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Incorporating history of mathematics in open-ended problem solving: An empirical study

Ioannis Rizos [1*], Nikolaos Gkrekas [1]

[1] Department of Mathematics, University of Thessaly, Lamia, GREECE

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Abstract

In this paper, we present a research project we conducted with 27 undergraduate students in a history of mathematics course in Greece during the academic year 2022-2023. In our study, we presented participants with an open-ended problem with historical background and evaluated their reactions and solving strategies. To reach findings we collected data via worksheets, questionnaires and interviews. We intended to focus on students' techniques for open-ended issues while also investigating whether and how History of Mathematics may be included into its instruction. The results showed that such type of problems is able to catch the participants' attention and support them in experimentation and development of multiple problem solving strategies. The students acquired a positive attitude towards the entire process, and they would like to repeat it in other university courses, too. This study might pave the way for a new curriculum that includes historically inspired open-ended assignments in school and university practice.

Keywords: open-ended problems, problem solving, hundred fowls problem, history of mathematics, mathematics education

INTRODUCTION

Every day, students are confronted with situations at school that are contradictory to them containing challenges that need to be overcome. In order to cope with these situations it is preferable to apply thought processes that enable the generation of knowledge required for the successful solution of the problems they confronted (Dostál, 2015). Each student may face different situations as problems, depending on various factors (sociocultural, academic, potential, etc.). However there seems to be no agreement among members of the scientific community on a clear definition of the term 'problem', especially with regard to mathematics.

A specific type of problems in mathematics is open-ended problems. Open-ended problems are those that may have several paths to resolution and may have more than one viable answer. The open-ended problem strategy's primary purpose is to give students the opportunity to face issues according to their own ideas (Ulinnuha et al., 2021, p. 23). Nevertheless, in many countries, such as Greece, open-ended problems appear not to be a significant part of curriculum and daily teaching practice.

Additionally, mathematical problems with a historical basis seem to have much to offer in teaching and learning of mathematics; it is at this moment when students participate in some activity in response to a sample of mathematics from the past, that mathematics and history intersect in a way that benefits mathematics education (Chorlay et al., 2022; Furinghetti, 2020). The integration of history of mathematics into their teaching invites us to consider mathematics from perspectives that extend beyond the established disciplinary subject boundaries, as well as to situate the development of mathematics in the scientific and technological context of a specific time, in addition to the history of ideas and societies (Jahnke et al., 2000, p. 292).

Furthermore, problem solving concerns the student's engagement on any mathematical task that is not judged procedural or the student does not have an initial overall idea how to proceed in solving the task and directly tied to thinking, learning, memory, perception and motivation (Mamona-Downs & Downs, 2013; Rohmah & Sutiarso, 2018). This perspective is essentially based on the classic work of George Polya, the father of modern

✉ ioarizos@uth.gr **(*Correspondence)** ✉ ngkrekas@uth.gr

Contribution to the literature

- The research proposes historically inspired open-ended tasks can have a significant effect on students' engagement.
- The research highlights the relationship between history of mathematics, mathematics education, open-ended tasks and problem solving.
- The research details the historical background and mathematical elaboration of "hundred fowls problem".

problem solving (Awofala & Ajao, 2021), *how to solve it* (Polya, 1945), and still includes heuristics, metacognition, identification of students' thinking patterns, and more. Problem solving has been a prominent subject in mathematics education for the past four decades, and research studies have focused heavily on the function of heuristics and their influence on students' problem-solving ability (Mousoulides & Sriraman, 2020).

Our research's main aim is to expose undergraduate university students to open-ended problems with historical background and give them the chance to approach this type of problems by developing their own strategies. We chose a classical issue from history of mathematics to deal with, the so-called "hundred fowls problem", which we extensively analyze in the third paragraph of this paper, because

(a) it has a great historical background and mathematical interest, since it leads into solving a linear system of Diophantine equations,

(b) it gives the students the chance to develop multiple different strategies, and

(c) it is an open-ended problem. As far as is known, there has not been much work on this subject, particularly in Greece.

Thus, this research might pave a part of the path for the development of a new curriculum in secondary or/and tertiary education.

In this paper, we present a research we conducted during the winter semester of the academic year 2022-2023 in two-hour sessions with subjects 27 undergraduate math students who attended the course "history of mathematics" in a Greek university department. The research combines components from history of mathematics and mathematical problem-solving processes. The data of the research were collected either in written form (from each participant's worksheets) and via interviews (verbal with each participant, during the solving, after the solution and overall feedback). The data was then classified using tables with characteristics such as different approaches, students' groups, frequency of replies and more, and are presented below in detail. We noticed that challenges like these were never before encountered by the specific students; it was an unknown path for them, and the uncertainty caused participants to switch between strategies and methods. This entire procedure has rattled them out of their previously recognized standard educational context, allowing them to experiment and form conjectures. The results demonstrated that such issues may capture participants' attention and lead to experimentation and the creation of numerous problem-solving solutions. The students praised the entire procedure and expressed a desire to replicate it in more academic courses. All the above, as well as some limitations and difficulties, are discussed extensively in the last two paragraphs of the paper.

THEORETICAL ISSUES

About Uncertainty

The state of uncertainty in general is considered to be an undesirable situation, which refers to the doubt that exists about whether or not a particular result will occur, is intertwined with ignorance and risk (Hogarth & Kunreuther, 1995; Keren & Gerritsen, 1999), and defines our times to a great extend (Scoones, 2019). In the mathematics classroom, *uncertainty* is the experience of students encountering questions or tasks that are unfamiliar to them or in which they are unsure of the solution type or solution method, regardless of whether they have previously received instructions for the task or the subject (Buckley & Sullivan, 2021). Uncertainty is a productive state, a *necessary* precondition for learning (Rowland, 1995). More specifically, according to Manz (2018), for uncertainty to be productive, it must be strategically designed into the learning environment and carefully managed by teachers.

The term "uncertainty" can also describe the behavior of a student in the mathematics classroom who feels uncertain about another student's opinion, a process, a concept, a method or a result, but this does not mean that his/her uncertainty is identical with ignorance. On the contrary, the feeling of uncertainty presupposes the existence of some knowledge of what exactly one feels uncertain about. The above view of uncertainty leads to exploration and speculation and places the students on the boundaries of their knowledge (Goos et al., 1999), thus making uncertainty a possible prerequisite for the activation of the zone of proximal development (Vygotsky, 1978, p. 86). However, a student may recognize the uncertainty inherent in an issue but choose to accept it and not seek to abrogate it. In this respect uncertainty is a *necessary* but not *sufficient* condition for the beginning of the learning process.

Zaslavsky (2005) considers that uncertainty is often associated with *cognitive conflict*, a psychological state involving a discrepancy between cognitive structures and experience or between various cognitive structures (Waxer & Morton, 2012), and consequently, if treated with care, can be linked to the development of new knowledge. Thus, she uses uncertainty as a guide for the construction of tasks aimed at cognitive conflict, while describes three interrelated types of uncertainty (competing claims, unknown path or questionable conclusion, and non-readily verifiable outcomes) entailed in certain mathematical tasks. According to Zaslavsky (2005, p. 302) the second kind of uncertainty (unknown path or questionable conclusion) is related with inquiry, exploration tasks and open-ended problems.

Open-Ended Problems

In the context of mathematics education, *open-ended problems* are generally non-routine problems open to the interpretation of situations that describe, amenable to multiple correct answers as well as to various problem-solving strategies. Pehkonen et al. (2013) classify tasks as *open*, if the starting or/and goal situation are not exactly given. Skovsmose (2020) describing the objective of critical mathematics education considers that open-ended situations must be created and in these, to let mathematics grow. Since "problem solving" means engaging in a task for which the solution method is not known in advance (NCTM, 2000, p. 52), open-ended problem solving provides a free and supportive learning environment for students to develop and express their exploration, collaboration with each other, discovery and mathematical understanding (Kosyvas, 2016). Researchers seem to agree that the open-ended learning model has a positive effect on students' improvement of their mathematical problem solving abilities (Hafidzah et al., 2021; Tanjung et al., 2020). Furthermore, it is widely accepted that open-ended problems can enable students to improve their mathematical creative thinking abilities (Fatah et al., 2016; Ulinnuha et al., 2021), as well as open-ended problems with multiple answers and problem-solving strategies are effective in increasing divergent thinking (Murwaningsih & Fauziah, 2022). Similarly, mathematical groups that utilize highly active mathematical thinking skills with open-ended problems are also effective in fostering creative problem-solving abilities, and students are prepared to cope with realistic situations they will have to face out of school, unlike many traditional classes that focus on closed problems (Bonotto, 2013, p. 53; Yuniarti et al., 2017, p. 662-663).

