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LEARNING MATHEMATICAL RULES WITH REASONING

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ABSTRACT. This research focused on students learning of mathematical rules with reasoning. A small group of students (age 11-12 yrs) was observed closely by the first author as she taught them fraction rules. The area of focus was fractions and activities were designed pertaining to the four rules of fractions: addition, subtraction, multiplication, and division. The study was undertaken from a constructivist perspective of learning according to which students' learn through active participation in the construction of knowledge (Glasersfeld, 1995). This was significant in the context of mathematics classrooms in Pakistan which usually subscribe to the objectivist epistemology i.e. knowledge of the ultimate reality is possible. An implication of adhering to this epistemology is the knowledge transmission view of teaching and learning (Halai, 2000). Key findings of the study were that there were two significant factors that enabled students to learn rules with reasoning were: Teachers questions and opportunities for students to explain thinking; and opportunity for students to engage with concrete and semi concrete materials. The study also provides some useful insights into the sequence of teaching the fraction rules and raises implications for mathematics teaching and teacher education.

KEYWORDS. Reasoning, Mathematical Rules, Understanding, Process of Learning Mathematics.

THEORETICAL FRAME WORK

A perspective which has greatly influenced the understanding of how students learn comes from Piaget's constructivism. Piaget (1959), provided a radical shift in the outlook toward the learners as active participants in the process of coming to know as opposed to passive recipients of knowledge. The ideas discussed above regarding learners as active participants in knowledge construction from Piaget (1959, 1969) were useful for me in making sense of students' learning and provided underpinnings for the teaching that I undertook as part of my research classroom.

Following from the work of Piaget, von Glasersfeld (1995, p.51) proposed the following two tenets of radical constructivism which claim:

1. knowledge is not passively received but is actively built up by cognising subject,

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2. the function of cognition is adaptive and serves the organisation of the experiential world, and not discovery of ontological reality.

A consequence of taking this view of learning is that individual learners construct unique and idiosyncratic personal knowledge when exposed to identical stimuli. There can be no transfer of knowledge from outside. The notion of personally constructed knowledge or constructivism offers a new set of assumptions about learning and adherents to this radical constructivist theory have interpreted it, and drawn from it, principles to set up teaching and learning situations in classrooms e.g. Cobb, Wood and Yackel (1995). On the basis of my experience as a mathematics teacher, and subsequently, as a researcher in mathematics classrooms, I found Glasersfeld's first tenet helpful to explain why individual students responded differently to the same teaching experience in the classroom.

The second tenet is radical in the sense that relinquishing the belief that knowledge must represent a reality that lies out side our experience is an enormous and frightening step (Glasersfeld, 1995). The enormity of considering the nature of knowledge as not fixed and objective was even more so for mathematics learning because the discipline of mathematics has been imbued with certainty. Acknowledging constructivist principles of knowledge and of coming to know would imply acknowledging the fallible nature of mathematics (Lakatos, 1976) with implications for classroom teaching and learning. For me an understanding of these theoretical perspectives and their practical manifestations meant, that, as a researcher I could recognise and appreciate the dilemmas arising during my teaching: for example, when students tried to construct their own meaning of mathematics while subscribing to the objectivist epistemology of mathematics i.e. there lies an objective body of (mathematical) knowledge which it is their aim to learn. An issue was that radical constructivism does not overtly emphasise the power of negotiation and social interaction on individual construction.

Confrey (1995), says that constructivism has the 'social' implicit in its theoretical position so that it is not necessary to have alternate theories to explain how the social and cultural elements are incorporated in its theory. For example she says that, "knowing is justified belief" (Confrey, 2000, p.12) i.e. for it to be regarded as knowledge the belief has to be justifiable to oneself and others which is in the essence of 'fit'. As students in my research engaged in their mathematics work and I observed them, I had some problems with the practical manifestation of the tenets of constructivism in settings where the genesis of learning was in the social interaction. For example, as students in my research classroom worked at mathematics problem solving tasks in small group settings and constructed their own mathematical understandings, they did not do so in isolation. Interactions with both other students and teacher gave rise to crucial learning opportunities. Thus, I found that collaborative work involved developing explanations that could be understood by others and trying to interpret and make sense of another's ideas and solution attempts as they evolved. I provided students with opportunities to give coherent explanations of their problems, interpretations, and solutions, and to respond to questions and challenges by their peers. They were also expected to listen to and try to make sense of explanations given by others, to pose appropriate questions, and to ask for clarifications. When students engaged in such a discourse, the nature of their mathematical activity was extended to encompass learning opportunities that had their roots and beginnings in social interaction. I reckoned that there was no contradiction here with radical constructivism.

Research Process

As part of the research process I taught and observed four boys Asif, Basit, Farhan and Rizwan (age 11-12 yrs.). Students stayed behind after school and the teaching took place in eight one hour sessions outside the routine classroom. I designed my lessons on four rules of fractions, pertaining to addition, subtraction, multiplication and division of fractions, and the order of dealing with operations, because fractions is an important topic included in the national curriculum of Pakistan.

I audio taped all the eight teaching sessions, and alter transcribed them. I also maintained a research journal where I described my observation, identified issues and emerging questions and planned further teaching in light of the ongoing analysis as a result of the journal recording.

INTRODUCTION OF FRACTIONS

I undertook the teaching of fraction rules from a constructivist perspective as discussed above. However, I wanted to look at the process of learning rules with reasons. Skemp (1991) makes a distinction between instrumental learning and relational learning.

