and proportionality, students were not able to make similar figures from others, and they did not identify the proportionality factor, either for an enlargement or a reduction.
4. In the diagnostic test, the absence of level 2 descriptors for the development of similarity, proportionality, and homothety was evidenced. The level 1 descriptors that were evidenced in the diagnostic test are N1.1, and N1.2, showing some difficulties in the three general aspects evaluated by the test.

Analysis of Teaching Unit

Activity 1

In task 1, Figure 8, students applied the principle of doubting what is seen. They moved the sliders, and they could verify that if the blue slider was moved, the blue segments were also moved, the same with the red slider. In this way, they were able to identify which are the corresponding sides.



1. Abre el archivo 01, los lados que tienen el mismo color se denominan lados correspondientes mueve los deslizadores ¿Cuál es la razón entre los lados correspondientes?

$\frac{AC}{A1C1}=\frac{AB}{A1B1}=\frac{B1C1}{CB}$ estas son las razones y la proporción.

Figure 10. Camilo’s production: Activity 1-task 1 (Source: Authors’ own elaboration)

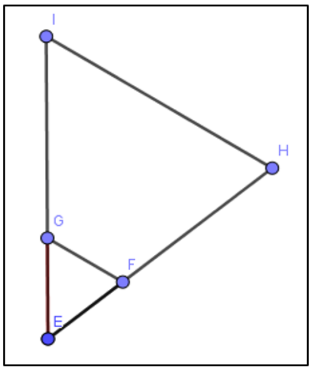
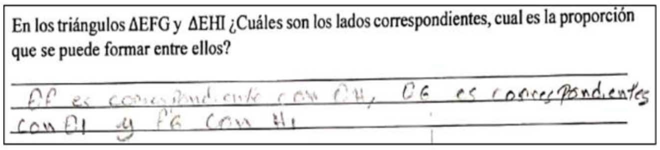


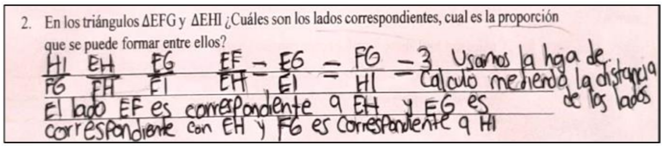
Figure 11. File 01: Task 1 (Source: Authors’ own elaboration)



En los triángulos ΔEFG y ΔEHI ¿Cuáles son los lados correspondientes, cual es la proporción que se puede formar entre ellos?

EF es correspondiente con EH, EG es correspondientes con EI y FG con HI

Figure 12. Herson’s production: Activity 1-task 2 (Source: Authors’ own elaboration)



2. En los triángulos ΔEFG y ΔEHI ¿Cuáles son los lados correspondientes, cual es la proporción que se puede formar entre ellos?

$\frac{HI}{FG}$ $\frac{EH}{FH}$ $\frac{EG}{EI}$ $\frac{EF}{EH}=\frac{EG}{EI}=\frac{FG}{HI}=3$ Usamos la hoja de Calculo mediendo la distancia de los lados

El lado EF es correspondiente a EH y EG es correspondiente con EH y FG es correspondiente a HI

Figure 13. Camilo’s production in activity 1-task 2 (Source: Authors’ own elaboration)

Figure 9 shows Andrea’s production, she argues that the red sides are corresponding, so AC is corresponding to A1C1: referring to the segments.

When answering what is the ratio between the corresponding sides, Camilo manages to state the ratio between the sides without considering the values of the length of the sides, see **Figure 10**. He does not recognize the difference between B1C1/BC and BC/B1C1.

Herson shows his arguments concerning task 2 (**Figure 11**); he succeeds in identifying the corresponding sides between the triangles, by expressing, referring to the sides of the triangles, “EF is correlated with EH, EG is correlated with EI and FG is corresponding with HI” (**Figure 12**).

In **Figure 13**, for the same task, Camilo correctly writes the proportion presented for the corresponding sides EF/EH=EG/EI=FG/HI.