Although much research has been done on open-ended problems over the last 30 years (Baba & Shimada, 2019; Cifarelli & Cai, 2005; Ibrahim & Widodo, 2020; Pehkonen, 2017) and mathematicians engage in problems that are full of uncertainty or wish to invite uncertainty into their classrooms (Beghetto, 2017), most curricular, textbooks and pedagogical approaches rarely offer students this open-ended view of mathematics and concentrate on students' creativity (Bicer et al., 2021). Especially in the Greek secondary education system open-ended problems are completely absent, while from the school year 2023-2024 it seems that there is the intention on the part of the Ministry of Education to be admitted to the junior and senior high schools' curricula. However, it remains to be seen exactly how this ambitious plan will be implemented in the context of teaching practice, since educators have neither experience in constructing and managing such problems nor the appropriate tools for assessing students (cf. Bobis et al., 2021; Randles et al., 2018). Some of the variables that can be considered for mathematical open-ended problem solving are students' mathematical knowledge, creativity, general intelligence and verbal ability (Bahar & Maker, 2015).

Notes on the History of Problem Solving

Although the term "heuristic" comes from the phrase "Eureka!" which, according to myth, Archimedes exclaimed when he discovered his principle and solved the problem posed to him by the king of Syracuse, the first rules of heuristics were set by Pappus of Alexandria. Pappus in Book 7 (*On the domain of analysis*) of his work *Mathematical collection* describes the method of "analysis and synthesis" in such a systematic and clear way that it still surprises us (Cuomo, 2007; Katz, 2009). Since then, many have meditated upon the problem-solving process, while many more have looked for the "rules of discovery" that were supposed to solve all possible mathematical problems, just as alchemists searched, in vain, for the "philosopher's stone" (see Polya, 1945).

In this long, and not always linear, path, we single out Descartes and Leibniz for the purposes of our study. For the former, each problem he solved in analytic geometry and algebra became a rule that was then used in the solution of other problems, as he himself mentions in his *Discourse on method*. For the latter, the calculus he devised was the method for solving scientific and philosophical problems (Grabiner, 1995) within the framework of a scientia generalis–a utopia that no one believes in anymore but without it, it is likely that theoretical computer science and artificial intelligence (AI) would not have evolved, at least as they have.

A turning point in the history of problem solving was the publication of George Polya's classic book *How to solve it* (Polya, 1945). This book contains a list of heuristics and has had a profound influence on mathematics education up to the present day. The books *Mathematics and plausible reasoning* (Polya, 1954) and *Mathematical discovery* (Polya, 1962) followed, thus dividing the world of problem solving into two eras: before and after Polya (Schoenfeld, 1987). Later, Schoenfeld (1985, 1992) refined the principles of problem

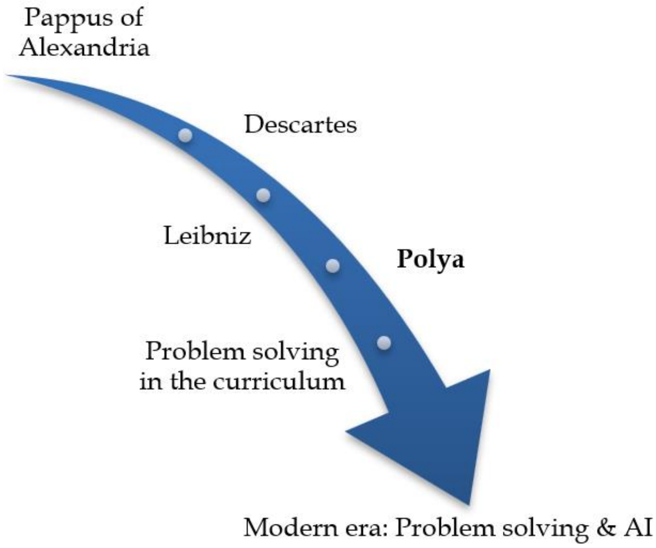

Figure 1. Milestones of problem solving (Source: Authors' own elaboration)

solving to the level of teaching practice and classified the strategies that students follow.

Another point of interest is the fact that since the late 1980s problem solving has been introduced into the curricula. According to the National Council of Teachers of Mathematics (NCTM), problem solving is a goal of learning mathematics and at the same time a major means of doing so (NCTM, 2000).

Today, problem solving can be divided into types (reflective, creative, etc.), depending on the problem (open-ended or closed), the teaching and learning objectives, the classroom management, the use of digital tools, etc. Especially digital technologies provide many possibilities for learners and open new perspectives in problem solving (Liljedahl et al., 2016), to such an extent that a computer-based problem-solving methodology can be created, where the computer is used as an assistant to the human brain–not the other way around (Rus, 2016). **Figure 1** depicts the milestones of problem solving.

HISTORICAL BACKGROUND AND MATHEMATICAL ELABORATION OF THE "HUNDRED FOWLS PROBLEM"

The "hundred fowls problem" is a famous open-ended problem with strong roots in history of mathematics. Is considered as one of the oldest examples of indeterminate analysis (Yong, 1997), specifically a system of linear equations which has more than one solutions under some constrains for the variables. The "hundred fowls problem", which is Zhang Quijian's final problem in *Zhang Quijian suanjing* (*Zhang Quijian's mathematical textbook*) about the second half of the 5th century AD, is about this (Libbrecht, 1973, p. 277):

> "A cock is worth five ch'ien (copper coins), a hen three ch'ien, and three chicks one ch'ien. With 100 coins we buy 100 of them. How many cocks, hens, and chicks are there?"

The above formulation, in modern symbolism, leads to following linear system of two equations with three unknowns:

$$x + y + z = 100, \qquad (1)$$

$$5x + 3y + \frac{1}{3}z = 100, \qquad (2)$$

where $x, y, z \in \mathbb{N}^*$ $(x < 20, y < 33, z < 300)$ are the cocks, hens and chicks, respectively. Zhang Quijian provides the answers: (4, 18, 78), (8, 11, 81), and (12, 4, 84), however it is unclear how he arrived at them.

Some variations of the hundred fowls problem can be found in Alcuin's of York (England, 8th century) *Propositiones ad acuendos juvenes* (Hadley & Singmaster, 1992) and Yang Hui's calculating methods (*Yang Hui Suanfa*, China, 13th century) where detailed explanations on how to find solutions are provided (Yong, 1977).

A similar but more complex hundred fowls problem appears in Mahavira's (India, 850 AD), *Ganita-sarah-sangraha* (*Compendium of the essence of mathematics*). A short explanation of a method for finding a solution based on the selection of relevant parameters is included in this book (Gupta, 2008, p. 1267-1268). The problem may be represented in contemporary terms as a system of two Diophantine equations with four unknown variables, with 16 solutions. Further on, Abu Kamil (Egypt, 9th-10th centuries) wrote *Kitab al-tair* (*The book of birds*), a brief disquisition that includes an introduction and six questions that are comparable to the "hundred fowls" issue (Sesiano, 2008). Abu Kamil was the first who focused on the fact that an issue like that might have so many answers, and that was why he authored *Kitab al-tair* (Sesiano, 2009).

