MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2023, 19(5), em2255
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/13127

Learning to promote students' mathematical reasoning: Lesson study contributions in initial teacher education

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Received 16 January 2023 ▪ Accepted 16 February 2023

Abstract

The aim of this study is to understand how prospective teachers can develop their knowledge about how to promote students' mathematical reasoning in a set of sequential lesson studies. The research follows a qualitative approach and addresses the case of a prospective teacher without teaching experience. Data was analyzed according to a set of principles regarding the characteristics of tasks and the teacher's actions that promote students' mathematical reasoning. Reflecting on the students' work in the lessons that the prospective teacher taught led her to understand the influence of the proposed tasks and her actions in promoting students' reasoning, thus developing her knowledge. Furthermore, the sequential structure of the lesson studies allowed her to prepare the following lessons based on the in-depth analysis of the previous lessons and to put into practice what she planned, leading her to rethink her teaching strategies and improve her practice.

Keywords: mathematical reasoning, initial teacher education, lesson study, mathematics teaching

INTRODUCTION

Mathematical reasoning is a key process in learning and understanding mathematics (NCTM, 2014; OECD, 2021). When we reason, "we develop lines of thinking or argument" (Brodie, 2010, p. 7) that enable us to support an idea, solve problems and make connections between several ideas to reach a conclusion. Due to its relevance for students' learning, reasoning has been increasingly highlighted in the Portuguese mathematics curriculum. The curriculum reforms suggest that teachers' actions should ensure that students have the "opportunity to discover, reason, prove and communicate mathematics" (DGE, 2018, p. 3).

However, promoting mathematical reasoning is a challenge for teachers as they need to understand what it is and also know how to promote it in the classroom (Brodie, 2010; Davidson et al., 2019; Lannin et al., 2011). For prospective teachers, it is particularly challenging since they are still developing their knowledge and only begin to have contact with students in their initial teacher education (Mendes et al., 2022; Stylianides et al., 2013). Hence, the creation of effective strategies is needed in initial teacher education so prospective teachers can develop their knowledge in this field.

Research has been moving forward intending to support teachers and prospective teachers to overcome this challenge (Buchbinder & McCrone, 2020; Davidson et al., 2019; Mata-Pereira & Ponte, 2017, 2018; Oliveira & Henriques, 2021). However, further research is still needed on how prospective teachers can develop their knowledge about students' reasoning and also on how this reasoning can be promoted through teachers' actions and the tasks they propose in the classroom (Buchbinder & McCrone, 2020; Oliveira & Henriques, 2021).

Reflecting on students' learning, based on classroom experiences and sharing ideas on those experiences, is an effective way of developing prospective teachers' knowledge (Ponte & Chapman, 2016; Potari & Ponte, 2017). As a student learning-oriented professional development process, lesson study enhances these reflective environments during initial teacher education (González et al., 2023; Leavy & Hourigan, 2016). Based on discussions with colleagues and teacher educators, the prospective teachers plan lessons in detail, rethink the tasks, anticipate students' solving strategies and

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Contribution to the literature

- This research highlights the contribution of lesson study activities in the development of prospective teachers' knowledge, namely the detailed planning in an environment of sharing and discussion of ideas, as well as the reflection focused on students' learning.
- It emphasizes the potential of reflection on practice to better understand the students' ways of learning and as a starting point for improving teaching practice. The potential of reflection was strengthened by the various opportunities to put into practice the planned strategies.
- The sequence *plan*, *teach*, and *reflect* carried out on three consecutive lessons enabled the prospective teacher to develop her knowledge and improve her practice through continuous reflection, with the support of her colleagues and teacher educator.

difficulties, and prepare their actions (Fujii, 2018). This enables them to put the strategies they defined into practice and to establish a connection between the theory learned during their initial teacher education courses and classroom practice. Thus, by analyzing particular lesson situations, they reflect on their own practice and define strategies to improve it, developing their knowledge (Ponte, 2017).

Acknowledging the importance of promoting students' reasoning in the mathematics lesson and the formative value of reflection in initial teacher education, this study seeks to understand how prospective teachers develop their knowledge about the way to promote students' mathematical reasoning, through an in-depth analysis of lesson situations, in a set of three lesson studies carried out in sequence.

THEORETICAL FRAMEWORK

Mathematical Reasoning

Although the term *reasoning* is used extensively in the teaching and learning of mathematics, the meaning and the purport of mathematical reasoning are not always clear (Jeannotte & Kieran, 2017; Mata-Pereira & Ponte, 2017). However, it is commonly accepted that *reasoning* is not limited to presenting mathematical ideas, but rather involves being familiar with and understanding the connections between these ideas in a given context and being able to organize them to construct a new mathematically valid idea (Brodie, 2010; Lannin et al., 2011). Reasoning allows students to "move beyond specific examples toward recognizing and supporting general relationships" (Lannin et al., 2011, p. 9). Thus, it is a crucial element for mathematics learning since it leads students beyond the memorization of concepts or the repetition of procedures, prompting them to understand why they are used, how they can be used, and what their results mean (NCTM, 2014).

In a simplified manner, reasoning is "making justified inferences" (Mata-Pereira & Ponte, 2017, p. 170). These inferences can be *inductive* when they generate new knowledge from the observation of several similar situations that lead to a generalization, or *abductive* if they generate knowledge from the in-depth consideration of a given situation. They can also be *deductive* if an idea is no longer considered probable to be considered true based on the justification of what validates or refutes the idea (Jeannotte & Kieran, 2017).

In practice, mathematical reasoning occurs through different inter-connected processes, where *conjecturing, generalizing* and *justifying* are considered the key processes (Mata-Pereira & Ponte, 2017, 2018; Stylianides et al., 2013). *Conjecturing* involves making statements that are assumed to hold true but have not yet been proven. *Generalizing* is formulating general conjectures about concepts, procedures, or properties for a set of objects (Mata-Pereira & Ponte, 2017). *Justifying* is presenting reasons to validate an idea through a logical sequence of statements that are already known to hold true (Jeannotte & Kieran, 2017; Lannin et al., 2011).