Before starting with the rules of the four basic operations on fractions, I tried to create a conducive environment in the group through introductory sessions. These sessions helped me to know about the children's prior knowledge of fractions. These sessions provided me a ground to build the later sessions, and helped me for further planning. These sessions also helped me to create an atmosphere of trust and relationships between the researcher and the participants. I think building relationships and creating an atmosphere of trust are the nuts and bolts between the researcher and the research participant, because with-out a conducive environment in the group research is difficult to carry out.

According to Piaget's theory of constructivism, using manipulative is physical knowledge and fraction pies and strips are often recommended to teach fractions, and students are also given paper strips to fold to see that 3/4 > 2/3. (Kamii,1999). To introduce fractions to the children, I selected a task taken from Burn (1992), to introduce fractions as part of a whole and to enable students to see patterns and relationships in fractions. They had to make fraction strips and manipulate them to discover the above mentioned four rules of fractions by themselves, rather then memorizing them. For example, the children were asked to work on paper strips and see how 1/2 +1/4 +1/4 = 1 or how many 1/8 are in one 1/2. I provided five 3-by-18 inch strips in five different colors, and a pair of scissors to each student. Having students cut and label the pieces helped them to relate the fractional notation to the concrete pieces, and to compare the sizes of fractional parts. For example, they could easily see that 1/4, is larger then 1/16, and they could measure to prove that two of the 1/8 pieces are equivalent to 1/4.

1/2	1/2

1/4	1/4	1/4	1/4

1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16

1

(Description about the fraction strips is provided in Appendix G).

After preparing the fraction strips, I provided worksheets (see Appendix H) to the children to complete by using fraction strips. The worksheets contained different questions related to fractions. For example, there were questions about equivalent fractions, concept of greater and smaller fractions. While doing these tasks, I observed that at the beginning students did not use fraction strips to solve the worksheet. They were doing the tasks mentally. I requested a number of times to explain their answers with the help of strips because I found that they were not able to explain how they got the answers through their mental work. As I was interested in their reasoning I wanted them to explain and believed that with the help of those strips they would be able to see and talk about the patterns and relationships among fractions.

A sample of the transcription is provided in box 1.

Box 1.

Shahida: How many 1/4 s are equal to one S ?
Farhan: 2.
Shahida: How?
Farhan: Miss, because, these two j are equal to one 1/2. When we multiply S Ч S then we get j.
Shahida: Why are we multiplying here?
Farhan: (silence).

(*Audio transcript*)

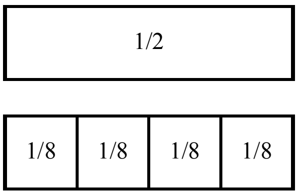
It is visible in the above mentioned conversation, that Farhan was able to give the correct answer. But when Basit used the fraction strips for the same question, he was not only able to give the correct answer but he was also able to justify his answer. He showed two strips of j and said, "when we join these two 1/4s, it is equal to one S". So, the possibility of joining the strips helped Basit. I was asking the children to put their strips in order on the table to enable them to see the different patterns, because my focus was also to see the process of learning mathematical rules with reasoning. I was observing that fraction strips were very helpful in giving reasons. In the following conversation, described in box 2, it is visible that fraction strips helped Rizwan to understand his question.

Box 2.

Shahida: Rizwan, how many eighths in S?
Rizwan: 8 no, no 2.
Shahida: Can you explain us how 2 or how 8 strips of 1/8 are equal to 1/2?
Rizwan: O.k. miss, four 1/8 s are equal to one 1/2.

(*Audio transcript*)

When I asked Rizwan to arrange paper strips according to the questions and see the arrangement of strips, he did so, and finally reached to the correct answer. He arranged his strips like this:



On the basis of my observation, I inferred that strips of fractions helped him to give a reason for his answer, and it also helped him to understand how four 1/8 s are equal to 1/2 because the strips were giving him a clear picture.

When the children were solving the tasks, I was trying to infer their understanding about fractions because I did not have much information about the children's prior knowledge of fractions. For example, the following conversation (Box 3) provided me information about Farhan's understanding of addition of fractions.

Box 3.

Shahida: Yes Farhan, can you explain us with the help of strips?
Farhan: These two 1/4 are making one 1/2.
Shahida: How?
Farhan: Because, these two 1/4 are equal to one 1/2. when we multiply 1/2 x 1/2 then we get 1/4.
(*Audio transcript*)

It was helpful making plans for further sessions.

I was asking them frequent questions to get the idea of fractions. Some of the questions were like the following:

- How many 1/2 strips are equal to 1 whole?
- How many 1/8 strips are equal to 1/4?
- Which is bigger 1/8 or 1/2?

This session provided me sufficient information about the children's understanding of fractions. I felt ready to move ahead and deal with the rule regarding addition of fractions.

ADDITION OF FRACTIONS $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

I designed a worksheet in order to know about the students' learning of the rule "addition of fractions" (see Appendix I). I designed the worksheet in such a manner that it would enable me to see the children's learning process easily. Specially, how the children learn with reasoning when they are involve in the rule $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

The worksheet was systematically designed from easy to complex. It consisted of three types of addition of fractions, such as addition of fractions with the same denominator, addition of fractions with different denominators, and the addition of improper fractions.