Leonardo of Pisa (Italy, 13th century) wrote the masterpiece *Liber Abaci*, which included not only algorithmic rules using the Hindu-Arabic numerals, but also a variety of practical problems known today as "recreational mathematics" (Devlin, 2011, p. 69). The hundred fowls problem is revisited in the last three questions of chapter 11. Although Leonardo's solution is neither elegant nor formal, the manner in which he presents it is. The mathematician and astronomer Jamshid al-Kashi (Persia, 1427 AD) published his most important work, *Miftah al-hisab* (*Key to arithmetic*), in which another form of the hundred fowls issue emerges. Clearly, the aforementioned problem has attracted a lot of attention in the Arabic world (Kangshen et al., 1999, p. 420-421).

Hundred fowls problem and similar recasts also appear in the works of Christoff Rudolff (Germany, 1525 AD) *Coss*, Seki Takakazu (Japan, 1683 AD) and Leonhard Euler's *Elements of algebra*, along with formal methods and rigorous rules for finding solutions (Campbell &

Table 1. Statistics for the research participants

Participants		Gender		Total
		Male	Female	
Academic year	1st	9	13	22
	2nd	3	2	5
Total		12	15	27

Higgins, 2019; Euler, 1828). A complete historical overview of the hundred fowls problem can be found in Rizos and Gkrekas (2022b).

Many solving techniques have been proposed for the hundred fowls problem, in addition to the different formulations of it. The "trial and error" method in Chinese mathematics (Brandenburg & Nevenzeel, 2007, p. 77) was probably the first basic approach. Another method, also credited to Chinese mathematicians, is to replace a variable with a parameter multiplied by a constant and then solve the resultant linear system.

Alcuin of York comes up with an intuitive one-of-a-kind solution using an arithmetic procedure and proper variable selections. This strategy works "backwards", in that we try to validate a solution that we already have in mind, which is clearly based on trials.

The dilemma of the hundred fowls is broken down into simpler connections by Leonardo of Pisa. He determines how many and which fowls may be purchased with a particular number of coins and then develops a linear combination of the aforementioned relationships to arrive at the given fowl numbers and coins. The solution is given verbally, with the relations to write down in the margins of the pages of *Liber abaci*.

In our research project, which we describe in the next paragraph, we suggested and discussed with our students three solving strategies for a problem completely similar to the "hundred fowls", based on elementary mathematics. In the first one we eliminate the variable z from Eq. (1) and Eq. (2) and thus a new equation,

$$y = -\frac{7}{4}x + 25, \quad (3)$$

is created, which is essentially a linear Diophantine equation with unknowns the natural numbers x and y. Since the first member of Eq. (3) is a natural number, the second member must also be a natural number. Therefore x must be a multiple of four. Only multiples of four that satisfy the condition $x<20$ are 4, 8, 12, and 16. Of these, only the first three lead to the valid solutions of the problem (4, 18, 78), (8, 11, 81), and (12, 4, 84).

According to the second solving strategy, setting $z=3t$ in Eq. (1) and Eq. (2) results a parametric system, which we solve using determinants, considering as before the constraints for the unknowns. Finally, in the third way, we consider the linear Eq. (3) as a line in the plane (specifically in the first quadrant) and look for the points with integer coordinates through which it passes. For this purpose we used the dynamic geometry software *GeoGebra*, as Meadows and Caniglia (2021) and Zengin (2018) did very successfully in a history of mathematics course.

THE PROJECT AND ITS RATIONALE

In the winter semester of the academic year 2022-2023, in a mathematics department in central Greece, we carried out a qualitative research project with 27 participants: 22 students from the first academic year and five from the second one (**Table 1**). The research was conducted face-to-face in class. The first author of this paper was the class teacher, while the second was an observer who took notes but did not interfere in the educational process. In the last academic year 2021-2022 the courses were also held face-to-face at the above University after three semesters of distance learning (Spring 2020 and 2020-2021), when all courses were taught remotely via the MsTeams platform and eClass learning management system, due to restrictive measures against the spread of COVID-19 coronavirus.

There were some important factors that we took into consideration before the implementation of the project. One of them was the existing curriculum and knowledge offered at the Greek secondary and tertiary education. The freshmen have been taught in high school two dimensional "analytic geometry" (vectors and lines in the plane) and "algebra" (equations and parametric equations of first and second degree, functions, trigonometry, polynomials and 2×2 and 3×3 linear systems of equations) and determinants and matrices in the compulsory course "foundations of mathematics" offered at university. The sophomores have in addition attended the compulsory course "linear algebra" (matrix analysis, determinants, vector spaces, and linear maps) and just one of them the elective course "introduction to number theory". Another one important factor was that the participants were our students, so we were able to know, at least to some extent, that they were of different educational and sociocultural levels, with various scientific interests and extracurricular activities. All of them were familiar with technology, had social media profiles and were able to use their mobile phones and tablets easily (Rizos & Gkrekas, 2022a).

In the context of the elective course "history of mathematics" offered at the first academic year at our university department, we had two two-hour meetings, which were four days apart in the same week, with 27 undergraduate students who had chosen this course. Using ICTs (presentation program, wikis, e-books, sites, etc.) we presented the life and work of Diophantus of Alexandria, mainly through the original sources *Anthologia Palatina* and *Arithmetica*, and we gave examples of Diophantine equations. We discussed the classic "Diophantus' riddle" about how many years Diophantus lived (see Pappas, 1989, p. 123), which we solved together with our students.

After that introduction of 45 minutes, we gave to the students a worksheet with a different expression of "the hundred fowls problem" entitled "The hundred amphorae problem" written by ourselves, accompanied by an image of three amphorae (provided as **Appendix A**). The problem we posed was the following:

> "In the time of Diophantus (3rd century AD), in Alexandria, as in the whole Greek world, the main means of transport and storage of goods such as olive oil, wine, cereals, etc. was the amphora. A workshop in Alexandria made three types of amphorae. Type A cost five coins, type B cost three coins, while with one coin you bought three types C amphorae. How many amphorae from each type, a total of 100, could one buy with 100 coins?"

We chose the above formulation instead of the typical of "hundred fowls", considering that this way our students will be involved in the educational process, without losing the essence of the original problem, because we had just introduced them to the life and work of Diophantus, thus making the historical context familiar. Besides, on the days of our teaching intervention the syllabus of the course "history of mathematics" provided for the introduction to the Diophantine equations, the mathematical methods in Hellenistic times and the contribution of Apollonius, Hipparchus, Ptolemy, and Diophantus to the evolution of mathematics and astronomy.

Greek students are not accustomed to deal with open-ended problems in secondary and tertiary education, so we assumed that such an original wording as "The hundred amphorae problem" could catch their attention. In other words, we sought to create a challenging *didactic situation*, that is a project organized so as to cause one or some students to appropriate some piece of mathematical reference knowledge (Brousseau & Warfield, 2014), using an open ended problem with respect to more than one solutions which can be approached with distinct techniques. Didactic situations favor the development of mathematical reasoning by students and at the same time provides the teacher with the appropriate environment for substantial mathematics education in the classroom. As it turned out, our choice was satisfactorily justified; one student at the end of the project stated that

> "the problem was interesting and original. I kept thinking it even after the class ended when I got home. It challenged my mind, which I liked a lot. It was an effective lesson."

According to the student-centered learning, which emphasize student responsibility and activity in learning (Attard et al., 2010; Slavich & Zimbardo, 2012; Todorovski et al., 2015) and considering that reciprocal teaching is an effective methodology able to create an interaction-rich and diverse environment especially when is used for problem solving (Palincsar & Brown, 1984; Tarchi & Pinto, 2016; van Garderen, 2004), we randomly divided the students into six groups of four and five and we gave them an hour and a quarter to deal with the problem. During that time, we roamed from group to group without intervening unless the students asked us to and noted students' ideas and approaches. Most of that approaches were based on intuition and the "trial and error" method, while others found multiple dead ends due to the "*lack of a third equation*", as several students wrote. Then, we collected the worksheets, and we analyzed them, together with our field notes based on students' reactions.