The tasks play an essential role in engaging students in the development of their reasoning (NCTM, 2014). It is by working on the tasks that students can be involved in "evaluating situations, selecting strategies, drawing logical conclusions, developing and describing solutions, and recognizing how those solutions can be applied" (OECD, 2021, p. 14). Therefore, it is important to propose tasks that foster the use of different solving strategies and representations, as well as questions with varying degree of mathematical challenge (Brodie, 2010). Moreover, tasks should engage the students in the *generalization* of mathematical ideas and in the *justification* of their answers (Brousseau & Gibel, 2005; Mata-Pereira & Ponte, 2017, 2018).

However, to promote students' reasoning, a standalone task with those characteristics is not sufficient. It is through the teacher's actions that student-teacher interactions can be established to make the work on the task visible and to highlight the reasoning processes (Brodie, 2010; NCTM, 2014). Teachers, therefore, need to create moments in the classroom for students to present and justify their answers, supporting them without reducing the degree of challenge of the task. In doing so, they should also highlight the reasoning processes involved to encourage the students to go beyond the task (Davidson et al., 2019; Mata-Pereira & Ponte, 2017, 2018). In practical terms, teachers can invite students to present, explain and justify their work, provide information, make suggestions or validate answers, thus keeping the discussion focused, support through questions or other interventions, both explicitly and implicitly, or challenge students to extend the work carried out (Ponte et al., 2017a).

The teaching approach and the way the lesson is structured are also important factors in the promotion of students' reasoning (Oliveira & Henriques, 2021). They need to understand the task before setting to work on it, as well as having the opportunity to solve it autonomously, in small groups, before sharing their ideas with the class. The *exploratory approach*, structured into three phases, stems from students' autonomous work on a challenging task and is centered on the discussion of this work, ending with a synthesis (Ponte et al., 2017a; Stein et al., 2008). It is during the whole-class discussion that students may establish connections between different mathematical ideas, through comparing solution strategies relationships between concepts, procedures and representations, and extend the work done, through the generalization and justification of their ideas, hence promoting their mathematical reasoning (Mata-Pereira & Ponte, 2018).

Developing Knowledge in Initial Teacher Education

In order to promote students' mathematical reasoning, teachers need to be familiar with its underlying processes, the characteristics of the task and the teacher's actions that foster its promotion, and the teaching approaches through which it is triggered (Davidson et al., 2019; Lannin et al., 2011). Thus, the development of teachers' knowledge in various domains is at the root of their preparation: on the one hand, *knowledge about students' learning* in which their interests, needs and socioeconomic and cultural characteristics are included, as well as their difficulties and ways of learning; on the other, *knowledge about teaching practice*, which involves being able to design tasks and plan lessons, organize students' work and promoting students' mathematical reasoning. Nevertheless, these domains are ineffectual without the *knowledge of the mathematical content* to be taught and *knowledge of the curriculum*, which are also fundamental for teaching practice (Ball et al., 2008; Ponte, 2012).

However, the development of prospective teachers' knowledge during initial teacher education is a challenge in terms of selecting and adapting tasks, planning lessons and anticipating students' work, and also teaching lessons and managing students' work (Martins et al., 2021; Mendes et al., 2022; Stylianides et al., 2013). In other words, integrating what they learn in theory throughout the course into teaching practice is a great challenge for prospective teachers.

Thus, finding strategies to promote the development of prospective teachers' knowledge is still an important theme. Indeed, it is crucial that prospective teachers reflect on their own practice so that they are able to recognize "important transversal issues in teaching, such as the need for careful planning and the value of reflection" (Ponte et al., 2017b, p. 301), which implies that they have the opportunity to observe and teach lessons. It is through these classroom experiences that they can understand how their actions influence students' learning (Buchbinder & McCrone, 2020; Oliveira & Henriques, 2021). Also, an in-depth analysis of lesson situations are important moments for prospective teachers to be able to foresee and define strategies to improve their practice (Ponte et al., 2017b; Ramos-Rodríguez et al., 2017). Additionally, the discussion and sharing of ideas between different people with different experiences is also essential for prospective teachers to develop their knowledge (Potari & Ponte, 2017), since they are led to consider other perspectives and reflect on the strategies they intend to adopt to improve their practice.

The potential of lesson study has been recognized in initial teacher education since it promotes the development of prospective teachers' knowledge based on experiences close to those they will encounter in their future practice (González et al., 2023; Leavy & Hourigan, 2016; Martins et al., 2023). Lesson study is recognized as a model for professional development whereby all the work is carried out collaboratively and based on reflections of students' learning (Fujii, 2018). The prospective teachers share their ideas and experiences with their colleagues, the teacher educator, and with the cooperating teacher they accompany during the internship, to plan a lesson in detail. Since they plan a lesson in this environment, being involved in designing the task, anticipating students' work and difficulties, and preparing their actions for the lesson, they have the opportunity to develop their knowledge about the students' learning and teaching practice (Willems & Bossche, 2019). By teaching the lesson they have prepared themselves, or observing their colleagues in this role, they are able to see in practice what they planned theory-based (Ponte, 2017). These experiences, for and in the classroom, also prompt them to reflect on the practice in order to find strategies to improve it, which is a privileged way to develop knowledge (Ponte & Chapman, 2016; Ramos-Rodríguez et al., 2017).

RESEARCH METHODOLOGY

This research follows a qualitative approach (Bogdan & Biklen, 2007) and is based on a set of three lesson studies carried out in sequence, during an initial teacher education course, in a Portuguese university.

Participants

Three prospective mathematics teachers (grades 7-12) participated in the lesson studies. By the time of the lesson studies, they were carrying out the curricular

Aspects to consider in pre-lesson written reflection

1. Justifying the task, the materials and the strategy.
2. Anticipating possible students' solving strategies. What difficulties may the students display? How can they be supported?
3. What challenges do you think you will experience? How do you expect to overcome them?

