MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2023, 19(5), em2266
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/13160

Learning trajectory of geometry proof construction: Studying the emerging understanding of the structure of Euclidean proof

Lathiful Anwar [1,2*], Martin J. Goedhart [2], Angeliki Mali [3]

[1] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Malang, Malang, INDONESIA
[2] Institute for Science Education and Communication, University of Groningen, Groningen, NETHERLANDS
[3] Department of Mathematics and Applied Mathematics, University of Crete, Crete, GREECE

Received 06 March 2023 ▪ Accepted 29 March 2023

Abstract

This paper presents a learning trajectory of geometry proof (LTGP) for Indonesian prospective mathematics teachers (PMTs) during their first year of studies at an Indonesian university. The trajectory aims at PMTs' progression of their understanding of the structure of proof and their proof construction abilities. We designed and implemented teaching materials with geometry problems based on the use of the flow-chart proof format and the model of understanding of proof structure from Miyazaki et al. (2017). We present an analysis of data from pre- and post-tests of proof construction problems, written answers to proof problems during intervention with 60 PMTs, and individual interviews with eight PMTs. We found that the intervention supports PMTs' understanding of the structure of proof and their proof construction abilities. Our findings contribute to knowledge about teaching strategies to support students in their understanding and construction of a proof. From our findings, we suggest the use of the flow-chart proof format together with other more formal proof formats in creating, reading, and rewriting proof of geometric propositions and the use of open problems to encourage students to think forward and backwards interactively to help students plan for proof construction.

Keywords: flow-chart proof, Euclidean geometry proof, proof construction, pre-service teachers, junior high school

INTRODUCTION

The ability to construct and comprehend proofs is a main learning goal of mathematics education at different levels and an important aspect for assessing students' performance (Selden, 2012; Weber, 2001). Constructing a proof is nevertheless more demanding than comprehending a "ready-made" proof, because one needs to know, select and use definitions and theorems in an appropriate way (Selden & Selden, 2017). Several difficulties students encounter in constructing proofs have been identified in mathematics education research (Antonini & Mariotti, 2010; Miyazaki et al., 2017; Stavrou, 2014; Weber, 2001, 2004).

In order to construct proofs, students need to understand the structure of proofs (Durand-Guerrier et al., 2012). Miyazaki et al. (2017) proposed a theoretical framework of structural understanding of proof. An earlier study (Miyazaki et al., 2015) claimed that the use of flow-chart proof with open problems could scaffold understanding of structure of proof by Japanese junior high school students (13-14 years old). However, the authors did not report on the progress of students' understanding over time. They only selected episodes discussing the validity of proof that contained logical reasoning, which is only a part of the whole progress of the understanding.

In this paper, we report on the design and evaluation of a learning trajectory of geometry proof (LTGP),

This study is a part of PhD research of the first author entitled "Fostering Indonesian prospective mathematics teachers' geometry proof competence" whose promotor and co-promotor were the second and the third authors, respectively.

✉ lathiful.anwar.fmipa@um.ac.id **(*Correspondence)** ✉ m.j.goedhart@rug.nl ✉ a.mali@uoc.gr

Contribution to the literature

- This paper offers an LTGP informing curriculum design and instructional development for effective teaching of deductive proof, particularly in an early stage of learning deductive proof in geometry.
- The study indicates how flow-chart proof formats support students' understanding of the structural relationship of a geometry proof.
- The study indicates how flow-charts with open problems help students to plan a proof preceding its construction.

offering opportunities for studying students' understanding of the structure of proof over time. LTGP aimed at supporting Indonesian prospective mathematics teachers (PMTs), aged 18-19 years old, in understanding the structure and in constructing geometry proof. LTGP was implemented in a first-year mathematics course focusing on learning geometry and geometrical proof. In Indonesia, proving is not part of the high school curriculum and PMTs are introduced to geometry proof for the first time. So, the tasks of our LTGP are on a level of geometry that is comparable to geometry for junior high school students in many other countries. As such, LTGP may also be used in the future to enhance the proving skills in junior high school students. In this paper we present our intervention, and we describe the progress of a group of 60 PMTs along with three individual trajectories; through these, we discuss the variation among students in understanding the structure of proof.

THEORETICAL FRAMEWORK

In this section, we account for the theoretical background of the study with reference to extant literature on proof construction, structure of proof and how the understanding of structure can support proof construction. In learning proof, students are taught a deductive method by drawing a conclusion from given premises and how definitions, theorems, and axioms are used. Proof construction involves a bridging process, which involves:

(1) understanding a given statement and its status,

(2) recognizing premises, argument, and conclusion,

(3) constructing intermediate proposition, and

(4) organizing these into an acceptable sequence (Heinze et al., 2008).

The bridging process includes two types of deductive reasoning: universal instantiation and hypothetical syllogism. In predicate logic, the universal instantiation deduces a singular proposition from an appropriate universal proposition (i.e., axioms, definitions, and theorems). In propositional logic, the hypothetical syllogism connects singular propositions, like premises, intermediate proposition, and desired conclusion in a logical way. When teaching proof, students should see a proof as a structured argument to understand:

(1) the components of proof and their connections,

(2) how a proof is composed of its components, and

(3) why a proof needs the structure that it has (Miyazaki et al., 2017).