When the children were adding fractions of the same denominator, they did not find any difficulty. Their prior knowledge was very strong in this case. They were even able to justify their work. An example is quoted in box 4.

Box 4.

> Shahida: What is 1/2 + 1/2 ?
> Farhan: 1.
> Shahida: How?
> Farhan: Miss, if we put two halves together it is equal to 1.
> (*Audio transcript*)

The children did not have much trouble with adding fractions with different denominators. Their justifications were based on figures.

I was looking at the process, that is, how they were arriving at the answer. They were solving on the paper very quickly. At that time it was difficult for me to see the process. When they finished, I posed different questions and asked them to explain the solution of their tasks. The children described the process as given in box 5.

Box 5.

> Basit: 3/5 + 2/3 = 5/8 because we add the fractions with different denominators and in the answer there will be different denominator as well. So, the answer will be 5/8. Because 3 + 2 = 5 and 5 + 3 = 8 there fore it will be 5/8.
> Shahida: Can you show me with figure?
> Basit: First we plus 3 and 2. So, it will be 5, then we plus 5 and 3 and it will be 8. So, answer will be 5/8. It means that there are total 8 boxes and 5 out of them are shaded.
> (*Audio transcript*)

When Asif used multi link cubes for his solutions, he described it in the following conversation, given in box 6.

Box 6.

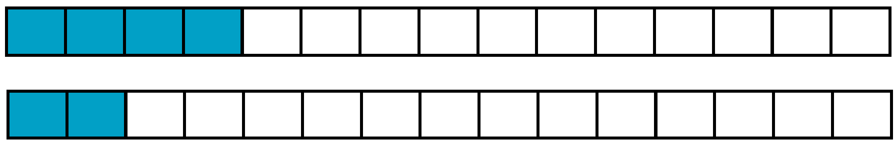
Shahida: Asif! Please can you explain your solution of the question 3/8 + 1/ 8?
Asif: White cubes are denominators and dark green cubes are numerators. I make 8 parts and 3 out of 8 are green. In second, their is 1 out of 8 is green.
Shahida: How many total greens do you have?
Asif: I have total 4 green cubes. So, 4/8 is answer.
(*Audio transcript*)

SUBTRACTION OF FRACTIONS $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

The next topic of the session was subtraction of fractions. I designed the worksheet for this session with a similar idea in my mind as I had for addition of fractions. I mean, the tasks started with the subtraction of fractions with the same denominators, with different denominator, and the subtraction of improper fractions respectively (see Appendix J).

In this session, I noticed a slight change in the students' explanations for justifications. For example, Asif started his explanations with drawings. I am reproducing a part of the conversation, while Asif was explaining the solution of the question 4/15 – 2/15 with drawing, as follows:

Asif: In my question 4/15 – 2/15. I minus 2 from 4 and the denominator is same. Therefore, 15 will be the same in answer. And the answer will be 2/15. I make figure like this:

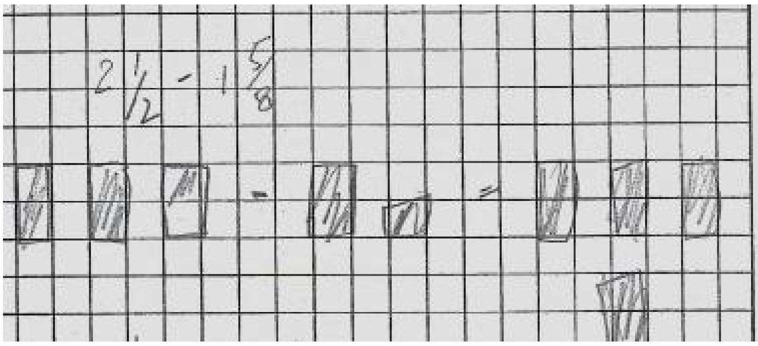


In first figure, I make 15 parts and shade four parts. in second figure, I make 15 parts and shade 2 parts. Miss is it minus? Oh, it's wrong. I added it. I take this 2 and minus from 4. I did mathematically right but on figure it is wrong.
(*Audio transcript*)

Here, it is apparent that the drawing helped Asif to understand the question and he himself realized his mistake with of the help of the drawing. It was difficult for him to describe with figures.

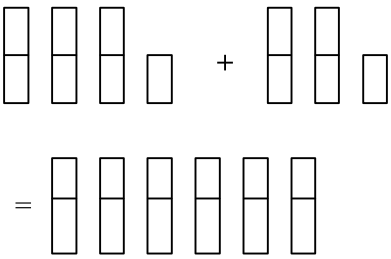
I asked the other children to describe their questions of subtraction of fraction of improper fractions with figures. For example, Basit's work (refer to figure 1). It was difficult for him to show 1 $^{5}/8$ even before showing the process of subtraction. He also had difficulty in the process of subtraction. Instead of subtracting, he was adding the fraction. So, it was quite difficult for children to subtract the improper fractions.

Figure 1.