Aspects to consider in post-lesson written reflection

1. Make a description of your lesson (strengths and weaknesses).
2. What did you intend for this class? What was accomplished? And what was not? Why?
3. What challenges did you experience in implementing what was planned? How were they overcome?
4. If you taught this lesson again, what would you change? Why? What outcomes do you draw from this lesson for your future practice? Justify.

Figure 1. Reflection guide (Source: data collection)

internship in a school context, accompanying one cooperating teacher at that school. They were the full group of prospective teachers supervised by a teacher educator who showed interest and willingness to participate this study. The prospective teachers had specific preparation in undergraduate mathematics for three years or more and had general preparation in different areas such as didactics of mathematics, sociology and psychology of education. Following the Portuguese structure of initial teacher education courses, in the first semester of this academic year, they observed several lessons taught by the cooperating teacher that they accompanied in the school. They also had already planned and discussed lesson plans following an exploratory approach.

For this research study, we selected the case of Sílvia (pseudonym) who, unlike her colleagues, had no teaching experience, either formal or informal. From the lessons that she observed, she chose to explore the contribution of questions for students' learning, in grade 10, to prepare her internship report in the school context.

Together with the cooperating teacher and, considering the curriculum of grade 10, Sílvia chose to work on the topic of functions. She decided to focus on solving problems involving quadratic functions for the first lesson (L1), introducing the function defined by branches to the second lesson (L2), and solving inequations with modules for the third one (L3). Sílvia planned her lessons following the structure of an exploratory approach, explaining that for her "it was important that they [the students] built their own knowledge" (initial interview).

The set of lesson studies were prepared considering this background of Sílvia's previous preparation. The researcher (first author) shared the role of facilitator with the teacher educator. Although he had just a superficial knowledge of lesson studies prior to his contact with the researcher, the latter already had some experience in this formative process.

The Lesson Studies

The lesson studies were prepared by the facilitators considering the organizational structure of the initial teacher education program of the university, and the ideas shared in the initial interviews with the prospective teachers, conducted by the researcher. Thus, the lesson studies were structured so that each prospective teacher planned, taught and reflected on three lessons in sequence, based on sharing and discussing ideas with their colleagues and facilitators.

Each lesson study consisted of five steps. The prospective teacher began by autonomously preparing a first version of the lesson plan, selecting tasks for the lesson, framing them following the curricular orientations, and justifying in the light of the established learning goal, following a reflection guide proposed by the teacher educator (**Figure 1**).

This document should also include the anticipation of students' difficulties, as well as the identification of the challenges foreseen for teaching the lesson. After writing this document, the prospective teacher shared it with facilitators and colleagues and the group met in the *lesson plan discussion* session. In this session, the tasks were then analyzed by the group according to the curriculum and adapted considering the learning goal and the possible students' difficulties. Then, the prospective teacher had the opportunity to improve the

Table 1. Data analysis categories

Category	Sub-category
Task's characteristics	[T1] include questions with different degree of mathematical challenge. [T2] allow different solving strategies and the use of multiple representations. [T3] ask for justification of answers and of solving strategies. [T4] prompt generalizations.
Teacher's actions	[A1] support students while they solve task, aiming not to significantly reduce its degree of challenge. [A2] invite students to share their ideas and explain their work, accepting and valuing incorrect or partial contributions, and support them to analyze, complement, or clarify their answers. [A3] invite students to explain "why" and support them to present alternative justifications. [A4] challenge students to identify valid and invalid justifications, emphasizing what may validate them. [A5] support students, aiming to highlight reasoning processes, and particularly generalizations. [A6] challenge students to go beyond the task by formulating new questions and generalizations.

lesson plan document according to the work of the previous session.

After sharing this new document with the group, they met again in the *lesson preparation* session, where the prospective teacher reshared the lesson with the group to prepare the teacher's interventions. During this session, the prospective teacher explained how she planned to lead the lesson considering the students' strategies and difficulties, so that the group could discuss effective strategies for leading the lesson according to the potential of the task, and to meet the learning goal. The prospective teacher then taught the *research lesson* while the rest of the group observed and took notes to later share. After the lesson, the group met to analyze the situations that, for the prospective teachers, had been more surprising during the lesson. In this *post-lesson discussion* session, the facilitators sought for the prospective teacher to analyze what had been achieved in light of what had been planned and its implications for students' learning, the greatest challenges faced, and what could improve in future practice, once again following the reflection guide (**Figure 1**). Finally, in autonomous work, the prospective teacher analyzed the videorecording of the lesson and the students' productions, considering what had been discussed in the previous session as well, drawing up a *written reflection*. Upon the work for the first lesson, the process began again with the preparation of the first version of the plan for the next lesson.

Data Collection and Analysis

The data collection includes participant observation, fieldnotes and audio recordings (Bogdan & Biklen, 2007). The tasks and their adaptations, the different versions of the lesson plans, and the written reflections were also collected. All data was collected after written permission from the interveners and all names used are pseudonyms.

Considering Mata-Pereira and Ponte's (2017, 2018) research on the principles that promote students' mathematical reasoning, in particular generalizations and justifications, regarding the proposed tasks and the teacher's actions, the data was analyzed according to **Table 1**, albeit with some adjustments. Regarding the characteristics of the tasks, (i) the use of multiple representations was considered in principle [T2], in addition to different solving strategies. For the teacher's actions, (ii) the verbs used by Ponte et al. (2017a) were adopted however, some of the original formulations were simplified, (iii) two of the originally proposed principles were joined to achieve the formulation of [A2], and (iv) we highlight generalization in the principles [A5] and [A6].

FINDINGS

Justifying

For the first lesson, Sílvia proposed a task (**Figure 2**) for students to use "their mathematical reasoning to interpret the question and solve the task ... [with] different [solving] strategies" (L1, lesson plan). For example, in the second question, students could calculate the vertex coordinates of the parabola or find the roots of the function and, considering the symmetry of a parabola, determine the asked value. This was the first time Sílvia expressed her concern with the promotion of students' reasoning by designing a task with different degrees of challenge [T1], which allowed different solving strategies and the use of multiple representations [T2]. Moreover, all the questions explicitly requested a justification for the answers [T3].