In the second meeting four days later, after informing the students about the history of "The hundred amphorae problem", we discussed with them their approaches to the problem and we presented successively to the classroom three different ways of solving it, taking steps together with the participants. Then we handed to the participants a questionnaire (provided as **Appendix B**) consisted of two closed-ended questions in 5-point Likert scale (Jamieson, 2004) and one open-ended question in order to assess the teaching intervention and give us feedback. The questions were, as follows:

1. The two lessons this week were interesting: not at all, a little, quite, much, very much.
2. I would like to encounter such problems as that of "The hundred amphorae", more often at University: strongly disagree, disagree, neither agree nor disagree, agree, strongly agree.
3. In one paragraph (approximately 100 words), please rate this week's teaching intervention.

In general, students' comments were positive, and it was clear that, regardless of whether or not they managed to solve the problem, they would like to relive such a "*different and uncommon*" learning experience, according to a student. At the end of the second meeting we had the opportunity to have a free discussion with our students about the whole process.

What we mainly wanted to investigate with our teaching intervention was:

(a) what strategies do Greek math students follow in order to deal with open-ended problems and

(b) if and how elements from the history of mathematics can be utilized in teaching and learning mathematics.

Our research has shared similar philosophy as other researchers in open-ended problems, and history of mathematics, focused tasks and riddles (Papadopoulos, 2020; Yazgan-Sag & Emre-Akdogan, 2016). That way we can compare some of our participants' reactions to the reactions of other participants in other projects and experiments. Similar research in groups of participants, in that case prospective math teachers, not undergraduate students, has been done and shown that facing difficulties and different situations in problems with historical background can change their path of thought and their pedagogical style (Clark, 2012; Furinghetti, 2007; Weldeana & Abraham, 2014). Another approach, closer to ours, is where the researchers suggested to the classroom that they cooperate in solving an open-ended problem, to improve their teamwork skills and come closer to a result (Suastika, 2017). In tandem with the history-focused tasks, math tasks played a crucial role in engaging teachers with teaching situations they are likely to face in the classroom and, through the discussion of said situations, provided opportunities to identify evidence of shifts in the teachers' mathematical and pedagogical discourses (Moustapha-Correa et al., 2021, p. 9).

In the second meeting we used a questionnaire for the participants to assess the process and their experience in our project (**Appendix B**). Researchers on the topic of problem solving in education seem to agree with the method of collecting data from the students via questionnaire. The technique of collecting the data in this study was a non-test of students' learning interests. The non-test used had passed a test with construct validity, content validity, and reliability, namely a questionnaire with 27 statements (Wulandari, 2021, p. 152).

Whatever content one tries to give to the term "mathematics education", this term will include references to the historical evolution of mathematics and its interaction with other sciences, to the complex phenomenon of classroom teaching in a specific social context, as well as in technology and human culture. Thus, our didactical approach combines problem solving together with history of mathematics in mathematics education. On the one hand we are interested in the different strategies that students adopt in order to solve an open-ended problem. On the other hand, as we saw in the previous paragraph, there is an exciting timeline of almost 1,800 years regarding to the "hundred fowls problem" and its variations, which we would like to keep alive in everyday school and university practice as we believe that, among other aspects, it can be a source of inspiration for students.

ANALYSIS OF RESEARCH DATA

The approach of triangulation was one of the fundamental methods we employed in our research in order to organize the evidence. In empirical research, triangulation refers to the combining of several aspects (theories, methodologies, data and observer viewpoints). We generally followed Denzin's (2009, p. 310-311) multiple triangulation technique, which advocates the use of numerous data collection methods over different time periods, as well as the participation of researchers with diverse expertise and roles. More specifically, methodological triangulation, a form of triangulation, refers to the combining of diverse approaches with the purpose of mutual validation of results or of obtaining a more sufficient and comprehensive image of the topic area through complementary outcomes (Kelle et al., 2019, p. 18).

In our situation, the three components are, as follows:

(a) the observer's point of view, in which an observer made notes during the experiment while without interfering with the procedure or the subjects (different viewpoints of two researchers),

(b) interviews, which were conducted by one researcher (the teacher) with some of the students, providing us with important pieces of dialogue, and

(c) the data, which consists of the worksheets that we received before (**Appendix A**) and after the experiment ended (**Appendix B**), on which each participant wrote down their methodology and thought process and their comments about the experiment.

The collected data in this paragraph are also organized in two tables (**Table 2** and **Table 3**).

Table 2. The categories of strategies by the participants

M	Description	F	P (%)
1	These students set additional variables in order to create equations to solve the problem. Specifically, they set A, B, C for each type of amphora and x, y, z for the corresponding number that one can buy. Although, they came to a dead-end because they had to solve a system with 6 unknowns.	2	7.40
2	These students tried to find a 3rd equation by using algebraic properties & combining Eq. (1) & Eq. (2).	3	11.11
3	These students tried random combinations of variables ("trial & error" method). Three of participants applied random combinations while another one applied "most costly amphorae" strategy (group C).	4	14.81
4	These students set an appropriate number in place of the variable z (a multiple of three) and solved the 2×2 resulting system of equations.	1	3.70
1+3	These students tried the first method & then they changed their strategy to "trial & error" method.	7	25.92
2+3	These students tried the second method & then changed their strategy to "trial & error" method.	7	25.92
4+3	These students tried the fourth method & then changed their strategy to "trial & error" method.	2	7.40
2+4	These students tried the second method & then changed strategy to the fourth method.	1	3.70
Total		27	100.00

Note. M: Method; F: Frequency; P: Percentage

Table 3. Students' answers to the first two questions of the questionnaire

Questions	SD	D	NAD	A	SA
This week's two lessons were interesting.	0 (0.00%)	2 (7.40%)	7 (25.90%)	9 (33.33%)	9 (33.33%)
I would like to encounter such problems as that of "hundred amphorae", more often at university.	2 (7.40%)	1 (3.70%)	3 (11.11%)	12 (44.44%)	9 (33.33%)

Note. SD: Strongly disagree; D: Disagree; NAD: Neither agree nor disagree; A: Agree; & SA: Strongly agree

The reason we chose this way of categorizing the evidence was mainly to provide a coherent account of the data and to have an–as much as possible–complete and certain image of the participants' reactions and strategies. This technique of using multiple ways of collecting data, such as worksheets with an open-ended problem and then interviews in groups of participants, has been integrated in similar research (Rahayuningsih et al., 2021).

The Task

In all the worksheets there were Eq. (1) and Eq. (2). However, almost every student in every group, at least in the beginning, was looking for "*a third equation*" in order to make the number of equations equal to the number of unknowns "*so that the system can be solved*", according to first-year student Ares[1].

Most students multiplied Eq. (1) by three to eliminate the unknown z and either they were led to a "dead end" because they emerged an equation with two unknowns ($7x + 4y = 100$ or, equivalent, $y = -\frac{7}{4}x + 25$ [3]), or they tried to give arbitrary values to x and y to satisfy it (the "trial and error" method). Next we describe the ideas and strategies of the members of each group.

Group A

The members of the group (three girls and one boy) did not have good cooperation with each other. Most of the time the three girls worked together and left the boy to work alone. After the first 15 minutes the group had difficulty figuring out how to define the variables (they set A, B, C for each type of amphora and x, y, z for the corresponding number that one can buy).

After everyone reached a dead end, they decided to seek the help of the teacher, in the middle almost of the available time. The teacher facilitated them by saying that they do not need so many variables and that the problem can be solved only by considering x, y, z. Ten minutes later everyone had come to Eq. (1) and Eq. (2), however it seemed that the confusion between the variables remained. Two students had the idea to write the extra relationship

$$y = 9z, \tag{4}$$

believing that they would eventually buy nine times the number of type B amphorae compared to type C amphorae, because the price of type B amphorae is nine times the price of type C amphorae. However, trying to solve the system of Eq. (1), Eq. (2), and Eq. (4) led again to a dead end.