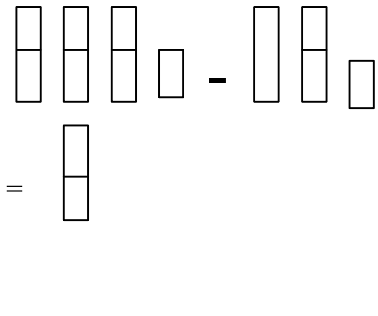
Kamii (1999) also shares the same idea and says, “adding and subtracting fractions tend to be quite difficult for children when unlike denominators are involved” (p.86). When I observed this issue, I tried to create a situation which would help them to show pictorial representations of subtraction of fractions, because I believed that pictorial representation would be helpful in order to learn the rule of subtraction of fractions with reasoning. I tried to relate subtraction of fractions with addition of fractions, because I noticed in the earlier session of addition of fractions that children were quite comfortable with the pictorial representation of addition of fractions. Burns (1992) also suggests, “classroom instruction should build on children’s previous experiences and help children clarify the ideas they have encountered” (p.212). I instructed the children to use the sign of subtraction instead of addition and solve accordingly. For example, I asked Basit to illustrate $3\ ^{1}/_{2} + 2\ ^{1}/_{2}$. He illustrated the process as described in figure 2.

Figure 2.



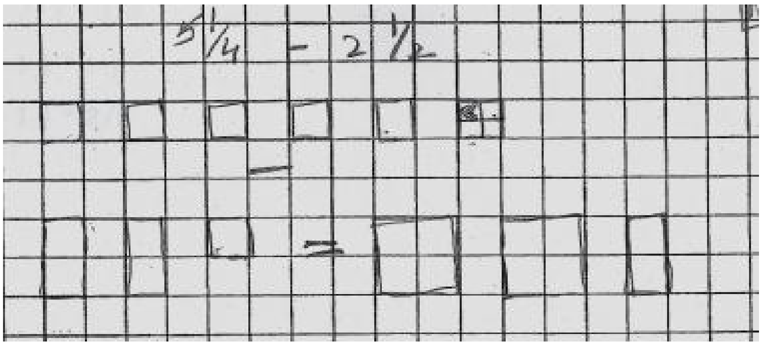
I told Basit to put the sign of minus instead of plus and solve. He did as I instructed.

Figure 3.



I asked the children to think and use other alternatives to solve the problems. I noticed that they started doing it with multi link cubes. Multi links cubes were available for them to use. The children put the cubes as shown in figure 4.

Figure 4.



It was a good exposure for children to check whether the use of manipulative and pictorial representations was workable for all rules of fractions or not. Burns (1992) also emphasizes the use of diverse traditions to introduce fractions, and says, " it is important to provide a variety of ways students can learn about fractions – with concrete materials, from a geometric perspective, with a numerical focus, and related to real life situations" (p. 213). It was also easy for me to explain to them with the help of manipulatives because it was quite visible for them to see the size of the cubes, which were not same for $2\ ^{1}/_{2}$ and $1\ ^{5}/_{8}$. Then I explained to them that the sizes of all cubes should be equal.

I noticed that meaningful interaction in the mathematics class also helped the children to clarify their arguments. In the particular session about subtraction of fractions, when the children were involved in the process of doing mathematics, they were discussing with each other. They were putting cubes in different manners to do subtraction of fractions. They were arguing with each other and at times they were looking for my help when they needed it. As Lindquist et al, m (1995) say, "… interaction allows students opportunities to talk about their ideas, get feedback for their thinking, and hear other points of view. Thus, students learn from one another as well as from teacher" (p.24).

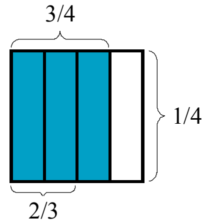
MULTIPLICATION OF FRACTIONS $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

The next rule was "multiplication of fractions". In this session, the students worked with the formula of multiplication, i.e. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. I planned some tasks (see Appendix K) to see the students' learning of the rule of multiplication of fractions with reasoning.

In this session, I observed a change. Students were solving the work sheet with pictures or by using multi link cubes, instead of solving mentally. They were trying to prove their solutions. Asif was working on the task 2/3 x 3/4. He explained it as follows (box 7).

Box 7.

Shahida: Can you explain your solution of the question (2/3 x 3/4)?
Asif : I have colored three boxes out of four.
Shahida: For 3/4, how many total parts have you made in the figure?
Asif : Four.
Shahida: How many you are taking from four?
Asif : Three.
Shahida: Yes, from these three parts what do you need?
Asif : 2/3 because it is 2/3 of 3/4.
Shahida: What is the remaining here?
Asif : One fourth and I also got 1/4 by doing mathematically.
Like : 2/3 x 3/4
= 6 /12
= 1/4.



(Asif's diagram)

(*Audio transcript*)

This scenario helped me to understand that Asif already had prior knowledge of symbolic representation of multiplication of fractions. At this step, he was relating that previous knowledge with the current one and justifying his solution with the figure. So, he explained his mathematical solution through pictorial representation.

I was thinking that children could easily make pictorial representations of all tasks related to fractions. But, I found that this claim was not always true. In a later session, children got stuck when they reached the question (1 $^{1}/2$ x 1 $^{1}/2$). It was difficult for them to make pictorial representation, as well as present concretely. I quote apart of the conversation in box 8.

Box 8.

Basit: Can you help me teacher for the question 1/2 x 1/2 ?
Shahida: Your question is 1 and 1/2 x 1 and 1/2.
Basit: 1 and 1/2 x 1 and 1/2.
Shahida: It means you have something one whole and half of that.
Basit: In this question we will take 1 and 1/2 and another 1 and 1/2.
Shahida: Why another? Are we going to add ?
Basit: Yes, miss.
Shahida: O.K. tell me that how will you add this?
Basit: We will multiply the denominators. So, 1/2 =2 and add 1. it will be 3. So, it will be 3/2.