One of the most surprising moments of the lesson identified by the group was the students' difficulty in understanding what was asked in the third question. In the post-lesson discussion, Sílvia recalled that the students "knew that there was a point when the temperature began to lower, but it was not obvious how they would determine it" (L1, post-lesson discussion). However, when she supported them in solving the task, suggesting representing the situation graphically [A1], she was surprised by "the students' ability to present arguments" (L1, post-lesson discussion), since they rapidly understood and subsequently justified their answers.

The way Sílvia led the whole-class discussion of the first question was also highlighted in the post-lesson

Task 1: André's fever

André woke up at 5 a.m. with some chills. He took the thermometer and found out that he had a fever. Following his mother's advice, he registered his temperature for the next five hours. Since he was studying the quadratic function, he noticed that his temperatures varied according to the function T, defined by $T(x) = -0{,}5x^2 + 2x + 38$, which begun to lower 20 minutes after taking a fever medicine. (T represents the observed temperature after x hours after the first registration).

1.1. What is the temperature observed at 5 a.m.? Justify your answer.

1.2. What is the maximum temperature reached during the observation period? Justify your answer.

1.3. At what time did André take the medicine? Justify your answer.

1.4. Show that the temperature remained above 39,5º C between 6 a.m. and 8 a.m.

Figure 2. Task for L1 (Source: lesson plan for L1)

discussion. Referring to this moment, she said that "when [the students] were faced with that problem, where the x [corresponding to the initial value] had to be zero and not five... they started to have some difficulties" (L1, post-lesson discussion), which led her to promote the whole-class discussion based on an incorrect solution:

> Sílvia: Do you think André's right? [A4].
>
> Fábio: No, he stopped measuring the temperature at 10 o'clock ... he started measuring the temperature at 5 o'clock and 5+5 is 10.
>
> ...
>
> Sílvia: What does it say there [in the task] about x? [A2].
>
> ...
>
> Diogo: Is not it from 0 to 5?!
>
> Sílvia: Why is it from 0 to 5? [A3].
>
> Bruna: Because at 5 o'clock it's [x=]0, which is the first measurement (L1).

Sílvia supported the students to clarify which values they should consider [A2], challenging them to identify which answers were or were not valid [A4], and to justify their answers [A3]. When she compared the lesson's videorecording with the students' productions, she concluded that, "although initially the students found it strange that ... [she] had asked them to explain and answer each other's doubts, as the lesson progressed, they began to intervene more frequently... and explain their reasoning, without having to be encouraged to" (L1, written reflection). Sílvia then realized that, besides the task, her actions had been determinant to promote students' reasoning. Thus, for the following lessons, she considered important to prepare "more questions... prompting a higher level of reasoning" (L1, written reflection).

In the following lessons, Sílvia remained intent on proposing tasks so that "the students can use different [solving] strategies" (L3, lesson plan). In particular, for the second lesson, she drew a task (**Figure 3**) with the support of the lesson study group including questions with different degrees of challenge [T1], proposing the algebraic and graphical representation of the situation [T2], and implicitly promoting the justification of the solving strategies [T3].

Moreover, following up on the reflection on the first lesson, to "assist [the students] in the construction of solving strategies" (L2, lesson plan), she thought on some questions to pose, such as, "How much are the fixed rate the habitants have to pay?" and "How can we divide the domain into intervals knowing that the price is different between certain water consumption values?"

In the post-lesson discussion, Sílvia concluded that the main difficulty of students was understanding that, since it was a function defined by branches, there were two expressions to calculate the price, depending on the water consumed:

> Sílvia: And what if Inácio used 11 m^3 of water? [A1].
>
> Diogo: We do 6+1.60×11.
>
> ...
>
> Sílvia: Do you agree? [A4].
>
> André: No, I would do 6+0.75×10+1.60.
>
> Sílvia: Explain what you're thinking [A2].
>
> André: As there are 11 m^3, up to and including 10 m^3, we have to pay 0.75€ [per m^3], that's why we do 10×0.75+1.60.
>
> Sílvia: Why cannot we do 6+1.60×11? [A3, A4].

Task 1: The water consumption

In the village of Lindoso, the price of water consumption per month is the sum of the following installments:

- 6€ for the counter rental;
- 0,75€ per cubic meter of water consumed up to and including 10 cubic meters;
- 1,60€ for each cubic meter of water consumed over 10 cubic meters.

To save water, each house habitant can only consume up to 100 cubic meters of water.

1.1. Inácio, an habitant of the village, consumes 8 cubic meters of water per month. How much does he pay in a year for the water he consumes?

1.2. Isabel, Inácio's friend, pays 15€ more than he does per month. What is Isabel's water consumption per month? (Present the result rounded to the nearest hundredth)

1.3. Define algebraically the function f that corresponds to the price paid, in euros, for each cubic meter of water consumed.

1.4. Represent the function f graphically.

Figure 3. Task for L2 (Source: lesson plan for L2)

Task 1: Graphical resolution of inequalities with modules

1.1. Fill in the following table:

	Meaning	Representation on the number line	Analytical resolution
$\lvert x\rvert < 2$			
$\lvert x\rvert > 1$			

1.2. Considering $k \in \mathbb{R}$, discuss the solutions of the inequality $|x| < k$.

1.3. Considering $k \in \mathbb{R}$, discuss the solutions of the inequality $|x| > k$.

Figure 4. Task for L3 (Source: lesson plan for L3)

André: Because it says it's 0.75 € [per m^3] up to 10 m^3 (L2).