In the design of our intervention, we adopted the use of flow-chart proof format developed by Miyazaki et al. (2012) for the following reasons. Firstly, our previous study (Anwar et al., 2021) confirmed that the flow-chart proof format supported PMTs' reading comprehension of geometry proof (RCGP). Secondly, Miyazaki et al. (2015) showed that the use of flow-chart scaffolded the development of understanding of the structure of proof by Japanese junior high school students. This finding was confirmed in our previous study (Anwar et al., 2021) with the same PMTs with a geometrical proof background similar to the high school students in Miyazaki's study. Also, the study by Selden et al. (2018) suggested the introduction of a structure as a way of framing a proof. Its use supports undergraduate mathematics students in writing correct, well-organized, and easy-to-read mathematical proofs, particularly the proof of a mathematical proposition in the form of an *if, then* statement (McKee et al., 2010).

Understanding the Structure of Proof

In this study, we use Miyazaki et al.'s (2015) model of understanding the structure of proof for the design of tasks and the analysis of PMTs' understanding. The model was based on the refinement and adjustment of RCGP model, originally developed by Yang and Lin (2008), and on their empirical insights gained from research among Japanese students into the difficulties of learning and teaching proofs. The model entails that students start to recognize the individual elements of a proof such as the premises, the conclusion, and the singular proposition(s), then recognize the relationships between these elements, and finally understand the relational network of a proof (**Table 1**). In this model, Miyazaki et al. (2015) distinguished three levels of understanding:

(1) pre-structural level,

(2) partial-structural level, with two sub-levels: partial-structural elemental and partial-structural relational, and

(3) holistic-structural level.

Table 1. Description of levels of understanding of structure of proof, adapted from Miyazaki et al. (2015, 2017)

Level of understanding			Description
Pre-structural level			Students see proof as a collection of symbolic objects without meaning in the context of proof. For instance, they know a geometric fact such as congruent sides but do not know how to use it in the proof.
Partial-structural level	Elemental sub-level		Students attend to components of proof (e.g., the premise, the conclusions, the singular propositions to be used), but do not know how to specify these components in accordance with universal proposition (e.g., theorems/axioms/definition) chosen as a reason to justify the proof.
	Relational sub-level	Universal instantiation	Students pay attention to components of proof and are able to use universal proposition (e.g., theorems, axioms, definitions) to specify each element of proof.
		Hypothetical syllogism	Students pay attention to components of proof and are able to connect logically all elements from premises to conclusion.
Holistic-structural level			Students can connect logically all elements via universal instantiation (connecting singular proposition with universal proposition) and hypothetical syllogism (connecting singular propositions). They can construct their own proof.

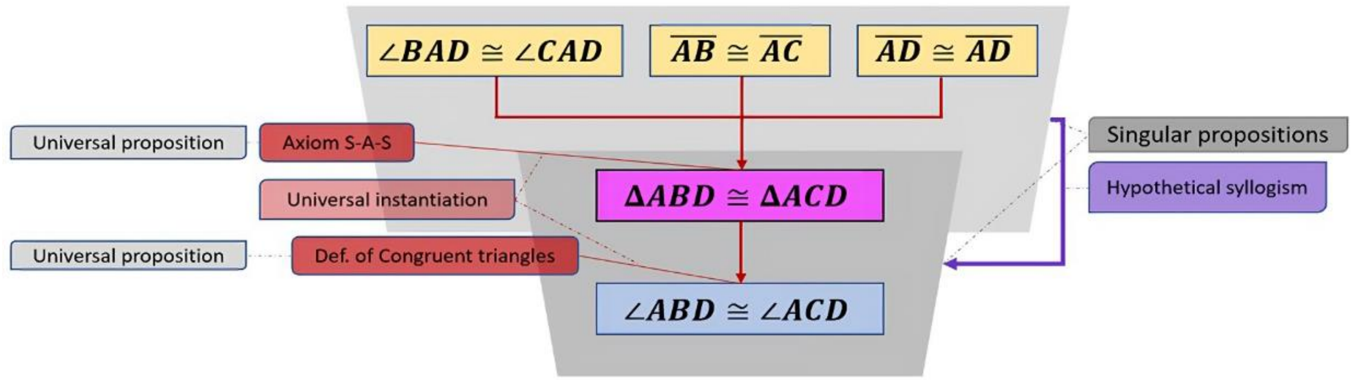


Figure 1. An example of a flow-chart proof (Adapted from Miyazaki et al., 2015)

At the pre-structural level, students see a proof as a collection of meaningless symbolic objects. They fail to identify these as components of proof. When students start to consider the components, they are at the second level, particularly the partial-structural elemental sub-level. However, knowing the components is not enough to understand structure or to construct a proof. At the partial-structural relational sub-level, students understand hypothetical syllogism and universal instantiation. For instance, if a student understands universal instantiation, then he/she is able to draw a conclusion of a given statement, such as 'in ΔABC and $\Delta ACD, AB \cong AC, \angle BAD \cong \angle CAD$, and $\angle ABD \cong \angle ACD$, with a conclusion that $\Delta ABD \cong \Delta ACD$ because of the axiom of congruent triangles (A-S-A).' At this sub-level, students may only understand either hypothetical syllogism or universal instantiation. Students who do not understand hypothetical syllogism will accept logical circularity: they draw a conclusion that is given as premise or as previous statement in the proof. Students who do not understand universal instantiation are able to draw a conclusion or connect premises to the conclusion, but they cannot identify the appropriate universal proposition to justify the conclusion. When students logically connect all singular propositions (premises, intermediate statements, and conclusion) and universal propositions via universal instantiation, and all singular propositions from premises to conclusion via hypothetical syllogism, they are at the holistic-structural level. After they reach this level, they can construct their own proof and are able to see the hierarchical relationship between theorems and the geometrical statement they prove.