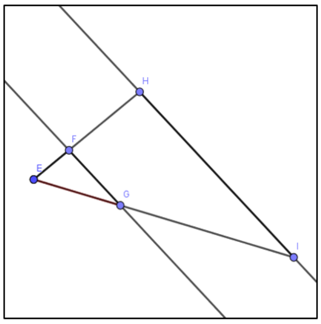
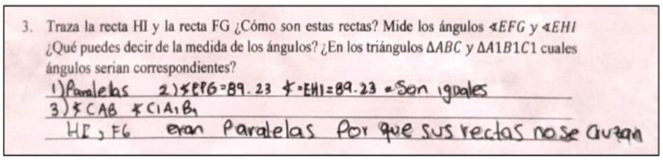


Figure 14. Herson’s construction on Geogebra: Activity 1-task 1 (Source: Authors’ own elaboration)



3. Traza la recta HI y la recta FG ¿Cómo son estas rectas? Mide los ángulos ∢*EFG* y ∢*EHI* ¿Qué puedes decir de la medida de los ángulos? ¿En los triángulos Δ*ABC* y Δ*A*1*B*1*C*1 cuales ángulos serian correspondientes?

1) Paralelas 2) ∢EFG=89.23 ∢EHI=89.23 Son iguales

3) ∢CAB ∢C1A1B1

HI, FG eran Paralelas Por que sus rectas no se cruzan

Figure 15. Herson’s production for activity 1-task 3 (Source: Authors’ own elaboration)

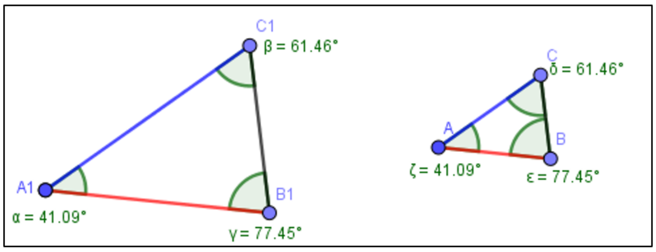


Figure 16. Andrea´s construction in activity 1-task 4 (Source: Authors’ own elaboration)

In task 3, using the second principle of seeing beyond what is seen, students can draw the lines HI and FG. Identifying that these lines are parallel (**Figure 14**).

Herson identifies that the constructed lines are parallel and uses the angle measuring tool in GeoGebra and measures the angles ∢EFG and ∢EHI from which he concludes their equality, furthermore he correctly relates the corresponding angles between the triangles ΔABC and ΔA1B1C1 from task 1. This reasoning is shown in **Figure 15**.

Figure 16 shows the construction made by Andrea for the development of task 4.

In **Figure 17**, Andrea exposes her reasoning around the same task; she identifies that the corresponding angles are congruent.

Figure 18 shows María’s production for the development of task 7. She measures the sides of the triangles and decides the similarity considering only the shape of the triangles and not mathematical aspects, such as the measure of the angles. This type of reasoning is assigned descriptor N1.1 [recognizing similar figures based on their appearance, i.e., using only visual strategies].

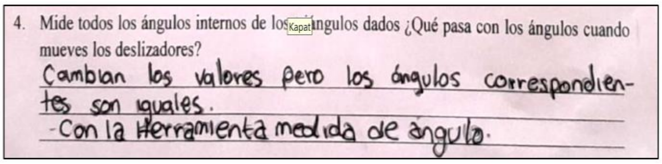

4. Mide todos los ángulos internos de los triángulos dados ¿Qué pasa con los ángulos cuando mueves los deslizadores?
Cambian los valores pero los ángulos correspondientes son iguales.
-Con la Herramienta medida de ángulo.

Figure 17. Andrea's production in activity 1-task 4 (Source: Authors' own elaboration)

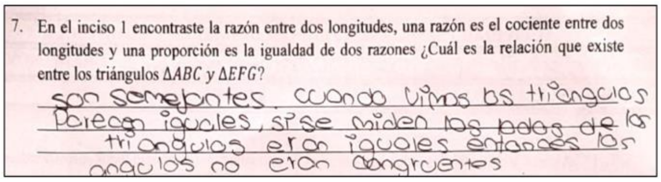

7. En el inciso 1 encontraste la razón entre dos longitudes, una razón es el cociente entre dos longitudes y una proporción es la igualdad de dos razones ¿Cuál es la relación que existe entre los triángulos ΔABC y ΔEFG?
son semejantes cuando vimos los triángulos parecen iguales, si se miden los lados de los triángulos eran iguales entonces los angulos no eran congruentes