By supporting the students to understand the task's context [A1], Sílvia promoted a sharing of ideas [A2] and challenge them to identify their colleagues' incorrect contributions [A4], thus promoting the justification of their answers [A3]. Encouraged by the teacher educator, reflecting on that, she said that "the questions I posed during the discussion... contributed a lot ... to the students' understanding regarding the topic under study" (L2, written reflection).

For the third lesson, the focus was the generalization of the solution-set of modulus inequalities [T4]. Sílvia also included questions in the task (**Figure 4**) with different degrees of challenge [T1], encouraging the use of multiple representations [T2], and implicitly promoting justification in the last two questions [T3]. To prepare the lesson, she drew on the findings of Almog and Ilany's (2012) article on students' most common difficulties in solving modulus inequalities to prepare the lesson, a suggestion from the researcher, and on NCTM (2014) guidelines regarding the teacher's role in leading the lesson, as suggested by the teacher educator.

During the lesson, Sílvia invited a student with an incorrect solution of the first inequality [A2], so that the difference between the conjunction "and" and the disjunction "or" of conditions could be clarified:

Elsa: I assumed that the x would be less than 2 or greater than -2.

Sílvia: Does everyone think it's "or"? [A2, A4].

Students: No, it's "and".

Sílvia: Why? [A3].

Students: Because it's less.

Sílvia: Do not we want an intersection of intervals? If we made a union of intervals, what would the solution-set be? [A4].

Elsa: ℝ.

Sílvia: And do you want ℝ? Why do not you want ℝ? [A2, A3].

> Elsa: Because the modulus of a value less than -2 or of a value greater than 2 is not less than 2 (L3).

By supporting the students to distinguish between conjunction and disjunction, she encouraged them to justify their answers [A3], prompting them to identify for themselves whether these justifications were valid [A4]. By recalling this episode, she emphasized the importance of reading the article as the starting point for anticipating the students' difficulties, valuing this work by saying that it "helped me a lot... because I got them to share their ideas and explain their reasoning" (L3, post-lesson discussion) through the questions she prepared and posed in the lesson to support them.

Generalizing

The discussion and analysis of the first lesson's situations during post-lesson discussion led Sílvia to express greater concern with the promotion of generalization. For the second lesson, she designed a task (**Figure 3Error! Reference source not found.**) with " the aim of directing students towards the construction of the function defined by branches, before they had learned it" (L2, lesson plan). The goal was for the students to be able to generalize, based on their prior knowledge of functions [T4], beginning by answering two less challenging questions [T1] and then drawing the algebraic and graphical representations of the situation [T2].

In the post-lesson discussion, the group pointed that the students' greatest difficulty was understanding that there are two expressions to calculate the price, depending on the volume of water consumed. Sílvia then recalled how she supported them in understanding the context of the task, through the questions she prepared [A1], before starting the whole-class discussion of the third question:

> Sílvia: What do we know? That the habitants pay 6€ for the water meter, right? So, do they always pay these 6€ or not? [A1].
>
> Pupils: Of course.
>
> Sílvia: Well then? ... What do they pay 0.75€ for? [A1].
>
> Pupils: For every cubic meter.
>
> Sílvia: Do they pay that forever? [A1].
>
> Pupils: No, only up to 10 m^3.
>
> Sílvia: And what then? [A1].
>
> Diogo: If they exceed 10 m^3 they pay 1.60€ (L2).

After this situation, Sílvia invited a student with an incorrect solution to share his work [A2] as she "knew that many of them [students] would make the same mistake" (L2, post-lesson discussion):

> Sílvia: Using André's expressions, what would you do to calculate f(11)? [A1].
>
> André: We use 10 here [pointing to the first expression] and 1 here [pointing to the second expression].
>
> Sílvia: Why? [A3].
>
> André: Because here it's up to 10 [m^3].
>
> Sílvia. But is not 11 greater than 10? … In that way, you are adding the two expressions ... [A1, A2].
>
> André: But we cannot do 6+11×0.75. Maybe the expression is wrong ...
>
> …
>
> Sílvia: Can anyone find an expression that, from 10 m^3, is always correct? [nods] So Vasco, tell me what you are thinking [A2].
>
> Vasco: I did 1.60×(x-10).
>
> Sílvia: Why (x-10)? [A3].
>
> Vasco: Because we only want to calculate the cubic meters from 10 m^3. Plus 13.50€ ... 6€ for the meter and 7.5€, which is the price of 10 m^3 of water (L2).

Sílvia supported the student to analyze and correct his answer, to find a correct algebraic expression [A1, A2], encouraging him to justify his answers [A3]. She also invited other students to share their ideas [A2], inciting them to find the algebraic expression and justify it [A3]. When recalling the students' work during the lesson, she acknowledged that it had been a challenge to support them in generalizing:

> Sílvia: [The students] can say this with numbers [concretizing], but transposing this to x … [generalizing] is a problem ...
>
> Researcher: Perhaps another example may be necessary ... [to] make the similarity [between objects] more understandable.
>
> Teacher educator: You could use inductive reasoning... It's not closing the question, it's just helping [the students] to interpret.
>
> Sílvia: Yes, and then perhaps... they will have understood a little better (L2, post-lesson discussion).

By realizing that the students had difficulties generalizing, and perceiving as a challenge the promotion of generalization, Sílvia reconsidered how she could promote this reasoning process in the next lesson.

To this end, for the third lesson, she thought of a task (**Figure 4**) for students to generalize the solution-set of inequalities $|x| < k$ and $|x| > k$ [T4]. Following the teacher educator and the researcher's suggestion, and with their support, she adapted her initial idea so that the students begin by solving two particular inequalities ($|x < 2|$ and $|x| > 1$) and represent the solution-set on a number line [T2]. She then introduced two questions more challenging [T1] so that the students could generalize the solution-sets of inequalities $|x| < k$ and $|x| > k$ based on these particular cases, according to the value of k [T4], keeping the justification implicit [T3]. Furthermore, and considering the relevance of her questions in previous lessons for students' learning, she once again prepared questions to support the students [A1], such as "Can you make a graphical representation of what is required?" and "If we think about a concrete number, can you solve that inequality?".