(*Audio transcript*)

Another response to the same question was in the following form:

Shahida: Asif, what do you think about 1 1/2 and 1 1/2?
Asif : I will make one whole and one half of that. When I will put these join, it will become half and half. One whole and I will also add two wholes. Then it will be total 3 wholes. But they easily solved the question mathematically. (Asif solved the question like this):

1 1/2 x 1 1/2
= 3/2 x 3/2
= 9/4
= 2 1/4.

(*Audio transcript*)

Here, I realized that the children did not have a clear concept of multiplication of fractions. They followed the rule of multiplication of fractions but could not explain its meaning. Burns (1992) describes the same idea and says, “giving students rules to help them develop facility with fractions will not help them understand the concepts. The risk is that when students forget a rule, they have no way to reason out through a process” (p.213).

When I was reflecting on the session, I was frequently asking myself, “Is teaching the rule of multiplication of fractions not appropriate for grade six children? Or there is any problem with my teaching?” Later, literature helped me to understand the problem. Kamii (1999) describes, “when teaching multiplication, we do not say “multiply” or use the symbol ‘Ч’ until well into our instruction. We believe that saying “of means to multiply” imposes words on children that do not make sense to them” (pp. 89-90). So, as teacher we do not start using the symbols at the beginning. Teachers have to make simple stories to explain mathematical concepts and the stories should be according to the children’s age and interest. When children get enough understanding then the teachers should use signs related to the concepts such as 2/3 of 3/4 or 3/4 x 4/5.

When the children were stuck, they asked me for help. I started to work it out in the class but the time was over. After the class, I sat down and tried to do pictorial representation of $1\ ^{1}/_{2}$ x $1^{1}/_{2}$. It was not easy as the other questions. I spent more then one hour but I could not find the solution. Then I consulted my colleagues, who were interested. They also tried but were not able to work it out through pictures. After all these efforts, I found out that not all the rules can be explained by pictorial representation and not all rules can be explained by materials.

DIVISION OF FRACTIONS $\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$

For the rule of division of fractions, I designed some tasks according to the level of grade six students. (see Appendix L). These were simple tasks, because in previous sessions I had found out that students faced difficulty in doing complex tasks. Through these tasks I wanted to know the children’s understanding of division of fractions. When I tried to know about their prior knowledge about division of fractions, the following response was received box 9.

Box 9.

Researcher: What do you mean by divide?
Asif: Making parts.
Researcher: So, what is 1/2 ÷ 3?
Asif: It means that how many parts will make three?

(*Audio transcript*)

It was essential for children to have a clear concept of division with whole numbers before they moved to the division of fractions. Kamii (1999) says, “to teach division with fraction, we begin with the simplest of problems. It is important to clarify the students' conceptual understanding of division with whole numbers before asking them to apply it to fractions" (p.88).

So, I tried to pick up a concept which was familiar to them. I asked them to share their understanding of 4 2. They provided me a clear explanation. For explaining 3/7, it was useful to relate it with 1/2 because the children were easily understanding the meaning of 1/2, and these could easily be represented by figures and by concrete materials. I used the strategy of 'known to unknown' in the beginning sessions, which was quite helpful in moving further, or helping the students to understand the concept. In the following box 10, it is apparent that the concept of 1/2 helped Basit to understand 3/7.

Box 10.

Shahida: What do you mean by 3/7 of this shape?
Basit: (……….) silence.
Shahida: What do you mean by 1/2?(Basit drew  this figure).
Shahida: Now, you have to do the same for 3/7. How many parts will you make?
Basit: 7 and I am shading 3 parts out of 7.
Shahida: How many parts are left?
Basit: 4 parts.
Shahida: Can you write this fraction?
Basit: Yes, it is 3/7.
= 2 1/4.
(*Audio transcript*)

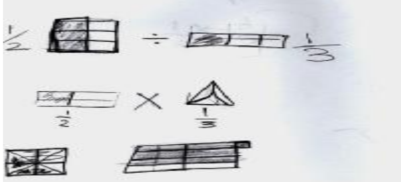
For example, one of the students said that it means four apples divided between two. Even in the conversation described in box 8, the children's knowledge about division was clear to some extent. I mean, Asif's response about division was: making parts of a thing. After having the students' knowledge about division of fractions, I provided them questions about simple division of fractions. At the beginning, I noticed that the children were able to symbolize division of fractions. For example, Basit described as shown in box 11.

Box 11.

Basit: 1/2 ÷ 1/3.
Shahida: How have you solved it, show me here?
Basit: Miss, I have divided 1/2 by 1/3 and then I came to the formulae and I changed it in to multiplication. I have done like this:
1/2 x 3/1. then 1 x 3 = 3 and 2 x 1 = 2. so, it is 3/2.
(*Audio transcript*)

When Basit described his solution, I asked him to show me with pictures, as it was difficult for me to see his understanding from his conversation. He tried to explain his solution with the following pictorial representation, shown in figure 5.

Figure 5.