Figure 18. María's production in activity 1-task 7 (Source: Authors' own elaboration)

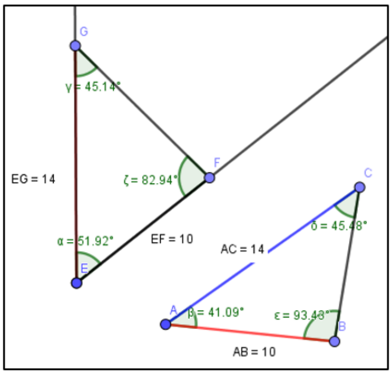


Figure 19. María's construction in activity 1-task 8 (Source: Authors' own elaboration)

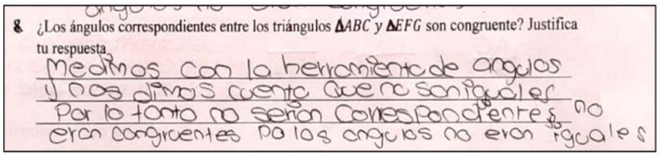

8. ¿Los ángulos correspondientes entre los triángulos ΔABC y ΔEFG son congruente? Justifica tu respuesta
Medimos con la herramienta de angulos y nos dimos cuenta que no son iguales por lo tanto no serian correspondientes no eran congruentes por los angulos no eran iguales

Figure 20. Marías production in activity 1-task 8 (She states, "we measured with the angles tool and realized they are not the same, then they would not be corresponding, they were not congruent because angles were not the same.") (Source: Authors' own elaboration)

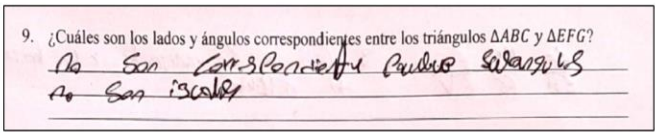

9. ¿Cuáles son los lados y ángulos correspondientes entre los triángulos ΔABC y ΔEFG?
No son correspondientes porque sus angulos no son iguales

Figure 21. Andres' production in activity 1-task 9 (He states, "they are not corresponding because their angles are not the same.") (Source: Authors' own elaboration)

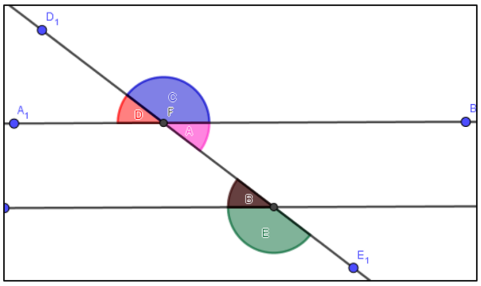


Figure 22. File 02: Activity 2 (Source: Authors' own elaboration)

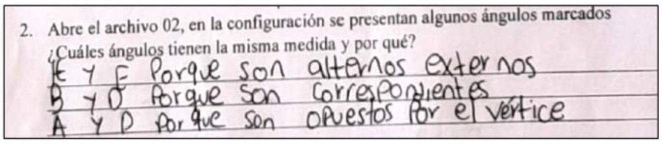

2. Abre el archivo 02, en la configuración se presentan algunos ángulos marcados ¿Cuáles ángulos tienen la misma medida y por qué?
E y F porque son alternos externos
B y D porque son correspondientes
A y D por que son opuestos por el vértice

Figure 23. María's production in activity 2-task 1 (Source: Authors' own elaboration)

Figure 19 shows GeoGebra construction made by María, where she used the second principle of dynamic geometry to measure the angles of the triangles.

Figure 20 shows María's production around the development of task 8, she managed to measure the angles of the triangles ABC and EFG, and to conclude that, although two of the sides of the triangles had the same length, their angles did not have the same measure, in that case, the triangles were not congruent.

Figure 21 shows Andres' argument on task 9, he concludes that, for two triangles to have corresponding sides, these triangles should have congruent angles.

From activity 1 it can be concluded that students could understand when two sides are corresponding, and that, if this correspondence between the sides of two triangles is present, the angles of the triangles must be congruent. In activity 1 there is evidence of reasoning with descriptor N1.3 [beginning to perceive some mathematical characteristics of similarity but still doing it in isolation].