Focusing on promoting generalization, and considering a suggestion from the teacher educator, the group planned to ask the students to make a summary-table with the conclusions drawn from the last two questions of the task [A5], highlighting the generalization. Sílvia also thought about posing the question "What strategies can we use to solve a modulus inequality of a quadratic function?" to challenge the students to extend generalization to other types of functions [A6].

In the post-lesson discussion, Sílvia mentioned that, while monitoring students' autonomous work, she had found that "the problem was that there was an [arbitrary] k" (L3, post-lesson discussion), so she decided to follow the teacher educator's suggestion and encourage them to consider particular cases:

> Sílvia: Perhaps dividing by cases might help ... Or, for example, we can assign a concrete value... Beginning with k>0, what could be the value of k? [A1].
>
> Rute: 1.
>
> Sílvia: So, let's represent the straight line y=1. When is the modulus less than 1? [A1].
>
> Elsa: From -1 to 1 (L3).

Given the students' difficulty to generalize, Sílvia focused on supporting them without reducing the degree of challenge of the task [A1] through the questions and the summary-table that she prepared. She referred she asked them to "assign different values to k more than zero ... [and to] give the solutions of the inequalities ... [and] repeat it for values [of k] less than zero. Following this strategy, several students appeared to have understood" (L3, written reflection). Moreover, when they "had several modulus inequalities to solve, they had no difficulties" (L3, written reflection):

> Vasco: This ($|x - 1| < -3$) is impossible, miss.
>
> Sílvia: Why is it impossible, Vasco? [A3].
>
> Vasco: There's no negative modulus.
>
> ...
>
> Sílvia: Telma said that $\left(\left|\frac{1}{3}x + 5\right| > -3\right)$ was a universal condition [A2, A4]. Why? [A3].
>
> André: Because all the moduli are greater than 0.
>
> Sílvia: All the modulus are positive. If the condition is universal, what is the solution-set? [A6].
>
> Rute: ℝ (L3).

By inviting the students to justify their statements [A3], Sílvia encouraged them to share their ideas [A2] and challenge them to identify the valid justifications [A4]. In highlighting the generalization [A5], she had the opportunity to challenge them to go beyond the task, seeking that they generalize the solution-set of a universal condition [A6].

Reflecting on the lesson, Sílvia said the students "seemed to have understood how to solve inequalities with modules, as well as their meaning ... [due] to the questions I posed... [and the] task" (L3, written reflection), recognizing the value of the proposed tasks within her actions during the lesson.

DISCUSSION

The tasks prepared by Sílvia and her colleagues, with the support of the facilitators, intended to foster the use of different solving strategies and representations, with questions with different degrees of challenge, as in Oliveira and Henriques (2021). In particular, the task proposed for the first lesson just focused on justifying solving strategies. However, by reflecting on this lesson, Sílvia also decided to consider generalizing. For the second and third lessons, the group thought about tasks that encouraged generalizing based on the students' prior knowledge, while implicitly requesting justifying answers.

Although Sílvia did not make explicit what she understood by a "higher level of reasoning", after the first post-lesson discussion, she focused on proposing tasks to promote both justifying and generalizing. Similar to the findings of Davidson et al. (2019), the post-lesson discussion session enhanced the prospective teacher's awareness of students' reasoning and how to promote it with tasks, which was fostered by the discussions of different ideas and the reflective nature of the lesson study.

Sílvia decided to prepare the lessons following an exploratory approach, which is in line with the principles for promoting students' reasoning (Mata-Pereira & Ponte, 2018). As suggested in national curricular documents (DGE, 2018), creating conditions for students to work in small groups and giving them opportunities to share their mathematical ideas, positively influenced the development of their reasoning processes. To prepare the lessons, in addition to consider the difficulties that the students showed in the previous lessons, Sílvia sought to find effective strategies to support them to justify and generalize, with the support from her colleagues and from the facilitators.

During the lessons, Sílvia supported students in understanding the context of tasks to analyze their answers, through the strategies she planned, and students were able to justify their answers. Lesson study entails detailed lesson preparation in a collaborative environment, focusing on students' learning (Fujii, 2018). Although it is complex to establish collaborative relationships between prospective teachers and teachers educators, because of their institutional differences and knowledge disparity (Ponte, 2017), it is possible to carry out collaborative work through lesson study. By thinking about teacher's actions and particular questions to pose to promote the students' reasoning, the group discussed and reflected on the flow of the lessons and on the aspects to be considered in preparing those lessons. In fact, the dynamic of the lesson study sessions, by preparing the teacher's strategies considering what happened in the previous lessons, provided her interactions with other participants with different experiences and knowledge, which is a powerful way to develop knowledge (Potari & Ponte, 2017).

Although the students' autonomous work moments emerged as fruitful opportunities for them to develop their reasoning, it was in the whole-class discussion that the justification and generalization processes were further explored in depth. While monitoring the students' autonomous work, Sílvia noted their main difficulties and mistakes, as well as the solving strategies and representations they were using. As Stein et al. (2008) suggest, this practice allowed her to organize the whole-class discussion.

By inviting the students to share their incorrect answers and supporting them to analyze wrong ideas, she promoted the justification of answers and the identification of the validity of these justifications. In addition, by inviting students to explain the "why", challenging them to identify the validity of these answers, she promoted the development of justification. As for generalization, although students begun by showing some difficulties, they were able to find the sought algebraic expressions. The prospective teacher encouraged them to analyze particular cases to conclude about the general case, highlighting the generalization, and challenged them to go beyond the task.

Following the suggestion of the facilitators to support students to generalize, Sílvia challenged them to extend properties to a wider set of objects. This was due to the task proposed, which prompted generalization, but the success resulted essentially from Sílvia's actions during the lessons. Indeed, Mata-Pereira and Ponte (2018) already advanced that following the principles of task design may contribute indirectly to students' reasoning. This idea is also in keeping with Brodie (2010) who argued that "putting learners into groups and leaving them to work without mediation from the teacher does not necessarily provide enough support for developing their reasoning" (p. 20).