Miyazaki et al. (2017) applied the model to analyze introductory geometry proof lessons in Japanese secondary schools in which students (aged 14) discussed validity of proof that contained logical reasoning, but this is only a part of whole progress of understanding. In this study we applied the model to capture the whole progress of students' understanding of the structure of proof, enabling them to construct their own proof.

Flow-Chart Proof Format and Open Problems

Formats that can be used in geometry proofs are two-column proof, tree proof, paragraph proof, and flow-chart proof. Details of these formats are found in Cirillo and Herbst (2011). Some communities accept or require a certain format, but that format might not be accepted in other contexts (e.g., two column proofs and flow-chart proofs are acceptable in school, but not in mathematics journals). Also, some communities see flow-chart proof as a 'pre-formal' proof, which can be introduced before students learn more formal proofs.

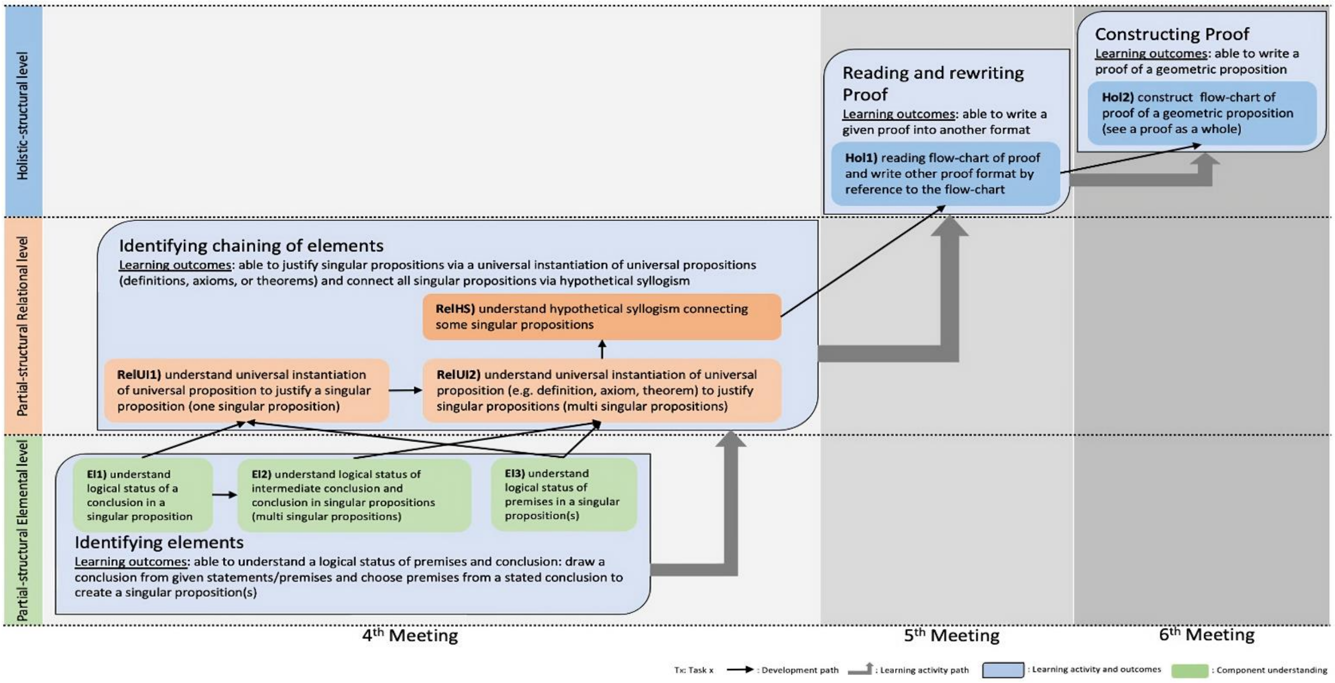


Figure 2. Levels & components of understanding of structure of proof in initial hypothetical learning trajectory (HLT) (prepared by Anwar)

The flow-chart proof displays the delineated structure of a proof using boxes and connecting arrows (Cirillo & Herbst, 2012). A representation of a flow-chart proof format is shown in **Figure 1**. Flow-chart presents deductive connections from premises to conclusions by identifying singular and universal propositions or supporting reasons (Miyazaki et al., 2015).

Figure 1 indicates that the flow-chart proof visualizes not only the components of the proof but also the connection between them via two types of logical reasoning. Therefore, in this study we introduce flow-chart proof as a format to represent proofs in geometry to help students' understanding of the structure of proof to enable them to construct geometry proofs. The study by Anwar et al. (2021) confirmed that the use of the flow-chart proof format supported students to comprehend geometry proof, especially to understand the logical status of components and the critical ideas of geometry proofs.