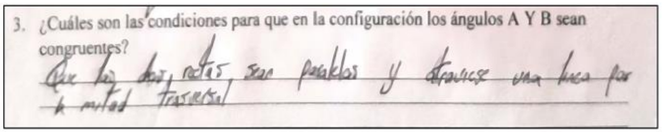
Activity 2

Figure 22 shows the GeoGebra file 02 for the development of activity 2, which sought to have students recognize the congruence of angles formed when two parallel lines are cut by secant lines.

Figure 23 shows María's production around task 2; she managed to clearly define the equality of angles and recognize the angles formed, classifying them according to their position in the configuration.

The following is a fragment of conversation between the researcher and María while she was developing task 2, from activity 2, and that are exposed in the field diary.

> Researcher: How do you know that they are internal alternates?
>
> María: We had already studied it before, the internal ones are inside, and the external are not...
>
> Researcher: Inside what?
>
> María: ... of the straight lines.
>
> Researcher: The parallel ones?


3. ¿Cuáles son las condiciones para que en la configuración los ángulos A Y B sean congruentes?

Que las dos rectas sean paralelas y atraviese una linea por la mitad transversal

Figure 24. Andres' production in activity 2-task 3 (Source: Authors' own elaboration)

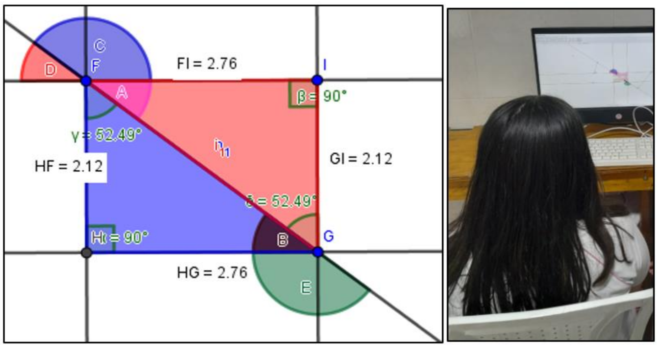
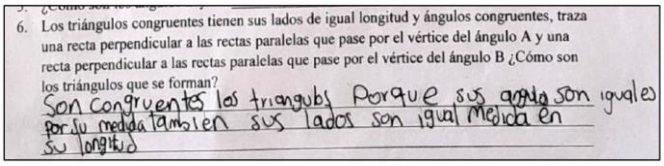


Figure 25. Mar's construction in activity 2-task 6 (Source: Authors' own elaboration)


6. Los triángulos congruentes tienen sus lados de igual longitud y ángulos congruentes, traza una recta perpendicular a las rectas paralelas que pase por el vértice del ángulo A y una recta perpendicular a las rectas paralelas que pase por el vértice del ángulo B ¿Cómo son los triángulos que se forman?

Son Congruentes los triangulos Porque sus angulos son iguales por su medida tambien sus lados son igual Medida en su longitud

Figure 26. Mar's production in activity 2-task 6 (Source: Authors' own elaboration)

María: Yes, those, the parallel ones, then those angles are the same (pointing to angles A and B).

Figure 24 shows Andres' production for the development of task 3. From this, it can be concluded that students manage to define a series of conditions for the angles to fulfill their congruence, Andres considers "A and B are congruent because it is formed when a transversal cuts two parallel straight lines".

Andres explains how he came to the above conclusion in the conversation with the researcher (conversation Andrés section 1 activity 2).

Researcher: Why are angles A and B equal?

Student: When I measured them, they were both 37.51, so they are the same.

Researcher: But why do you think they measure the same?

Student: Because they are internal alternates, like before, they are inside the straight lines, and all the angles that are inside are equal.

Investigator: How so?

Student: If the angles are inside the parallel lines and pass this line (referring to the transversal) then the angles are equal (referring to angles A and B).