It should be noted that these actions were prepared in detail by the lesson study group following Sílvia's reflection on the first lesson. This reflection was based on lessons that she taught. Having the opportunity to put into practice the tasks and the teaching strategies to support students in reasoning was a key element in the development of her knowledge about teaching practice. Realizing that promoting generalization is, in fact, a challenge, led the prospective teacher to focus on improving her practice on this regard.

Not least, the use of different data sources on students' work played a prominent role in the written reflections. Sílvia was able to review the lesson, through the video recordings, and compare it with the students' productions, thus reflecting on their learning in a more sustained manner. These written reflections led her to search for information on how to improve her practice, preparing the following lessons based on research articles (Almog & Ilany, 2012) and guiding documents for teaching practice (NCTM, 2014), thus also developing her knowledge about teaching practice.

Reflecting on their own teaching practice allows teachers to understand it, considering their students' particular ways of learning. This activity of continued reflection on practice leads them to restructure their teaching practice and further their professional development (Ramos-Rodríguez et al., 2017). Thus, as pointed out by Davidson et al. (2019), the reflection on specific lesson situations, as the lesson study provides, was an effective means for Sílvia to develop her knowledge, in particular about students' difficulties and ways of learning, and also about how to lead the lesson to promote generalization.

CONCLUSIONS

Lesson study implies detailed lesson planning, with careful selecting, analyzing, and adapting tasks, based on the discussion of different teaching ideas. Sílvia prepared her lessons based on the analysis of situations from the previous lessons, considering the difficulties that the students had shown, the interconnection with empirical and theoretical papers, and the greatest challenges she had faced.

This meticulous work, based on sharing and discussing teaching perspectives, helped the prospective teachers in the development of their knowledge about students' learning and teaching practice, equipping them with the tools to be better prepared for leading the lessons (Potari & Ponte, 2017; Willems & Bossche, 2019).

It should be noted that all this work on lesson planning was facilitated by the teacher educator and the researcher. Their suggestions as well as reading and discussing research articles on students' most common difficulties, brought conceptual elements to the work of the lesson study group and contributed to the development of Sílvia's knowledge about students' learning and teaching practice.

Sílvia also had the opportunity to carry out these lesson plans, leading her to focus on developing strategies to improve her practice. Teaching the planned lessons, by applying the designed tasks and the prepared strategies was fundamental to understanding the potential of tasks and the consequences of the teacher's actions in promoting students' reasoning (Buchbinder & McCrone, 2020; Oliveira & Henriques, 2021).

Indeed, classroom experiences may lead prospective teachers to focus on students' learning as a reflection of their own practice and allow them to connect academic knowledge with practical knowledge (González et al., 2023; Leavy & Hourigan, 2016). In this set of three lesson studies, the post-lesson discussion sessions of lesson study were fundamental for the prospective teacher to rethink how she could improve her practice in the following lessons, based on the knowledge she was developing about students' learning.

In initial teacher education, the prospective teachers tend to do a descriptive analysis of students' work and focus on logistic aspects of classroom management (Ramos-Rodríguez et al., 2017). However, the work carried out during the lesson study sessions, with the opportunity to prepare, teach and reflect on three consecutive lessons, led the prospective teacher to reflect on particular situations of those lessons and think about how she could improve her practice. The sequential nature of this set of lesson studies, not usual in initial teacher education, providing the opportunity to go several times through all lesson study phases, contributed to the development of the prospective teacher's knowledge about how to promote students' reasoning processes.

Preparing teachers to be able to develop their knowledge in light of the needs of a constantly changing society is a great challenge for initial teacher education (Ponte & Chapman, 2016). In this research, we show that lesson study has a high potential to prepare prospective teachers for their professional life, providing them with practical knowledge to be active agents in the development of their knowledge.

Author contributions: MM: wrote first version of theoretical framework & research methodology & collected & made a preliminary analysis of data & **MM, JPP, & JM-P:** designed & discussed all parts of article & reviewed final version of the manuscript. All authors have agreed with the results and conclusions.

Funding: This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia [Foundation for Science and Technology] by a grant to Micaela Martins (SFRH/BD/143869/2019), Project REASON Mathematical Reasoning and Teacher Education (PTDC/CED-EDG/28022/2017) and from UIDEF – Unidade de Investigação e Desenvolvimento em Educação e Formação [Research and Development Unit in Education and Training] (UIDB/04107/2020) of Instituto de Educação da Universidade de Lisboa.

Ethical statement: Authors declared that all participants were informed about the research aims and all permissions for data collection were requested. To ensure participants' anonymity, their names are pseudonyms. The authors considered the ethical principles indicated by AERA (2011).

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

AERA. (2011). Code of ethics. *Educational Researcher, 40*(3), 145-156. https://doi.org/10.3102/0013189X11410403

Almog, N., & Ilany, B. (2012). Absolute value inequalities: High school students' solutions and misconceptions. *Educational Studies in Mathematics, 81*, 347-364. https://doi.org/10.1007/s10649-012-9404-z

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bogdan, R. C., & Biklen, S. K. (2007). *Quality research for education: An introduction to theory and methods*. Pearson.