Group B

The members of the group (three girls and one boy) did not cooperate very effectively. Firstly, they did not really show any interest in solving the problem or working as a team. After 20 minutes of attempting, a member gave up. Then, verbally, they had an intuitive understanding about "Pythagorean triplets", but they rapidly changed their minds. Following that, they begin to work more systematically, only to reach a dead end, producing apparent dissatisfaction and uncertainty. To discover a proper solution to the problem, all of the participants focused on applying the "trial and error" technique. Unfortunately, they were unable to locate the triplet.

Group C

This group's members (three boy and one girl) collaborated just mediocrely. They spent 15 minutes debating the issue and conversing with one another. After a half-hour, they discovered a slew of dead ends and loops in their thinking. They found two equations and continued to search for the third in order to solve the linear system, which they already knew how to do. 10 minutes later, a participant recommended removing the first two equations to obtain the third, but this yielded no results. One of them was unconcerned about the situation and just copied his colleague. The other began with trials on the costliest of the three amphorae (type A) and using approximation he attempted to identify the other two variables assuming he had already spent 100 coins. Another student attempted to fix this using several formal transformations, and the team followed suit, but with little success.

Group D

This group consists of one boy and four girls. They instantly requested help from the teacher, but he did not give it to them, encouraging them to cooperate. After about five minutes, they began utilizing the "trial and error" technique and choosing random values for the variables. They gave up after half an hour of attempting solving the problem. They merely considered making some restrictions concerning values (on maximum and minimum) before giving up for good. Some students

[1] We give to the students fiction names based on Greek mythology in order to preserve their anonymity.

began to use random values for the variable z (amphorae type C), such as three. Then they ran into some inequalities concerning the number of each form of amphora ($5x < 100 \Rightarrow x < 20$ and $3y < 100 \Rightarrow y < 33{,}33\ldots$). Finally they attempted other numbers and obtained an answer with 100 coins but not 100 amphorae.

Hera, first-year student, arbitrarily set $z = 30$ in equations (1) and (2),

> "because with the data of the exercise we get two equations and our unknowns are three, we will inevitably put a random variable equal to a constant number in order to find solutions that verify it,"

without however worrying about the fact that the ordered pair (x, y) she came up with did not consist of natural numbers (specifically $x = -60$).

Group E

This group (three boys and two girls) was quite functioning and effective. The members felt free to share their thoughts and challenge one another. Within the first 10 minutes, they began looking for the third equation. Then they divided into subgroups with diverse methods and ideas. When one participant quickly utilized the "trial and error" technique, another member thought formally and continued trying to obtain the third equation. They appeared to be interested in answering the mystery, and as time passed, they all, as a team, went through a number of psychological phases (anger, stubbornness, etc.), and their mood fluctuated frequently. Using his intuition, one of the participants came up with the solution (6, 20, 30) using 100 coins but not 100 amphorae. Another member picked at random to set the variable $z = 0$ and then attempted to solve the problem, but to no avail. After the girl's remarks on the issue, they attempted a new way and intended to apply formulae from the course Linear Algebra; therefore they began the following discussion with the teacher:

> Athena (second-year student): There should be a third equation!
>
> Teacher: Why is that obligatory?
>
> Athena: I know I must have as many equations as unknowns.
>
> Teacher: Can you solve this equation $(x-2)^2 + (y+3)^2 = 0$? [the teacher writes on Athena's worksheet]
>
> Athena: Yes, I think I can.
>
> Teacher: Why? This is one equation with two unknowns, similar to the riddle.
>
> Athena: This is quite easy. We have an obvious solution here… Wait a minute! We could maybe solve the riddle with the use of formulas from linear algebra.
>
> Teacher: What kind of formulas? If I gave you them, could you solve it?
>
> Athena: Yes, we could! We would apply them and just get the solution!

It seems that Athena considered the counterexample brought by the teacher as a "special case", still believing that in order to find n unknowns she must have n equations. She thus resorted to the *certainty* provided by the formulas–specifically those of linear algebra.

Group F

The last team (three boys and two girls) was the only one who partially solved the problem. The last variable was attempted to be expressed as a linear combination of the two others after one girl's suggestion. She reasoned that there could not be a solution with 100 amphorae, thus she proposed that the riddle did not have one. Then, the team experimented with identifying the multiples of a variable and then "filling in the gaps" with trials for the other two variables. They discovered all the multiples they could and began attempting, but to no avail. After that the participants started using the "trial and error" method and after a whole hour, one of them found one of the three valid solutions, namely the $(x, y, z) = (8, 11, 81)$ with eight amphorae type A, 11 amphorae type B and 81 amphorae type C, costing 100 coins totally. When the member of the group that suggested this method presented the solution to the teacher, the teacher asked him a question in order to understand his aspect:

> Teacher: Is this the only valid solution?
>
> Hermes (first-year student): I guess so, but I cannot prove it. Maybe there could be a combination in the formula $7x + 4y = 100$ …

We knew in advance that the task would not have a high "success rate", in the strict sense, because the vast majority of the participants had not been taught number theory or Diophantine equations. Therefore it would be very difficult for them to find a proper methodology that would lead them to a complete solution. However, the primary concern for us was not just "to find the solution" but, observing our undergraduate students, to focus on the solving strategies they develop and the extent to which they engage when exposed to open-ended problems with a historical context.

We divided the participants into groups because they have gone through one and a half year of isolation due to the pandemic outbreak (March 2020-August 2021) and we thought it would be beneficial for their social evolution and collaboration with each other. Indeed, not only they had social interaction, which was very needed by the time, but they also got to experience a very different environment of teamwork, which is not the case in Greek universities. Furthermore, working in a group can boost collaboration and idea sharing among students since they may feel more ready to share work with their peers (Deep et al., 2019, p. 18).

In **Table 2**, we can see the final distribution of the different aspects and approaches of the students.

Students' Opinions about the Project

In order to investigate the opinions of the students who participated in the project about the problem we posed and the whole teaching intervention, we asked them to fill in a questionnaire with three questions. The answers we received from the above questionnaire, together with the notes we took from both meetings and the discussion we had with our students, provided us with the necessary feedback. The implications of these views will be discussed in the next paragraph. The data we collected from the answers in the first two multiple choice questions in the feedback questionnaire (**Appendix B**) are presented below in a 5-point Likert scale:

From the comments we received as well as from the notes we kept from the discussion that took place at the end of the process, we quote some indicative student views.

Comments

> Rhea: I did not expect it because we know that Mathematics leads to specific results. In the problems [that we solved at school] we always found one or two solutions, but we rejected one e.g. if we found a negative length or time we rejected them. We did not find many pairs or triplets like here.

> Athena (second-year student): The exercise caught my interest and I worked hard, but I did not come up with a solution.

> Helen: I was impressed by the story and how many solutions you can find if you search […]. I would love to be a part of this experience again and a similar project.

Following the conclusion of the teaching intervention, we considered various requests from students to provide them the solution (and multiple ways of solving the riddle). So we demonstrated the three ways we described at the end of paragraph 3 (see more in Rizos & Gkrekas, 2022b, p. 4-6).

DISCUSSION

In our research, we wished to examine undergraduate students' strategies on open-ended problems and at the same time to investigate if and how history of mathematics can be integrated in its teaching. The answer to our first research question is

(a) students understood, at least to some extent, that a problem can have different solutions without any of them being "appropriate" or the most acceptable and

(b) students partially lost their certainty that "mathematics always leads to certain results".

The answer to second research question is that history of mathematics can play the role of an environment that can activate the students and in parallel give them the ability to evolve their different strategies. That notion about historical mathematical challenges is close enough to history-focused tasks of Moustapha-Correa et al. (2021).

No one of participants had ever seen a similar problem and therefore, they did not know how to approach it. This whole experience was an unknown but challenging path for the participants. The uncertainty inherent in the "hundred amphorae problem" led the students to develop alternative solution strategies, at least compared to those they had been taught at school and challenged somewhat their entrenched notions of pre-existent knowledge of the methodology and the uniqueness of the solution. However, the query of whether we wish to present such difficulties to our students or pupils, given that they are absent from the daily school and university practice, remains open and critical. But what has research nowadays shown about students' adaptability in changes in level of education? Researchers who also used open-ended questions, discovered that, while students' transition experiences appear to be uniform on the surface, significant differences emerge for motivation to pursue mathematics degrees, motivations for changes in perceived competence, and the impact of university mathematics formalism (Di Martino et al., 2022).