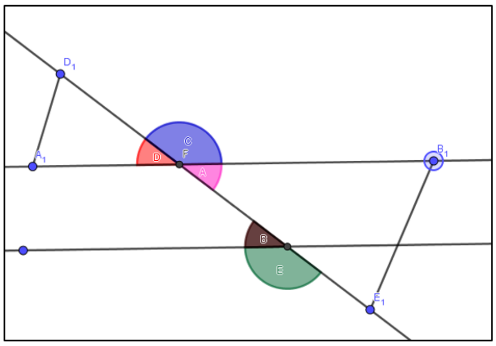
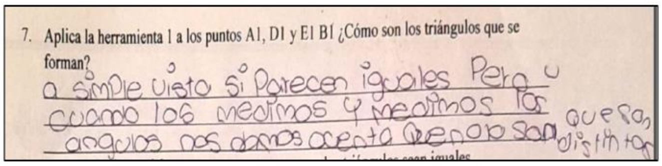


Figure 27. María's construction in activity 2-task 7 (Source: Authors' own elaboration)


7. Aplica la herramienta I a los puntos A1, D1 y E1 B1 ¿Cómo son los triángulos que se forman?

a simple visto si parecen iguales pero cuando los medimos y medimos los angulos nos damos cuenta que no son que son distintos

Figure 28. María's production in activity 2-task 7 (Here María states: "At first glance, they look the same, but when we measure the angles we realize that they are not, that they are different") (Source: Authors' own elaboration)

Figure 25 shows the construction of Mar for task 6, evidencing that students follow the construction proposed to them, which clarifies the conditions to be corroborated using GeoGebra.

Figure 26 shows Mar's production for task 6. She can understand the congruence of the triangles that were formed, constructing a definition of congruence between triangles: "triangles are congruent because their angles are congruent and also their sides are of equal length".

Conversation with Mar section 1 activity 2:

Researcher: When are two triangles congruent?

Mar: It says here that when they have the same sides, and angles.

Researcher: How did you verify that these two triangles are congruent (pointing to the triangles formed earlier).

Mar: I measured them, and I knew they were equal (referring to the sides and angles).

In task 7, students explore on their own the construction of task 7 shown in **Figure 27**. They make their conclusions in the following way: at first, students considered that the triangles are equal, however, they applied the second principle, seeing beyond what you see, and measure with the help of GeoGebra the lengths.

Figure 28 shows María's production, she manages to use GeoGebra and correctly constructs an argument

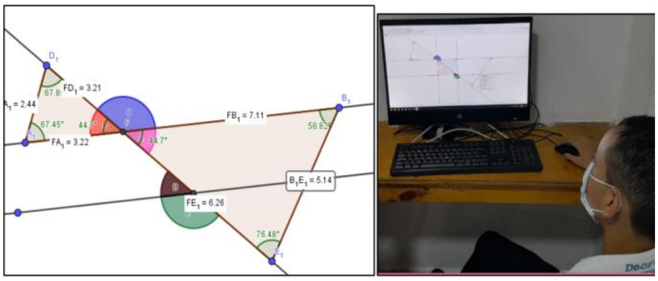
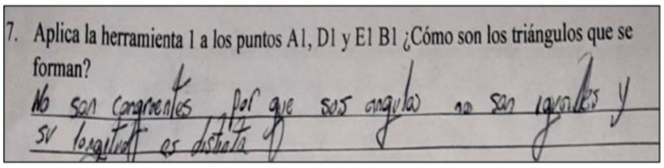


Figure 29. Andres' construction in activity 2-task 7 (Source: Authors' own elaboration)


7. Aplica la herramienta 1 a los puntos A1, D1 y E1 B1 ¿Cómo son los triángulos que se forman?

No son congruentes por que sus angulos no son iguales y su longitud es distinta

Figure 30. Andres' production in activity 2-task 7 (Source: Authors' own elaboration)

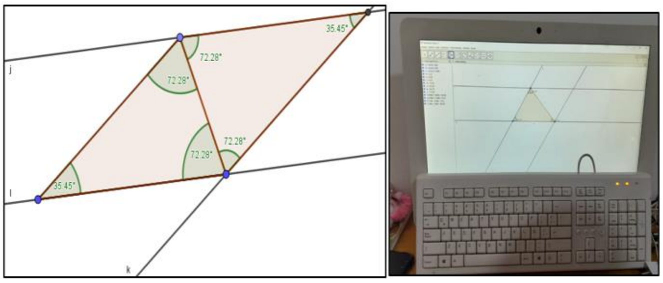
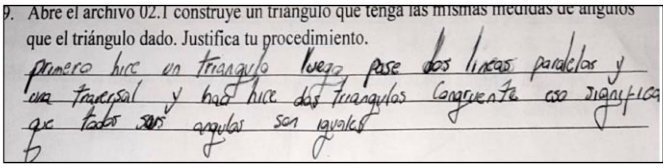