Brodie, K. (2010). *Teaching mathematical reasoning in secondary school classrooms*. Springer. https://doi.org/10.1007/978-0-387-09742-8

Brousseau, G., & Gibel, P. (2005). Didactical handling of students' reasoning processes in problem solving situations. *Educational Studies in Mathematics, 59*, 13-58. https://doi.org/10.1007/s10649-005-2532-y

Buchbinder, O., & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *Journal of Mathematical Behavior, 58*, 100779. https://doi.org/10.1016/j.jmathb.2020.100779

Davidson, A., Herbert, S., & Bragg, L. A. (2019). Supporting elementary teachers' planning and assessing of mathematical reasoning. *International*

Journal of Science and Mathematics Education, 17, 1151-1171. https://doi.org/10.1007/s10763-018-9904-0

DGE. (2018). *Aprendizagens essenciais de matemática A–10º ano* [*Essential learning in mathematics A–10th grade*]. Ministério da Educação e Ciência [Ministry of Education and Science]. http://www.dge.mec.pt/sites/default/files/Curriculo/Aprendizagens_Essenciais/10_matematica_a.pdf

Fujii, T. (2018). Lesson study and teaching mathematics through problem solving: The two wheels of a cart. In M. Quaresma, C. Winslow, S. Clivaz, J. P. Ponte, A. Ni Shuilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 1-21). Springer. https://doi.org/10.1007/978-3-319-75696-7_1

González, G., Villafañe-Cepeda, W., & Hernández-Rodríguez, O. (2023). Leveraging prospective teachers' knowledge through their participation in lesson study. *Journal of Mathematics Teacher Education, 26*, 79-102. https://doi.org/10.1007/s10857-021-09521-4

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*, 1-16. https://doi.org/10.1007/s10649-017-9761-8

Lannin, J., Ellis, A., & Elliott, R. (2011). *Developing essential understanding of mathematical reasoning for teaching mathematics in prekindergarten-grade 8*. NCTM.

Leavy, A. M., & Hourigan, M. (2016). Using lesson study to support knowledge development in initial teacher education: Insights from early number classrooms. *Teaching and Teacher Education, 57*, 161-175. https://doi.org/10.1016/j.tate.2016.04.002

Martins, M., Mata-Pereira, J., & Ponte, J. P. (2021). Os desafios da abordagem exploratória no ensino da matemática: Aprendizagens de duas futuras professoras através do estudo de aula [The challenges of the exploratory approach in mathematics teaching: Preservice teachers' learning through lesson study]. *Bolema: Boletim de Educação Matemática* [*Bolema: Mathematics Education Bulletin*], *35*(69), 343-364. https://doi.org/10.1590/1980-4415v35n69a16

Martins, M., Ponte, J. P., & Mata-Pereira, J. (2023). Preparing, leading, and reflecting on whole-class discussions: How prospective mathematics teachers develop their knowledge during lesson study. *European Journal of Science and Mathematics Education, 11*(1), 33-48. https://doi.org/10.30935/scimath/12432

Mata-Pereira, J., & Ponte, J. P. (2017). Enhancing students' mathematical reasoning in the classroom: teacher actions facilitating generalization and justification. *Educational Studies in Mathematics, 96*, 169-186. https://doi.org/10.1007/s10649-017-9773-4

Mata-Pereira, J., & Ponte, J. P. (2018). Promover o raciocínio matemático dos alunos: Uma investigação baseada em design [Promoting students' mathematical reasoning: A design-based research]. *Bolema: Boletim de Educação Matemática* [*Bolema: Mathematics Education Bulletin*], *32*(62), 781-801. https://doi.org/10.1590/1980-4415v32n62a02

Mendes, F., Delgado, C., & Brocardo, J. (2022). Challenges faced by preservice teachers in planning and exploring tasks that promote mathematical reasoning. *Acta Scientiae* [*Journal of Science*], *24*(4), 147-182. https://doi.org/10.17648/acta.scientiae.7123

NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

OECD. (2021). *PISA 2021 mathematics framework*. https://www.oecd.org/pisa/pisaproducts/pisa-2021-mathematics-framework-draft.pdf

Oliveira, H., & Henriques, A. (2021). Preservice mathematics teachers' knowledge about the potential of tasks to promote students' mathematical reasoning. *International Journal of Research in Education and Science, 7*(4), 1300-1319. https://doi.org/10.46328/ijres.2472

Ponte, J. P. (2012). Estudiando el conocimiento y el desarrollo profesional del profesorado de matemáticas [Studying the knowledge and professional development of mathematics teachers]. In N. Planas (Ed.), *Teoría, crítica y prática de la educación matemática* [*Theory, criticism and practice of mathematics education*] (pp. 83-98). Graó.

Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169-181. https://doi.org/10.1108/IJLLS-08-2016-0021

Ponte, J. P., & Chapman, O. (2016). Prospective mathematics teachers' learning and knowledge for teaching. In L. English, & D. Kisrshner (Eds.), *Handbook of international research in mathematics education* (pp. 275-296). Routledge. https://doi.org/10.4324/9780203448946

Ponte, J. P., Quaresma, M., & Mata-Pereira, J. (2017a). The challenge of mathematical discussions in teachers' professional practice. *Didacticae* [*Didactic*], *1*, 45-59. https://doi.org/10.1344/did.2017.1.45-59

Ponte, J. P., Santos, L., Oliveira, H., & Henriques, A. (2017b). Research on teaching practice in a Portuguese initial secondary mathematics teacher education program. *ZDM, 49*, 291-303. https://doi.org/10.1007/s11858-017-0847-7

Potari, D., & Ponte, J. P. (2017). Current research on prospective secondary mathematics teachers' knowledge. In M. E. Strutchens, R. Huang, L. Losano, D. Potari, M. C. C. T. Cyrino, & R. M. Zbiek (Eds.), *The mathematics education of prospective secondary teachers around the world* (pp. 3-15). Springer. https://doi.org/10.1007/978-3-319-38965-3_2

Ramos-Rodríguez, E., Flores Martínez, P., & Ponte, J. P. (2017). An approach to the notion of reflective teacher and its exemplification on mathematics education. *Systemic Practice and Action Research, 30*, 85-102.https://doi.org/10.1007/s11213-016-9383-6

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340. https://doi.org/10.1080/10986060802229675

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education, 11*, 1463-1490. https://doi.org/10.1007/s10763-013-9409-9

Willems, I., & Bossche, P. Van den. (2019). Lesson Study effectiveness for teachers' professional learning: A best evidence synthesis. *International Journal for Lesson and Learning Studies, 8*(4), 257-271. https://doi.org/10.1108/IJLLS-04-2019-0031