Perspectives on mathematics education have shifted over the last decades to stress conceptual understanding, higher-level problem-solving processes, and children's internal constructions of mathematical concepts in instead of, or in addition to, mechanical and algorithmic learning (Leavy & Hourigan, 2020). There does not appear to be any compelling evidence that problem-solving teaching has a detrimental influence and as a result, issue posing may be a good tactic for boosting students' problem-solving skills for teachers wishing to at least update their teaching approach and offer more contemporary methodologies (Calabrese et al., 2022). A significant part in problem solving seems to be the ability to understand when to change strategy and having a feeling of uncertainty when approaching the problem. In general, uncertainty seems to be a very important characteristic of scientific progress. Uncertainty can lead participants to a safer, more natural response to an unknown problem, which is the "trial-and-error" method. Our research participants were less meticulous in their estimations when trying to solve the problem, and their strategies were similarly trial-and-error in character (Puspitasari et al., 2019). It also seems that our data confirm that approach. In the vast majority of the participants, the "trial and error" method was used sooner or later (see methods 3, 1+3, 2+3, and 4+3 in **Table 2**).

In our research open-ended problems with historical context were shown to tend to reframe uncertainty in mathematics classroom, so they could play the role of appropriate teaching material or/and instructional technique for the benefit of mathematical teaching and learning (cf. Buckley & Sullivan, 2021). The usage of such issues has the benefit of posing the challenge of whether a solution or a list of solutions obtained in this manner is comprehensive. Observing our students' attempts and documenting their conjectures while trying to eliminate any ambiguity on this resulted in a number of valuable reports of their solution behavior like Poulos and Mamona-Downs (2018) observed. Thus, history of mathematics had a positive effect on students' interest in mathematics. With that conclusion, other researchers seem to agree (e.g. Arthur et al., 2022). Utilizing history of mathematics may be highly beneficial in mathematics education. A historical foundation may aid in comprehending differences and linkages between fields, transforming "the enormous corpus of knowledge" into a "live stimulus" for teaching (Jahnke et al., 2022, p. 1429). In that thought, we combined elements from open-ended problem solving and history of mathematics. From the answers in our assessment questionnaire and the interviews, it follows that the open-ended task we gave has got the participants' engagement, not only because of its originality, but also because of the great historical journey through the ages. In this specific issue, our research findings seem to converge with similar research (cf. Arcavi et al., 1987; Furinghetti, 2020).

Hundred fowls problem is a challenge that has tormented the mathematical traditions of both Eastern and Western civilizations for hundreds of years, and it may serve as a symbol that brings peoples and cultures closer together. At the same time, it emphasizes mathematics' complicated importance through its connection with other sciences such as history, literature, and technology. Apart from that, it seems to be of particular interest to mathematics education. This research could set the basis for a reframed new curriculum, which would contain the problem solving process and the topics of open-ended problems and problems with historical background. This new curriculum might give students the opportunity to evolve their problem solving abilities, which could help them apply them in their everyday problems and maybe approach mathematics differently.

CONCLUSION

Following the conclusion of our research experiment, several important results appear to emerge. We are certain that we have a full picture (worksheets, interviews, observer, feedback, monitoring the entire process, controlled environment) of the participants' reactions and strategies when faced with open-ended tasks and apply numerous problem-solving methodologies in the process. There were eight distinct techniques or combinations of strategies in the 27 replies, implying that the great majority, sooner or later, employed the "trial and error" method to solve the problem due to the "certainty" of finding a solution. The results of our research also showed that problems from the history of mathematics and open-ended problems can really initiate the feeling of engagement in the participants and probably help them to upgrade their problem-solving skills. This research might open a gateway for a formation of a new curriculum, containing problems inspired by the history of mathematics, open-ended tasks, and problem-solving in general.

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REFERENCES

Arcavi, A., Bruckheimer, M., & Ben-Zwi, R. (1987). History of mathematics for teachers. *For the Learning of Mathematics, 7*(2), 18-23.

Arthur, Y. D., Appiah, S. K., Amo-Asante, K., & Asare, B. (2022). Modeling student's interest in mathematics: Role of history of mathematics, peer-assisted learning, and student's perception. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(10), em2168. https://doi.org/10.29333/ejmste/12458

Attard, A., Di Iorio, E., Geven, K., & Sánta, R. (2010). Student-centered learning: Toolkit for students, staff and higher education institutions. *The European Students' Union*. https://www.esu-online.org/wp-content/uploads/2017/10/SCL_toolkit_ESU_EI.compressed.pdf

Awofala, A. O. A., & Ajao, E. A. (2021). Trends in research in problem solving in mathematics. *International Journal of Social Sciences & Educational Studies, 8*(3), 183-199. https://doi.org/10.23918/ijsses.v8i3p183

Baba, T., & Shimada, I. (2019). Socially open-ended problems for enriching student learning with mathematical models and social values. In P. Clarkson, W. T. Seah, & J. Pang (Eds), *Values and valuing in mathematics education: Scanning and scoping the territory* (pp. 171-183). Springer. https://doi.org/10.1007/978-3-030-16892-6_12

Bahar, A., & Maker, C. J. (2015). Cognitive backgrounds of problem solving: A comparison of open-ended vs. closed mathematics problems. *EURASIA Journal of Mathematics, Science and Technology Education, 11*(6), 1531-1546. https://doi.org/10.12973/eurasia.2015.1410a

Beghetto, R. A. (2017). Inviting uncertainty into the classroom. *Educational Leadership, 75*(2), 20-25.

Bicer, A., Marquez, A., Colindres, K. V. M., Schanke, A. A., Castellon, L. B., Audette, L. M., Perihan, C., & Lee, Y. (2021). Investigating creativity-directed tasks in middle school mathematics curricula. *Thinking Skills and Creativity, 40*, 100823. https://doi.org/10.1016/j.tsc.2021.100823

Bobis, J., Russo, J., Downton, A., Feng, M., Livy, S., McCormick, M., & Sullivan, P. (2021). Instructional moves that increase chances of engaging all students in learning mathematics. *Mathematics, 9*(6), 582. https://doi.org/10.3390/math9060582

Bonotto, C. (2013). Artifacts as sources for problem-posing activities. *Educational Studies in Mathematics, 83*, 37-55. https://doi.org/10.1007/s10649-012-9441-7

Brandenburg, R., & Nevenzeel, K. (2007). *The nine chapters on the history of Chinese mathematics*. https://www.astro.rug.nl/~nevenzeel/Study/PGvdW_t=9C_HCM_a=RB,KN.pdf

Brousseau, G., & Warfield, V. (2014). Didactic situations in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-94-007-4978-8_47

Buckley, S., & Sullivan, P. (2021). Reframing anxiety and uncertainty in the mathematics classroom. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-021-00393-8

Calabrese, J. E., Capraro, M. M., & Thompson, C. G. (2022). The relationship between problem posing and problem solving: A systematic review. *International Education Studies, 15*(4), 1-8. https://doi.org/10.5539/ies.v15n4p1

Campbell, D. M., & Higgins, J. C. (2019). *Mathematics: People, problems, results*. Chapman and Hall/CRC. https://doi.org/10.1201/9781351074315

Chorlay, R., Clark, K. M. & Tzanakis, C. (2022). History of mathematics in mathematics education: Recent developments in the field. *ZDM Mathematics Education, 54*, 1407-1420. https://doi.org/10.1007/s11858-022-01442-7

Cifarelli, V. V., & Cai, J. (2005). The evolution of mathematical explorations in open-ended problem-solving situations. *The Journal of Mathematical Behavior, 24*, 302-324. https://doi.org/10.1016/j.jmathb.2005.09.007

Clark, K. M. (2012). History of mathematics: illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics, 81*, 67-84. https://doi.org/10.1007/s10649-011-9361-y

Cuomo, S. (2007). *Pappus of Alexandria and the mathematics of late antiquity*. Cambridge University Press.