In order to teach students how to construct a proof, van Engen (1970) suggested to give more explicit attention to strategies of proof construction including how to start writing proof. For instance, to prove a statement, '*if P, then Q*', students start from premises P to reach a conclusion Q by going through a logical sequence of steps in a forward process. In fact, it is not always easy to start with the premises and continue with a logical sequence of steps. Another, and in many cases more fruitful, way is to follow a backward process. Here, students can call on the information provided by the premises, together with universal propositions. This process allows students to focus on information that seems essential to them. To write proof they follow the reversed process starting from the premises and write towards the conclusion Q via a logical sequence of steps.

A study by Miyazaki et al. (2015) showed that open problems supported students in proof construction by encouraging them not only to deduce a conclusion from given assumptions (forward process) but also to choose assumptions to prove a conclusion (backward process). In this study, we define an open problem as a problem with multiple possible solutions.

Hypothetical Learning Trajectory

In this study we used the methodology of design research, and more specifically, we used a hypothetical learning trajectory (HLT) as a sequence of student tasks with specified learning goals, and predictions of students' answers for each task (Bakker, 2018; Mckenney & Reeves, 2012; Plomp, 2013; Prediger et al., 2015). As such, it gives a detailed, stepwise scenario for the overall intended learning goals.

Following Bakker (2018), our HLT has three different functions. Firstly, HLT was used as a guideline for designing student tasks through a progression of the levels of understanding of structure of proof. Secondly, HLT functioned as a guideline for the researcher; based on evidence from the implementation of the design during class meetings, the researcher revised the instructional activities of HLT for the next class meeting. Lastly, HLT functioned as a guideline for analyzing data by contrasting the learning goals of each task with PMTs' learning outcomes. In design research, this analysis might lead to revision of HLT, but in this study we report about our experiences with the initial HLT.

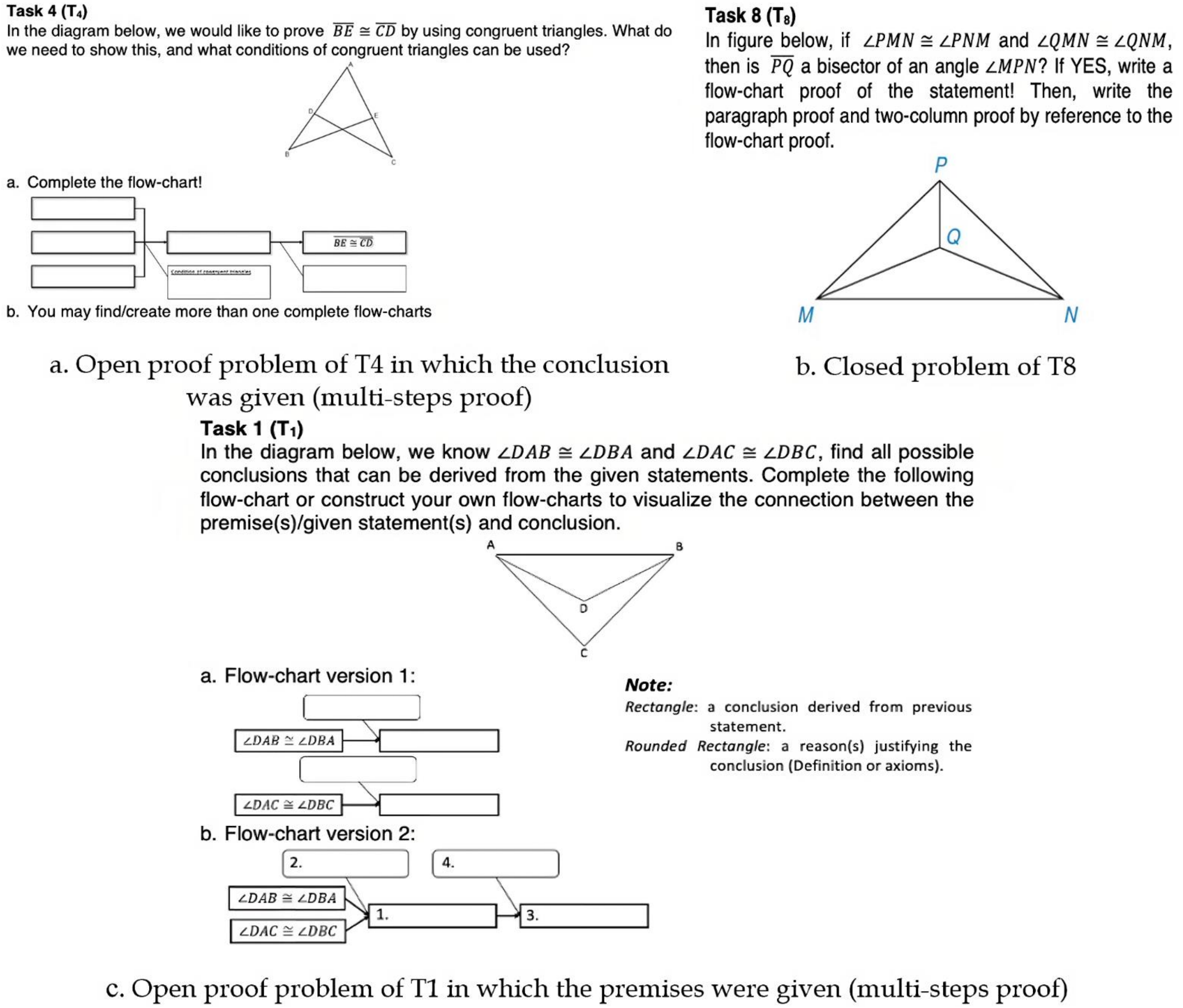


Figure 3. Open and closed proof problems (prepared by Anwar)

Figure 2 depicts the intended progression of students' levels of understanding of the structure of proof in the initial HLT during three class meetings through four main learning activities. Arrows identify the predictions of the students' progression to the levels of understanding (e.g., partial-structural elemental sub-level) and their components [e.g., elementary 1 (El1), relational universal instantiation 1 (RelUI1), relational hypothetical syllogism (RelHS)] over time.