Figure 31. Andres' construction: Activity 2-task 9 (Source: Authors' own elaboration)


9. Abre el archivo 02.1 construye un triangulo que tenga las mismas medidas de angulos que el triángulo dado. Justifica tu procedimiento.

primero hice un triangulo luego pase dos lineas paralelas y una trancersal y hay hice dos triangulos congruente eso significa que todos sus angulos son iguales

Figure 32. Andres' production: Activity 2-task 9 (Source: Authors' own elaboration)

about the similarity of the triangles formed. This reasoning is assigned descriptor N1.6 [identifying mathematical properties to decide the congruence of figures, but not relating it to similarity]; for example, they can decide the congruence between two triangles by measuring their angles and sides, which have the same value, but they do not see this characteristic necessary to decide similarity.

Conversation with María section 1 activity 2:

Researcher: How did you develop this task?

María: First I moved points B1 and the other one, and I saw that the triangles were different, because one side was longer than the other, so they were not congruent and that's it.

Researcher: So what are they?

María: Similar?

Researcher: Are you sure? what should you do to verify that?

María: Measure the sides if they are the same then they are similar.

Researcher: The sides or the angles? because you already said the sides are different.

María: The angles.

Students use GeoGebra to make the necessary measurements for their conclusions (see Figure 29), managing to define criteria to verify the congruence of triangles.

Figure 30 shows the production of Andres, he concludes that the triangles that are formed are not congruent because they do not meet the criteria to be so.

In task 9, students must construct a triangle that has the same angle measures as the given triangle. Students enrich the initial figure with other geometric elements to create a triangle congruent to the given triangle, since they recognize that congruent triangles have congruent angles.

Figure 31 shows Andres' construction to develop a triangle congruent to another given triangle.

Figure 32 shows Andres' production, where he explains the step-by-step solution of task 9.

The lines J and I are parallel and the transversal to which he refers is parallel to the side of the triangle, thus forming two congruent triangles.

Figure 33 shows the explanation given by Andres to his group of classmates.

Conversation with Andres, section 1, and activity 2:

Researcher: Explain to me, how did you build the triangle?

Andrew: First I did it wrong, I made a triangle, and I measured the angles, and it did not come out, then, I remembered about the internal alternates, and I passed this line (pointing to line J) and then I passed the other parallel and that's it (referring to line K).

Researcher: Why did you make line I?

Andrew: To better see the property of the internals.

In activity 2, students were able to identify the relationships between the angles formed in configurations, where parallel lines are cut by a secant line. Students were able to identify the correspondence of angles and sides of two congruent triangles, and also

Figure 33. Andres is socializing with the group the development of activity 2 (Source: Authors' own elaboration)

define the criteria necessary for two triangles to be congruent, although they were not able to understand congruent triangles as a special case of similar triangles. The advantage of using GeoGebra is evident, since the students were able to verify the veracity of their constructions through dragging, thus confirming that the invariant properties of the elements were maintained. This type of reasoning was assigned the descriptors: N2.1 [constructing or drawing figures similar to a given one, explicitly considering mathematical aspects such as the measure of the angles or the lengths of the sides] and N1.6 [identifying mathematical properties to decide the congruence of figures without referencing similarity].