In the mentioned figure, it is apparent that it was tricky for Basit to picturize the solution of the question with a figure. I found that he described symbolically but it was not easy for him to picturize. Wu (1999) says,

> *But we should not make students feel that the only problems they can do are those they can visualize… but this does not mean they cannot do the problem! Or that more complex problems like this one are not essential (p. 2).*

It is not necessary that children can draw figures for all the problems. Teachers can start with simple fractions for division to make pictorial representations easier because it will give children practice for the particular concept. As Wu (1999) states, "it is good to start with simple fractions that children can visualize, and they should do many such problems, until they have a firm grasp of what they are doing when they divide fractions" (p.1).

But they need to have a deeper understanding of division and it needs lots of practice in mathematics.

Many times children were referring to their prior knowledge in order to relate with the current one. For example,

Box 12.

Shahida: Can you explain us that how you divide it into 15 parts?
Asif: First I, make these five lines. Then I think about five table, and it is 3 x 5 is 15. Then I cut the five parts into three and get 15 parts.
Rizwan: I think that 1, 2, 3, ….10 and 3 x 10 = 30. Half of the 30 is 15. I have made two parts, two parts and two parts. it is equal to fifteen and now shaded two. Because 1, 1, 1, 1,1…..1 equals to 15.
(*Audio transcript*)

While working with the children on the four rules of fraction, I found out that teachers need to start explaining mathematical concepts with concrete material because of the abstract nature of mathematics. They can move to the pictorial representations and then the symbolic representations. Teachers also have to arrange activities from easy to complex. Before starting a new concept, teachers have to be sure that the children have understood the current concept very well. For example, before going to do division of fractions , children must be competent with the concept of multiplication of fractions.

In this chapter, I described the activities which lead me to next write about the factors which enhanced the students' learning of mathematical rules with reasoning.

LEARNING MATHEMATICAL RULES WITH REASONING

Field data shared above led to several key conclusions about the process of students' learning mathematical rules with reasoning. These are discussed below.

Mathematical Reasoning: Role of Teacher's Questions & Student Talk

"Questions are a vital element of the learning process" (Lindquist et.al, 1995, p.25). During my field work I found what Lindquist and other researchers have described as true. I found that questions were significant in a number of ways. First, questions that the teacher raised played a key role in students' learning by providing an opportunity to explain their thinking and provide the teacher with some evidence of what they did or did not know. Second, it was the talk with peers that the students engaged in questions that students raised themselves.

While working in particular sessions, which I designed for the children, I found that different types of questioning was very helpful in providing them opportunities to justify their work. I mostly asked questions like: Why do you think so? How did you come to this answer? Could you elaborate on it? Why do you agree with him? The students were reluctant at the beginning and were unable to answer these questions because the 'why' and 'how' were somehow challenging words for them. But later, they were explaining and justifying their work. And these 'how' and 'why' questions helped them to think about their work. As it was evident in the previous chapter, in box 4.11, I asked Asif to elaborate his question, and his explanation specified his understanding.

However, I recognized that my own questioning skills had to be developed for them to be effective for students' learning. At the beginning of the sessions, I asked questions but was not very clear why I asked those questions. As I engaged for the purpose of the research in focused and deliberate reflection I found out that questioning is a good way of eliciting students' ideas, and it also helped me to know about the student's prior knowledge. It was very helpful to construct new ideas on the basis of that prior knowledge. In the later sessions, I tried to ask more critical and well organized questions. During the ongoing analysis of data, I noticed that in all the sessions pre planned and well organized questions played a key role in learning mathematical rules with reasoning. For example, the questions mentioned in box 4.4, helped Farhan to explain the process of solving the problem. While working with the four students, evidence suggested that my questions helped them think about their work, and because of questioning they started giving justifications for their work, and these justifications led the children to learn with understanding.

Second dimension of the questions were those questions which emerged as a result of students talk among themselves, and between students and teacher. For example, when students got stuck, others helped him. For example, when I asked Asif to explain his solution of the question to the whole group and he did so. There were some mistakes in his solution, and the other children confidently raised questions like, why did you say this?

During this study one of my strategies in the group was to ask children to explain their solutions to the whole group, as I mentioned earlier. By doing so, I realized that I was giving them a chance to share their ideas and thinking. As Hart (1993) says, “we tend often to assess the progress of a child by stating what he does that is correct and what he does that is incorrect rather than asking ourselves why he is correct or why he is wrong” (p.213).

However, it was not easy to engage the students so that they asked questions. Initially, students responded to my questions with silence or with one word answers. I recognized that as a new teacher I needed to set up an environment based on trust and friendliness, so that the emotional environment was that of confidence. I observed that this friendly environment helped children to reason their work. For example, children were asking questions without any hesitation. They were not feeling shy to ask questions. While working in the group, they were arguing with each other and raising questions when they needed clarifications. I tried to create a desirable atmosphere for learning with understanding.

I reflected on the reasons that made it difficult for me to ask open questions that would lead to learning with reasoning, and which made it difficult for students to engage with questions in a meaningful way. I do not have direct evidence to substantiate, but it is likely that students’ and teacher’s difficulty in dealing with questions which were open and broad could be because classrooms in Pakistan are characterized by mathematics teaching which focuses on memorization and rote learning (----). Students do not necessarily have opportunities to engage in open ended questions. Hence, a strong implication is that teacher education courses need to focus more strongly on the role that good open questions can play in enabling learning of mathematics where students learn rules with reasons and through rote memorisation.

Learning by doing: Role of Concrete and Semi concrete Materials

Askey (1999) says, “Having students work with concrete objects or drawings is helpful as students develop and deepen their understanding of operations” (p.7). During the particular sessions, which I designed for my study, I found the above mentioned quote true to some extent.