Intervention and Task Design

The intervention consisted of six meetings of 150 minutes each. This study focuses on the last three meetings (4th, 5th, and 6th meeting) on proof construction. The first three meetings were designed to support students' understanding of definitions of geometric concepts (e.g., a midpoint, bisector of a segment, vertical angles, etc.), including axioms and theorems, and to create conjectures (geometric propositions) in the form of *if, then* statements related to axioms of congruent triangles. The teaching in the last three meetings aimed to initiate PMTs' understanding of the structure of geometric proof to such a level that they develop abilities to construct proofs in this course and tackle the more complicated proofs they will encounter in subsequent courses.

As said, we adopted Miyazaki et al.'s (2017) learning trajectory including learning goals and learning activities to develop our HLT. One of the features of HLT is the use of flow-chart proofs. The main learning goals intend that students will be able to:

(1) understand structure of proofs in geometry and

(2) construct geometric proofs.

The intervention design consisted of three learning phases conducted in three 150-minute meetings. We designed tasks with open (e.g., tasks T1, T2, T3, and T4) and closed (e.g., task T5, T6, T7, and T8) problems. Tasks introduce students to flow-chart proof format, adapted from Miyazaki et al. (2015). The term "open" in our open

Table 2. Main learning activities, learning goals, targeted component understandings, & tasks of the course meetings

Meeting & activities	Learning goals	Targeted component understanding of structure of proof *(Code)*	Tasks (T)
Meeting 4: Constructing flow-chart proof with open problem	1. Able to understand logical status of elements of proof	1.1. Understand logical status of a conclusion of a singular proposition *(El1)*	T1a
		1.2. Understand logical status of intermediate proposition & conclusion of "multiple" singular propositions *(El2)*	T1b
		1.3. Understand logical status of premises of a singular proposition(s) *(El3)*	T2a, b, T3a, b, T4a, b
	2. Able to understand connection/chaining elements	2.1. Understand universal instantiation of universal proposition (i.e., definition, axiom, & theorem) to justify a singular proposition (one singular proposition) *(RelUI1)*	T1a, 1c, T2a, 2b
		2.2. Understand universal instantiation of universal proposition (e.g., definition, axiom, & theorem) to justify "multiple" singular propositions *(RelUI2)*	T1b, T3a, b, T4a, b
		2.3. Understand logical chaining (i.e., hypothetical syllogism) connecting some singular propositions *(RelHS)*	T2a, T3a, T4a, T1c, T2b, T3b, T4b
	3. Able to apply forward/backward process to construct their proof	3.1. Apply forward thinking to complete/create flow-chart proof	T1a, b, c
		3.2. Apply backward thinking to complete/create flow-chart proof	T2a, b, T3a, b, T4a, b
Meeting 5: Reading & constructing proof	4. Able to transform their flow-chart into other formats & vice versa	4.1. Rewrite a proof in other formats (two-column or paragraph) by reference to given flow-chart proof *(Hol1)*	T5
		4.2. Rewrite a proof in flow-chart proof by reference to given paragraph, & two-column proof, format *(Hol1)*	T6
Meeting 6: Constructing & rewriting proof	5. Able to construct flow-chart proof & transform proof into another form of proof: A paragraph or two-column	5.1. Construct flow-chart proof of a geometric proposition (see proof as a whole) *(Hol2)*	T7b
		5.2. Rewrite a proof in other formats (two-column or paragraph) by reference to constructed flow-chart proof *(Hol1)*	T7c
		5.3. Construct flow-chart proof of a geometric proposition (see a proof as a whole) *(Hol2)*	T8a
		5.4. Rewrite a proof in other formats (two-column or paragraph) by reference to constructed flow-chart proof *(Hol1)*	T8b

problem refers to a situation, where PMTs could give more than one answer or suitable proof. An example of an open problem is shown in part a in **Figure 3**. In contrast, closed problems refer to proof problems similar to the typical form appearing in textbooks, as shown in part b in **Figure 3**. The open problems here are designed to encourage PMTs' creativity and flexibility in choosing premises, intermediate propositions or conclusions and their connections.

We detailed the learning goals of the tasks in terms of components of understanding, as shown in **Table 2**. These components include student's understanding of the logical status of premises and conclusions, two types of reasoning (universal instantiation and hypothetical syllogism) and thinking processes to construct proofs (forward thinking from premise to conclusion, see part c in **Figure 3**, and backward thinking from conclusion, see part a in **Figure 3**). For instance, a component of understanding is the universal instantiation of a specific universal proposition (e.g., axiom of congruent triangles: S-A-S). If one understands universal instantiation, when faced with a question such as "In ΔPMQ and ΔPNQ, with $\overline{PM} \cong \overline{PN}$ given, which additional premises are needed to prove $\Delta PMQ \cong \Delta PNQ$?", one can answer by stating "In order to use the condition of congruent triangles, $\angle PMQ \cong \angle PNQ$ and $\overline{MQ} \cong \overline{NQ}$ are needed."