Deep, S., Mohd-Salleh, B., & Othman, H. (2019). Study on problem-based learning towards improving soft skills of students in effective communication class. *International Journal of Innovation and Learning, 25*(1), 17-34. https://doi.org/10.1504/IJIL.2019.096512

Denzin, N. K. (2009). *The research act: A theoretical introduction to sociological methods*. Transaction Publishers Rutgers.

Devlin, K. (2011). *The man of numbers. Fibonacci's arithmetic revolution*. Walker Books.

Di Martino, P., Gregorio, F., & Iannone, P. (2022). The transition from school to university mathematics in different contexts: affective and sociocultural issues in students' crisis. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-022-10179-9

Dostál, J. (2015). Theory of problem solving. *Procedia-Social and Behavioral Sciences, 174*, 2798-2805. https://doi.org/10.1016/j.sbspro.2015.01.970

Euler, L. (1828). *Elements of algebra*. Longman, Rees, Orme and Co.

Fatah, A., Suryadi, D., Sabandar, J., & Turmudi. T. (2016). Open-ended approach: An effort in cultivating students' mathematical creative thinking ability and self-esteem in mathematic. *Journal on Mathematics Education, 7*(1), 11-20. https://doi.org/10.22342/jme.7.1.2813.9-18

Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics, 66*, 131-143. https://doi.org/10.1007/s10649-006-9070-0

Furinghetti, F. (2020). Rethinking history and epistemology in mathematics education. *International Journal of Mathematical Education in*

Science and Technology, 51(6), 967-994. https://doi.org/10.1080/0020739X.2019.1565454

Goos, M., Galbraith, P., & Renshaw, P. (1999). Establishing a community of practice in a secondary mathematics classroom. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 36-61). Falmer Press.

Grabiner, J. (1995). Descartes and problem-solving. *Mathematics Magazine, 68*(2), 83-97. https://doi.org/10.2307/2691183

Gupta, R. C. (2008). Mahavira. In H. Selin (Ed.), *Encyclopedia of the history of science, technology and medicine in non-Western cultures*. Springer https://doi.org/10.1007/978-1-4020-4425-0_9267

Hadley, J., & Singmaster, D. (1992). Problems to sharpen the young. *The Mathematical Gazette, 76*(475), 102-126. https://doi.org/10.2307/3620384

Hafidzah, N. A., Aziz, Z. & Irvan, I. (2021). The effect of open ended approach on problem solving ability and learning independence in students' mathematics lessons. *Indonesian Journal of Education and Mathematical Science, 1*(1), 11-18. https://doi.org/10.30596/ijems.v2i1.6176

Hogarth, R. M., & Kunreuther, H. (1995). Decision making under ignorance: Arguing with yourself. *Journal of Risk and Uncertainty, 10*, 15-36. https://doi.org/10.1007/BF01211526

Ibrahim, I., & Widodo, S. A. (2020). Advocacy approach with open-ended problems to mathematical creative thinking ability. *Infinity, 9*(1), 93-102. https://doi.org/10.22460/infinity.v9i1.p93-102

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., da Silva, C. M. S., & Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel, & J. van Maanen (Eds), *History in mathematics education: The ICMI study* (pp. 291-328). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47220-1_9

Jahnke, H. N., Jankvist, U. T., & Kjeldsen, T. H. (2022). Three past mathematicians' views on history in mathematics teaching and learning: Poincaré, Klein, and Freudenthal. *ZDM Mathematics Education, 54*, 1421-1433. https://doi.org/10.1007/s11858-022-01376-0

Jamieson, S. (2004). Likert scales: How to (ab)use them. *Medical Education, 38*(12), 1217-1218. https://doi.org/10.1111/j.1365-2929.2004.02012.x

Kangshen, S., Crossley, J. N., & Lun, A. W. C. (1999). *The nine chapters on the mathematical art: Companion and commentary*. Oxford University Press.

Katz, V. J. (2009). *A history of mathematics. An introduction*. Pearson Education, Inc.

Kelle, U., Kühberger, C., & Bernhard, R. (2019). How to use mixed-methods and triangulation designs: An introduction to history education research. *History Education Research Journal, 16*(1), 5-23. https://doi.org/10.18546/HERJ.16.1.02

Keren, G., & Gerritsen, L. E. M. (1999). On the robustness and possible accounts of ambiguity aversion. *Acta Psychological, 103*(12), 149-172. https://doi.org/10.1016/S0001-6918(99)00034-7

Kosyvas, G. (2016). Levels of arithmetic reasoning in solving an open-ended problem. *International Journal of Mathematical Education in Science and Technology, 47*(3), 356-372. https://doi.org/10.1080/0020739X.2015.1072880

Leavy, A., & Hourigan, M. (2020). Posing mathematically worthwhile problems: Developing the problem-posing skills of prospective teachers. *Journal of Mathematics Teacher Education, 23*, 341-361. https://doi.org/10.1007/s10857-018-09425-w

Libbrecht, U. (1973). *Chinese mathematics in the thirteenth century*. MIT Press.

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problems solving in mathematics education*. Springer. https://doi.org/10.1007/978-3-319-40730-2

Mamona-Downs, J., & Downs, M. (2013). Problem solving and its elements in forming proof. *The Mathematics Enthusiast, 10*(1), 137-162. https://doi.org/10.54870/1551-3440.1263

Manz, E. (2018). Designing for and analyzing productive uncertainty in science investigations. In J. Kay, & R. Luckin (Eds.), *Rethinking learning in the digital age: Making the learning sciences count*. International Society of the Learning Sciences.

Meadows, M., & Caniglia, J. (2021). That was then … This is now: Utilizing the history of mathematics and dynamic geometry software. *International Journal of Education in Mathematics, Science, and Technology, 9*(2), 198-212. https://doi.org/10.46328/ijemst.1106

Mousoulides, N., & Sriraman, B. (2020). Heuristics in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_172

Moustapha-Correa, B., Bernades, A., Giraldo, V., Biza, I., & Nardi, E. (2021). Problematizing mathematics and its pedagogy through teacher engagement with history-focused and classroom situation-specific tasks. *The Journal of Mathematical Behavior, 61*, 100840. https://doi.org/10.1016/j.jmathb.2021.100840

Murwaningsih, T. & Fauziah, M. (2022). The development of open-ended questions test to measure divergent thinking skills: The development of open-ended questions test. In

Proceedings of the 5th International Conference on Learning and Quality Education (pp. 1-7). Association for Computing Machinery. https://doi.org/10.1145/3516875.3516876

NCTM. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Palincsar, A. S., & Brown, A. L. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction, 1*(2), 117-175. https://doi.org/10.1207/s1532690xci0102_1

Papadopoulos, I. (2020). Using tasks to bring challenge in mathematics classroom. *Journal of Pedagogical Research, 4*(3), 375-386. https://doi.org/10.33902/JPR.2020063021

Pappas, T. (1989). *The joy of mathematics*. Wide World Publications/Tetra.

Pehkonen, E. (2017). Open tasks in mathematics: Experiences with one problem field. *Magistra Iadertina, 12*(2), 9-19. https://doi.org/10.15291/magistra.1487

Pehkonen, E., Näveri, L., & Laine, A. (2013). On teaching problem solving in school mathematics. *CEPS Journal, 3*(4), 9-23. https://doi.org/10.26529/cepsj.220

Polya, G. (1945). *How to solve it: A new aspect of mathematical method.* Princeton University Press. https://doi.org/10.1515/9781400828678

Polya, G. (1954). *Mathematics and plausible reasoning*. Princeton University Press. https://doi.org/10.1515/9780691218304

Polya, G. (1962). *Mathematical discovery*. John Wiley.