During my field work, I observed that concrete materials and semi concrete materials such as pictures and paper cutouts played a significant role in learning mathematical rules with reasoning. On many occasions, when children were not able to justify their solutions, concrete materials helped them a lot. For example, in the introductory sessions, I prepared paper strips to help children to understand ‘fractions as a whole’. During those sessions I noticed that fraction strips were helpful to the children in providing reasons for their work. I also found out that if teachers were to provide multiple opportunities to the children, it could be helpful for their learning and more children could benefit from it. I also found that concrete materials were very useful for students to learn fractions because I observed on many occasions, that when children were unable to understand mathematics, because of its abstractness, then concrete materials provided them with a physical manifestation of the abstract ideas.

Difficulty with Improper Fractions and Issues in the Rule of Multiplication of Fractions

Working with the students, using innovative styles, and a focus in mind created a different picture about the rules of fractions in my mind. For example, during this study I found two major issues while observing the process of students' learning. The first one was about difficulty with improper fractions, and secondly, dealing with the rule of multiplication of fractions.

The difficulty with improper fractions was that students were having difficulty in showing the solutions of improper fractions through pictorial representation, as well as through use of concrete material. In the previous chapter, figure 4.1 is evidence of the statement. In the question 2S - 15/8, children faced difficulty in representing 15/8. The same in figure 4.4. The children had problems with putting cubes in order to show 5j. But they easily showed the pictorial representations of proper fractions, and they were quite confident in showing the same with cubes. Research also indicates the similar and I observed that they could easily solve the questions symbolically. I thought, may be, the problem was that they were not used to presenting fractions through figures or materials. I decided to support their thinking by posing questions, and created situations which facilitated them to find the solutions on their own. According to Kamii (1999),

> *The teacher's job is not to explain mathematics but to facilitate critical thinking and the honest, respectful exchange of ideas among the students. When students explain their reasoning to others, they clarify their own thinking and learn to communicate clearly". (p.88)*

So, children must be aware of multiple ways of solving problems. They should not stick just to one way while dealing with a variety of problems.

Another issue arose when I was observing the process of rule of multiplication of fractions. Asif perceived it as a multiplication of whole numbers. This could be because children in elementary grades are taught that multiplication is repeated addition. Usually, mathematics teachers relate multiplication with addition like 3 x 3 is same as 3 + 3 + 3. Hart (1993) acknowledges the same situation in these words,

The meaning of multiplication is firmly rooted in the child's experience of whole numbers where the operation can always be replaced by repeated addition. If the child sees 4 x 3 as four groups of three subjects … the meaning he attaches to 1/3 x 6/7 is unclear (p.80).

When I read the relevant literature, I found the same situation mentioned in a number of books. In addition to the one quoted above, another similar comment has been made by Skemp (1991), who says,

> *Multiplication is often taught to young children as repeated addition. For the natural number this causes no problem, and is probably the easiest for them to understand. But it causes problems later, for example, when we ask them to learn how to multiply fractions. Here, the concept of repeated addition has no meaning" (p.84).*

As I found the same issue in different contexts, I started thinking that children, and their level of thinking, is more or less the same, not completely but to some extent. Even though their environment and other factors are different, which enhance or hinder their learning, their level of understanding is the same the world over.

It was somehow challenging to explain the rule of multiplication of fractions with reasoning. In the limited time, I found 'rule learning' most suitable for teaching multiplication of fractions. As Hart (1993) acknowledges, "Multiplication of fractions cannot be dealt with by the use of naпve and intuitive methods and is therefore based very much more on 'rule learning' than some other aspects of mathematics" (p.80).

Students learn the basic facts in three stages: the manipulative stage, the pictorial stage, and the symbolic stage. Teachers need to start teaching mathematics from concrete materials, then pictorial representations and then help them to symbolize, because of the abstract nature of mathematics. Each stage builds upon the previous stage to help students master their basic facts.

CONCLUDING REFLECTIONS

As I was looking at the process of students' learning mathematical rules with reasoning, I found out some factors which enhance the process of reasoning. There are a number of ways which enhance students' learning of mathematical rules with reasoning. In order to develop the reasoning ability in students, teachers must design such questions which may help the children to think and justify their answers. Another strategy can be creation of a conducive environment in the class, which can help children to express themselves without any hesitation. Students' prior knowledge is very helpful in order to reason. Teachers' guidance and interactions with peers also enables children's ability of reasoning.

My study provided me precious insights, which can enhance students reasoning ability, and guidelines for the teachers to practice in their mathematics classrooms, to adopt learning with reasoning.

Because of the abstract nature of mathematics, role of concrete materials is obvious. It helped children to proceed from concrete to abstract. The concrete materials played a vital role in enhancing the students' mathematical reasoning. As per the old Chinese saying, "when I see I forget, when I hear I remember and when I touch I understand". It is the same in mathematics.

While conducting my study, I found that the role of teachers questioning is very important. As teachers ask questions, children explain their ideas. By doing this teacher get an awareness of their student understands. Questioning can also be considered as a tool for assessment i.e. in order to see how much the children understand. When children explain their thinking the teachers easily evaluate them.

Friendly environment in the class was also supportive to reasoning, because in such an environment children argue, raise questions, and describe their thinking without any hesitation. A friendly environment creates a good relationship among teachers and students, and amongst students themselves. By doing so, they learn from each other. In a conducive environment a teacher's guidance is also taken positively. So, a friendly environment in the class is an essential element for learning with understanding.