Aims of the Study

The purpose of this HLT is to scaffold students' understanding of the structure of proof and their abilities to construct a geometry proof. The purpose of the study is to understand why the intervention helps students understand and construct geometry proofs. Our intervention was guided by Miyazaki et al.'s (2017) levels of proof understanding and our research particularly intended to identify the roles of flow-chart proof formats in student understanding. The corresponding research question is: *How does the intervention support students' understanding of the structure of proof and their performance in geometry proof construction?*

RESEARCH METHODS

Design of the Study

This is an intervention-based study in the area of proof construction. The term intervention denotes an action aimed to change a teaching and learning situation, here geometry proof (Stylianides et al., 2017). We designed an HLT and implemented and tested HLT in two classrooms, class 1 (C1) and class 2 (C2). HLT was used in one class, and a (slightly) revised version was used in the other class of students. The minor revisions included details in the formulation of tasks and additional instruction by the teacher. We collected several data before, during and after the intervention, including pre-test, students' written answers to tasks, students' utterances in class discussions and student pair discussions, task-based interviews, and post-tests. We analyzed these data by comparing the actual learning for each task with the aims specified in the initial HLT.

Participants

Participants were 60 PMTs during their first year of the university of four-year curriculum, aged 18-19: 32 PMTs (six males, 26 females) in C1 and 28 PMTs (five males, 23 females) in C2. This proportion reflects the number of females and males in Indonesian teacher education programs, particularly in mathematics education. PMTs begin learning formal proofs in geometry. PMTs were introduced to the fundamentals of plane geometry at their secondary school, including the properties of polygons (triangles, rectangles, squares, etc.), similarity, and congruence applied to issues involving measurement and computation, but they were not exposed to proof.

The lecturer in this intervention was the first author, hereafter called a lecturer-researcher, whose role is twofold (Mills, 2014). As a lecturer, he provided students with tasks and allowed the students to explore the tasks in pairs while discussing ideas with each other. The lecturer supported the students by probing their thinking, asking them to explain and justify their strategies, and encouraging the free exchange of ideas. As a researcher, he audio-recorded group and classroom discussions. The lecturer-researcher also wrote reflection notes during or after each class.

Data Collection

All students took a pre-test and a post-test (a week before and a week after the intervention, respectively), testing their understanding of structure of proof as an indication of the effect of the intervention, see the **Appendix A**.

The tasks tested proof reading comprehension and proof construction. But, in this article we focus on proof construction. Pre- and post-test questions were similar in terms of complexity and type of task, but they differed in the proposition to be proven.

During meetings all students' written answers to the tasks were collected, and classroom discussions and discussions in groups were audiotaped. The written answers to the tasks and students' utterances during whole-class discussions and group discussions were used to interpret students' understanding.

We combined the data collected during intervention with data from interview sessions for triangulation purposes and for gaining more detailed information about the individual learning trajectories. Six students were selected for post-interviews based on their ability to communicate actively, gender diversity (three males and three females), diversity in expected proof construction performance, and their willingness to participate. The interviews were individual think-aloud sessions in which students worked on proof construction problems, conducted a day after the last course meeting. Each interview lasted for about 60 minutes.

In the interview sessions, interviewees solved a proof construction problem, and they were requested to speak out loud and explain as much as they could to investigate their thinking. During the interviews, the interviewer gave minimal directions. He also explained that any time he asks "why?" it did not mean that the participant was wrong, but that he was seeking to understand his/her thinking.

Data analysis

Analysis of pre- and post-test

We categorized students' written answers to pre- and post-test to determine the level of understanding of structure of proof at the initial and end stage of the intervention. We used a rubric for identifying PMTs' levels of understanding, see **Appendix B**.

Analysis of student answers to tasks in class meetings

We scored students' answers to each task to determine whether they reached the targeted components of understanding of structure of proof. Students' answers (written in flow-chart, see the problem in **Figure 3**) were scored one when each component of understanding targeted by the task was present in the answer, following the criteria of each component of understanding in **Table 3**, and zero if the answers did not meet the criteria.

A second rater who had not participated in the prior coding activities independently scored 25% of students' written answers (pre-test, tasks, and post-test) using the scoring rubric (**Table 3**). We calculated the percentage of agreement between raters to test the inter-rater reliability (Mchugh, 2012). The percentages of agreement of pre-test, task, and post-test were 80%, 93%, and 80% (that is, one point difference in scores of about two to three students).