Poulos, A., & Mamona-Downs, J. (2018). Gifted students approaches when solving challenging mathematical problems. In F. Singer (Ed.) *Mathematical creativity and mathematical giftedness*. Springer. https://doi.org/10.1007/978-3-319-73156-8_12

Puspitasari, L., In'am, A., & Syaifuddin, M. (2019). Analysis of students' creative thinking in solving arithmetic problems. *International Electronic Journal of Mathematics Education, 14*(1), 49-60. https://doi.org/10.12973/iejme/3962

Rahayuningsih, S., Sirajuddin, S., & Ikram, M. (2021). Using open-ended problem-solving tests to identify students' mathematical creative thinking ability. *Participatory Educational Research, 8*(3), 285-299. https://doi.org/10.17275/per.21.66.8.3

Randles, C., Overton, T., Galloway, R., & Wallace, M. (2018). How do approaches to solving open-ended problems vary within the science disciplines? *International Journal of Science Education, 40*(11), 1367-1390. https://doi.org/10.1080/09500693.2018.1503432

Rizos, I., & Gkrekas, N. (2022a). Teaching and learning sciences within the COVID-19 pandemic era in a Greek university department. *U.Porto Journal of Engineering, 8*(1), 73-83. https://doi.org/10.24840/2183-6493_008.001_0008

Rizos, I., & Gkrekas, N. (2022b). The historical background of a famous indeterminate problem and some teaching perspectives. *Journal of Mathematics and Statistics Studies, 3*(1), 1-8. https://doi.org/10.32996/jmss.2022.3.1.1

Rohmah, M., & Sutiarso, S. (2018). Analysis problem solving in mathematical using theory Newman. *EURASIA Journal of Mathematics, Science and Technology Education, 14*(2), 671-681. https://doi.org/10.12973/ejmste/80630

Rowland, T. (1995). Hedges in mathematics talk: Linguistic pointers to uncertainty. *Educational Studies in Mathematics, 29*(4), 327-353. https://doi.org/10.1007/BF01273910

Rus, T. (2016). Milestones for computing future. *Journal of Software Engineering and Applications, 9*, 52-56. https://doi.org/10.4236/jsea.2016.92003

Schoenfeld, A. H (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). Macmillan.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.

Schoenfeld, A. H. (1987). Polya, problem solving, and education. *Mathematics Magazine, 60*(5), 283-291. https://doi.org/10.2307/2690409

Scoones, I. (2019). What is uncertainty and why does it matter? *STEPS Center*. https://steps-centre.org/publication/what-is-uncertainty-and-why-does-it-matter/

Sesiano, J. (2008). Abu Kamil. In H. Selin (Ed.), *Encyclopedia of the history of science, technology and medicine in non-Western cultures*. Springer. https://doi.org/10.1007/978-1-4020-4425-0_9198

Sesiano, J. (2009). *An introduction to the history of algebra: Solving equations from Mesopotamian times to the Renaissance*. American Mathematical Society.

Skovsmose, O. (2020). Critical mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_34

Slavich, G. M., & Zimbardo, P. G. (2012). Transformational teaching: Theoretical underpinnings, basic principles, and core methods. *Educational Psychology Review, 24*, 569-608. https://doi.org/10.1007/s10648-012-9199-6

Suastika, I. K. (2021). Mathematics learning syntax model using open-ended problem solving to develop students' creativity. *Pancaran Pendidikan FKIP Universitas Jember* [*Education Radiation FKIP University of Jember*], *6*(4), 105-112. https://doi.org/10.25037/pancaran.v6i4.118

Tanjung, D. F., Syahputra, E. & Irvan, I. (2020). Problem based learning, discovery learning and open ended models: An experiment on mathematical problem solving abilities. *Jurnal Teori dan Aplikasi Matematika* [*Journal of Mathematical Theory and Applications*], *4*(1), 9-16. https://doi.org/10.31764/jtam.v4i1.1736

Tarchi, C., & Pinto, G. (2016). Reciprocal teaching: Analyzing interactive dynamics in the co-construction of a text's meaning. *The Journal of Educational Research, 109*(5), 518-530. https://doi.org/10.1080/00220671.2014.992583

Todorovski, B., Nordal, E., & Isoski, T. (2015). Overview on student-centered learning in higher education in Europe: Research study. *The European Students' Union*. https://www.esu-online.org/wp-content/uploads/2016/07/Overview-on-Student-Centred-Learning-in-Higher-Education-in-Europe.pdf

Ulinnuha, R., Budi Waluya, S., & Rochmad, R. (2021). Creative thinking ability with open-ended problems based on self-efficacy in gnomic blended learning. *Unnes Journal of Mathematics Education Research, 10*(A), 20-25.

van Garderen, D. (2004). Focus on inclusion reciprocal teaching as a comprehension strategy for understanding mathematical word problems. *Reading & Writing Quarterly, 20*(2), 225-229. https://doi.org/10.1080/10573560490272702

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Waxer, M., & Morton, J. B. (2012). Cognitive conflict and learning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning*. Springer. https://doi.org/10.1007/978-1-4419-1428-6_280

Weldeana, H. N., & Abraham, S. T. (2014). The effect of an historical perspective on prospective teachers' beliefs in learning mathematics. *Journal of Mathematics Teacher Education, 17*, 303-330. https://doi.org/10.1007/s10857-013-9266-z

Wulandari, A. (2021). The effectiveness of the application of learning models creative problem solving and problem based instruction: The impact on students mathematics learning interest. In *Proceedings of the 2nd International Conference on Technology and Educational Science* (pp. 150-155). Atlantis Press. https://doi.org/10.2991/assehr.k.210407.229

Yazgan-Sag, G., & Emre-Akdogan, E. (2016). Creativity from two perspectives: Prospective mathematics teachers and mathematician. *Australian Journal of Teacher Education, 41*(12), 25-40. https://doi.org/10.14221/ajte.2016v41n12.3

Yong, L. L. (1977). *A critical study of the Yang Hui Suan Fa*. Singapore University Press. https://doi.org/10.1007/BF00374594

Yong, L. L. (1997). Zhang Qiujian Suanjing (the mathematical classic of Zhang Qiujian): An overview. *Archive for History of Exact Science, 50*(3-4), 201-240.

Yuniarti, Y., Kusumah, Y. S., Suryadi, D., & Kartasasmita, B. G. (2017). The effectiveness of open-ended problems based analytic-synthetic learning on the mathematical creative thinking ability of pre-service elementary school teachers. *International Electronic Journal of Mathematics Education, 12*(3), 655-666. https://doi.org/10.29333/iejme/640

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*, 297-321. https://doi.org/10.1007/s10649-005-0606-5

Zengin, Y. (2018). Incorporating the dynamic mathematics software GeoGebra into a history of mathematics course. *International Journal of Mathematical Education in Science and Technology, 49*(7), 1083-1098. https://doi.org/10.1080/0020739X.2018.1431850

APPENDIX A: THE WORKSHEET

"The Hundred Amphorae Problem"

Name: …

Date: …

In the time of Diophantus (3rd century AD), in Alexandria, as in the whole Greek world, the main means of transport and storage of goods such as olive oil, wine, cereals, etc. was the amphora. A workshop in Alexandria made three types of amphorae. Type A cost five coins, type B cost three coins, while with one coin you bought three type C amphorae. How many amphorae from each type, a total of 100, could one buy with 100 coins?

APPENDIX B: THE QUESTIONNAIRE

The following questions are about the evaluation of the teaching intervention that took place this week (Diophantine equations, the hundred amphorae problem, approaches and solutions etc.). The answers are confidential, and their honesty will help improve the educational process.

Name: …….

Date: ……

1. This week's two lessons were interesting:

▢ Not at all

▢ A little

▢ Quite

▢ Much

▢ Very much

2. I would like to encounter such problems as that of "hundred amphorae", more often at university:

▢ Strongly disagree

▢ Disagree

▢ Neither agree nor disagree

▢ Agree

▢ Strongly agree

3. In one paragraph (approximately 100 words) please rate this week's teaching intervention.

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