In conclusion, students were able to understand the concept of similarity, proportionality and homothecy, and the acquisition of mathematical language was evidenced. In activity 1, students were able to understand when two sides are corresponding, and the fact that if this correspondence is present between the sides of two triangles, the angles of the triangles must be congruent. In activity 2, students were able to identify the relationships between the angles formed in configurations, where parallel lines are cut by a secant line. Students were able to identify the correspondence of angles and sides of two congruent triangles, they also define the necessary criteria for two triangles to be congruent, although they were not able to understand congruent triangles as a special case of similar ones. In activity 3, students made a list of mathematical criteria, apart from shape, to recognize similarity between triangles. They also recognized congruence between triangles as a special case of similarity. In activity 4, it is evident that students were able to define a concept of proportionality and similarity, defining mathematical aspects, such as the proportionality of segment lengths and the equality of angle measures. In addition, they understood and defined the similarity criteria angle-angle, side-angle-side, and side-side-side. Students concluded that all similar triangles meet the similarity criteria. In activity 5, students were able to understand the concept of homothecy and related it to similarity. They related the ratio of homothecy with the ratio of similarity, although they do not use the word scale, they understand that, if the ratio of proportion is greater than one, the polygon resulting from the homothecy will be an enlargement and if it is between zero and one, a reduction. In activity 6, students are able to understand when a Thales configuration is present, and they defined the similarity between triangles that are in Thales's position. They also saw the similarity of triangles as a Thales configuration and as a homothecy, with the center at the vertex that shares the triangles. In activity 7, students made constructions of similar figures such as rectangles and circles, understanding properties that these figures shared and using homothecies, as well as recognizing the similarity factor. They are able to make conjectures about the ratio between the areas and perimeters of similar figures. In activity 8, students applied the concept of similarity and related it to other mathematical concepts to solve problems and to find unknown distances.

In this sense, it can be inferred that students reached level 2 of geometric reasoning, around the study of similarity, proportionality, and homothecy. In addition, the use of GeoGebra allowed students to learn more quickly some concepts without having to perform a definition as such, for example, in the concept of angle, students were able to measure angles formed by the sides of triangles and identified the difference between an external and an internal angle by themselves. Students showed, at the beginning of the activity, a positive disposition in spite of the fact that the class was held in an unusual environment, the board and notebook were changed for a computer, which generated motivation to learn.

The use of van Hiele's (1986) model for the design of teaching units is very useful for working with GeoGebra; it allowed students to explore and learn geometry in a meaningful way, taking advantage of all of the software's features.

It is necessary to propose activities, where the software can be left aside because students tend to rely on the program to check some properties they already learned; for example, when they had to decide the congruence of angles formed by parallels and a transversal, although they already knew that they were congruent, for them it was still essential to make the measurements with the program.

DISCUSSION AND CONCLUSIONS

The objective of this work was to analyze the learning process of ninth graders of an educational institution while they were studying the concepts of similarity, proportionality and homothecy using GeoGebra as a learning tool. In order to perform this analysis, the learning trajectories were characterized through the levels of geometric reasoning of van Hiele's (1986) model and the elements of visualization in which the students' thinking processes were evidenced while they developed the teaching unit.

As in the studies of Aravena and Gutiérrez (2016), Gualdrón (2011), Gualdrón et al. (2020), Gutiérrez et al. (2021), and Santos (2014), this research managed to identify the importance of developing teaching units based on van Hiele's (1986) model, since in the different tasks that were proposed in the development of the sections, reasoning in van Hiele's (1986) levels 1 and 2 were emerging, this implies the acquisition of the concepts of proportionality, similarity, and homothecy. The students went from not having a clear notion of similarity to constructing a definition for similarity, proportionality, and homothecy; from not identifying criteria for similarity to identifying and understanding the mathematical properties that remain invariant in similar figures; from not using visualization skills such as mental rotation (MR), and not communicating their arguments using an appropriate mathematical language to using visualization skills and processes to identify, for example, alternate internal angles or configurations in homothetic position, in addition to expressing their reasoning in a symbolic manner.

This study, as presented by Bueno and Valencia (2016), Chávez (2012), and González (2018), recognizes the potential of using dynamic geometry, since it fosters the development of visualization skills in the teaching and learning of geometry. By using GeoGebra, students were able to identify the similarity of figures, for example, by rotating and superimposing them. In addition, they discovered that the position of similar figures is irrelevant, that is, it is not necessary that similar figures have the same position in space.

To conclude, it is important for mathematics teachers to reflect on the activities they bring to the classroom, because depending on how the practice and educational management are carried out, adequate development of learning can be guaranteed or not. This study provides teachers who wish to develop the concepts of proportionality, similarity and homothecy with a well-structured teaching unit based on a current and relevant theoretical model for the teaching of geometry, which also links the use of GeoGebra, which enhances visualization skills that are part of the geometric reasoning that should be developed at school.

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