Analysis of classroom discourse

We transcribed students' utterances from audio recordings during classroom and pair discussions. Then, we coded and classified the transcriptions using criteria

Table 3. Criteria used to determine students' understanding of structural understanding of proof (on Miyazaki et al., 2017)

Component	Criteria
El1	The flow-chart mentions a correct conclusion
El2	The flow-chart consists of all correct intermediate proposition(s) and a conclusion
El3	The flow-charts consist of all correct premises and intermediate proposition(s)
RelUI1	The single-step proof flow-chart consists of all correct intermediate proposition(s) and a conclusion and all appropriate universal propositions (i.e., definition, axiom, or theorem)
RelUI2	The multi-step proof flow-chart consists of all correct intermediate conclusion(s) and conclusion; and all appropriate universal propositions (i.e., definition, axiom, or theorem)
RelHS1	The completed flow-charts connect all singular propositions logically from premises to conclusion or do not indicate PMTs accept circular reasoning
RelHS2	The constructed flow-charts connect all singular propositions logically from premises to conclusion or do not indicate PMTs accept circular reasoning

Table 4. Percentage of students who reached a certain level of understanding of structure of proof in pre- & post-test (n=60)

Levels of understanding	Before intervention (pre-test)	After intervention (post-test)
Pre-structural	68	3
Partial-structural elemental sublevel	27	22
Partial-structural relational sublevel	5	25
Holistic-structural	0	50

Table 5. Percentages of PMTs who reached components of understanding of structure of proof (partial-structural elemental sub-level) in written answers to tasks T1, T2, T3, & T4

Component understanding	Task no. (in percentage)								
	T1a	T1b	T1c	T2a	T2b	T3a	T3b	T4a	T4b
(El1) Understanding logical status of a conclusion in a singular proposition	100								
(El2) Understanding logical status of conclusions in all singular propositions		92	92						
(El3) Understanding logical status of premises in singular				96	81	100	89	93	76
(RelUI1) Universal instantiation in one-step proof	92		88	93	81				
(RelUI2) Universal instantiation in multi-step proof		92				95	84	82	72
(RelHS1) Hypothetical syllogism (complete flow-chart)				98		96		95	
(RelHS2) Hypothetical syllogism (create flow-chart)			92		96		89		100

in **Table 3** to investigate whether they provided evidence of reaching the components of understanding of structure of proof in HLT.

Our analysis focused on some parts of classroom and pair discussions as it aimed to support and triangulate our claims of analysis of students' tasks.

Analysis of interviews

We transcribed the interviewees' utterances during interview sessions. Next, we scored their answers to the interview tasks using criteria in **Table 3** to determine whether components of understanding of structure of proof targeted by the task emerged.

FINDINGS

Students' Levels of Understanding of Structure of Proof

Table 4 gives the percentages of students who reached the levels of understanding of structure of proof based on their performances on a proof construction problem in the pre- and post-test. **Table 4** shows that 50% of PMTs reached the highest level of understanding of structure of proof after the intervention. In contrast, no PMTs were at the holistic-structural level in the pre-test. Because of the hierarchical nature of the levels, it can also be concluded that 97% of PMTs reached the partial-structural elemental level, and 75% of PMTs reached the partial-structural relational level in the post-test. So, the intervention supported the vast majority of PMTs in understanding the structure of a proof.

In the pre-test, all PMTs wrote their proof of a proposition in a narrative way as a paragraph proof. In contrast, in the post-test, only 10% of PMTs wrote the proof in the form of paragraph proof, and the other 90% wrote the proof in the form of flow-chart or combinations of flow-chart with other proof formats (i.e., two-column proof, paragraph proof). In addition, almost all PMTs (27 of 30) who were at the holistic-structural level after the intervention wrote the proof in the form of a flow-chart proof or combined the flow-chart proof with other proof formats. This is an indication that the use of flow-chart proof helped students understand the structure of proof, and then, construct a proof.

Partial-Structural Elemental and Relational Level of Understanding

Table 5 presents the percentages of PMTs who identified the components of proof (e.g., premises,

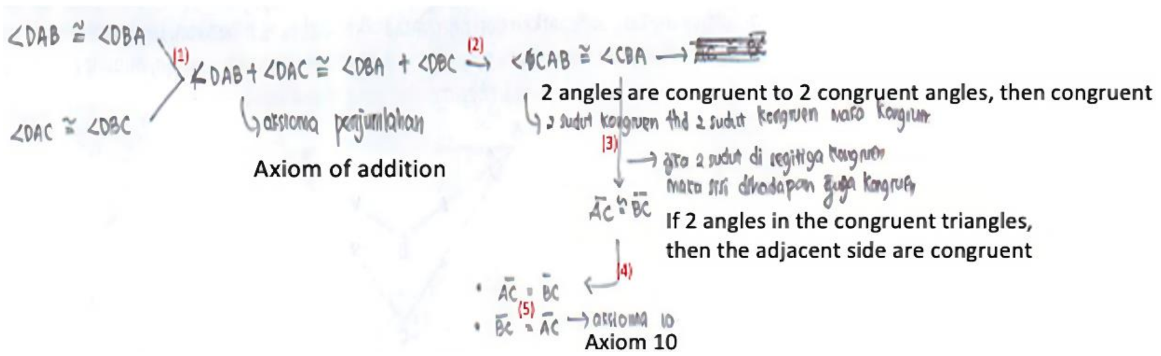
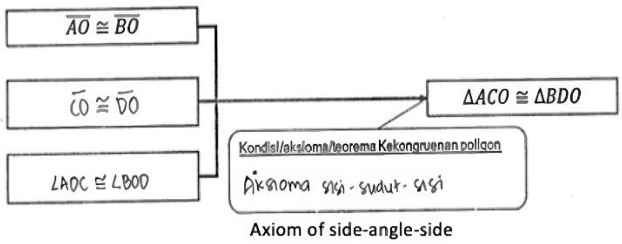
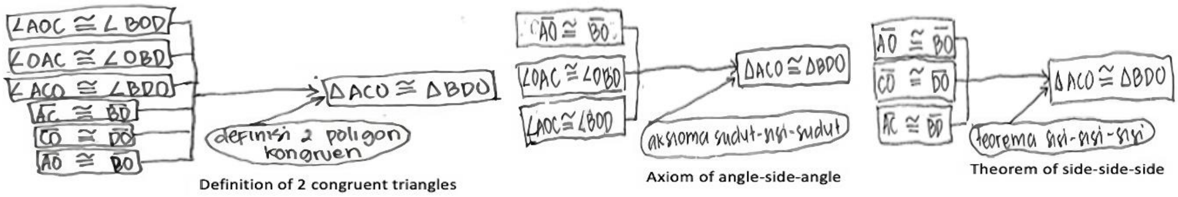