While conducting this small scale study, I found that teachers play a major role in enabling student's learning of mathematics. The teacher must choose activities which provide opportunities for children to communicate their understanding. I found out that children learn better if they learn to solve problems, to communicate mathematically and to demonstrate reasoning abilities. These attributes will improve the children's understanding of mathematics and will enhance their interest in math concepts and thinking. It is obligatory for all mathematics teachers to always think of innovative strategies of teaching, and to create situations in the classrooms which may enhance learning with conceptual understanding.

Findings of the study raise significant implications for teacher education and for mathematics teachers. One, for students to learn in this case fraction rules with reasons, students need opportunities to talk about their mathematical thinking and explain it to others. Second, to be able to cope with the abstract mathematical concepts, pictures, paper cutouts and other concrete materials help learning by providing opportunities to manipulate, and see. Third, planning tasks and teaching based on constructivist principles of learning by doing and through social interaction had limitations as the mathematics became more complex.

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APPENDIX G

PURPOSE

- This activity introduces students to fractions as part of a whole.
- Students will prepare their own fraction kit for a number of fraction activities.

Instructions for fraction strips

Give students 6 strips of five different colors (white, blue, red, yellow and green).

Ask students to take a strip, fold it and then cut it into halves. Then ask them to label each piece 1/2.

Ask students to take a strip, fold that strip into halves and again fold that folded strip. Cut the strip into four. Ask students to label each piece j.

Ask students to take another strip, repeat the previous action of folding. This time fold one time more. You will get eight folded pieces. Cut the strips into eight equal parts. Ask students to label each part 1/8.

Ask students to take another strip and fold and cut the strip into sixteen equal parts. Then ask students to label each part 1/16.

Ask students to keep the last strip in rows and then compare the fractional parts.

e.g. 1/2 and 1/4. Find which fractional part is larger and smaller then the other.

Give everyone an envelope to keep their stripe and these will be the students' fractional kits.

APPENDIX H

Name ---------------------

Date ----------------------

The purpose of the activity was to introduce fractions as part of a whole.

Use your fraction strips to complete the following tasks:-

1. Place a 'greater than' sign '>' or a 'less than' sign '<' between each set of two fractions.

a. 1/2 □ 1/4

b. 1/4 □ 1

c. 1/4 □ 1/8

d. 1/16 □ 1

e. 1/8 □ 1/4

2. Use fraction stripe and solve the following questions:

a. 1/2 + 1/2 = ---------

b. 1/2 +1/4 + 1/4 = --------

c. 1/2 + 1/4 + 1/8 + 1/8 = -------

d. 1/2 + 1/4 + 1/8 + 1/16 + 1/16 = -------

3. Write five equivalent fractions of 1/2.

1/2 = ------- = ------- = ------- = ------ = --------

APPENDIX I

ADDING FRACTIONS

Name: _____________

Date: ______________

PURPOSE

The main objective of the session was to develop a situation where the students deal with tasks related to the addition of fractions, which will be helpful in order to enhance students' understanding of the topic, and will help me to generate data for my study.

Add the fractions with the same denominator:

1. 3/7 + 1/7
2. 3/8 + 1/8
3. 1/5 + 3/5

Add the fractions with different denominators:

1. 3/5 + 2/3
2. 5/8 + 1/10
3. 1/3 + 1/6

Add the improper fractions:

1. $1\ ^{1}/_{2} + 1\ ^{1}/_{3}$
2. $1\ ^{3}/_{4} + 2\ ^{1}/_{2}$
3. $4\ ^{2}/_{5} + 3\ ^{1}/_{2}$

APPENDIX J

SUBTRATION OF FRACTIONS

Name: _____________

Date: ______________

PURPOSE

To help students justify their solutions for subtraction of fractions.

Subtract the fractions with the different denominators:

1. 9/10 – 7/10
2. 4/7 – 3/7
3. 4/15 – 2/15

Subtract the fractions with different denominators:

1. 5/6 – $^1/_2$
2. 8/9 – $^3/_4$
3. 4/5 – $^3/_4$

Subtract the improper fractions:

1. 2 $^1/_2$ - 1 5/8
2. 3 $^1/_2$ - 1 $^3/_4$
3. 5 $^1/_4$ - 2 $^1/_2$

APPENDIX K

MULTIPLYING FRACTIONS

Name: _________

Date: __________

PURPOSE

Students will be able to understand the rule of multiplication of fractions.

Multiply the following and use cubes to solve the problems:

1. 3/7 x 15
2. 7/8 x 16
3. 4/5 x 30
4. 5/7 x 21
5. 1/8 x 32
6. $^{1/}2$ x 1/3
7. 2/3 x 3/4
8. 1 $^{1/}2$ x 1$^{1/}2$
9. 1 $^{3/}4$ x 2/5

APPENDIX L

DIVIDING FRACTIONS

Name: -----------

Date: ------------

PURPOSE

Students will be able to understand the rule of division of fractions.

Divide the following:

1. $1/2 \div 3$
2. $3/4 \div 4$
3. $6/7 \div 3$
4. $4/5 \div 9$
5. $1/2 \div 3$
6. $3/4 \div 4$
7. $6/7 \div 3$
8. $4/5 \div 9$

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