Figure 4. PMT32's flow-chart of task 1c (a scan of the participants' answers, reprinted with permission)



a. Completing flow-chart



b. Creating own flow-charts

Figure 5. PMT5's flow-chart proofs of tasks T2a & T2b (a scan of the participants' answers, reprinted with permission)

conclusion, and singular proposition), to instantiate a universal proposition to deduce a singular proposition and to connect logically all singular propositions from premises to conclusion by hypothetical syllogism. The percentages are calculated from the flow-charts written in PMTs' answers to the tasks (i.e., T1, T2, T3, and T4).

Regarding the components of understanding El1, El2, and El3, **Table 5** shows that almost all PMTs (i.e., over 92%) chose correct premises, intermediate propositions and conclusions to produce correct singular propositions in tasks T1a, T1b, T2a, T3a, and T4a. This data indicates that most PMTs understood the logical status of components of proof when completing a given flow-chart. For example, as we see in **Figure 4**, PMT32 created five singular propositions in task T1c.

The flow-chart shows that PMT32 understood the logical status of premises and conclusion in the singular proposition. Particularly, she understood how the conclusion can be derived from given premises, so she understood the singular proposition as an element of proof. In tasks asking PMTs to create their own flow-charts (i.e., T1c, T2b, T3b, and T4b) percentages are somewhat lower. The El3 score of T2b is lower than the El3 score for T3b, and the El3 score for T4b is lower than the one for T3b. This may be explained by the different degrees of difficulty of the tasks.

The percentages of RelUI1 in **Table 5** show that most PMTs (over 81%) chose the appropriate axiom, definition, or theorem and placed it in the rounded rectangle in the flow-chart (blank rounded rectangles asked for a single singular proposition to be completed). Most PMTs (92% in T1b, 95% in T3a, 84% in T3b, 82% in T4a, and 72% in T4b) chose an appropriate axiom, definition, or theorem in the flow-chart. Thus, most of PMTs were able to instantiate an appropriate universal proposition (i.e., axiom/definition/theorem) to justify each singular proposition in a flow-chart proof. As an example, in task T2a, PMT5 wrote a correct universal proposition "axiom of side-angle-side" in the rounded rectangle to justify a singular proposition "if $\overline{AO} \cong \overline{BO}, \overline{CO} \cong \overline{DO}$ and $\angle ADC \cong \angle BOD$, then $\Delta ACO \cong \Delta BDO$", as shown in part a in **Figure 5**.

In T2b (part b in **Figure 5**), PMT5 created her three own flow-charts consisting of a singular proposition with correct premises and universal proposition to justify each singular proposition. PMT5's flow-charts indicate that she understood the relational connection between a singular proposition and a universal proposition via universal instantiation.

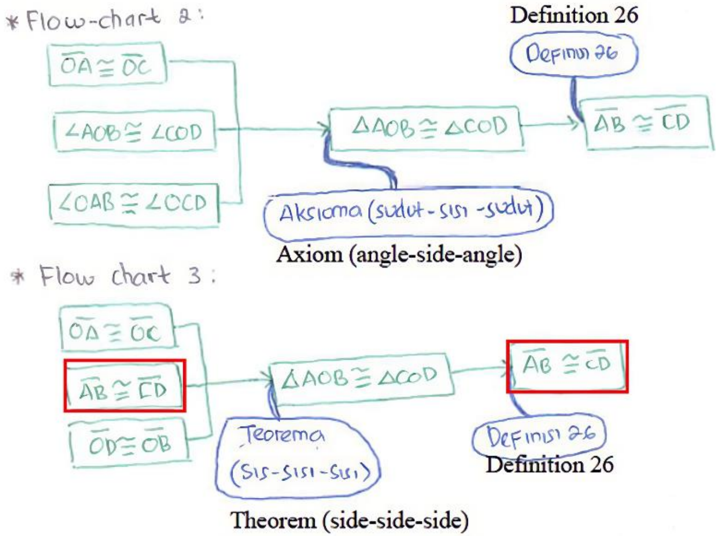


Figure 6. PMT38's flow-chart proof of task T3b (a scan of the participants' answers, reprinted with permission)

Table 6. Percentage of PMTs who reached holistic component of proof understanding in students' written answer to tasks T5, T6, T7, & T8

Component understanding	Task			
	T5 (1-16)	T6 (1-17)	T7 (1-3)	T8
Hol1 rewriting/writing proof by reference to a given/constructed proof	75	70	83	83
Hol2 proof construction			91	85

Statements	Reasons	
1. $\overline{AM} \cong \overline{BM}$	Given	*Line 1*
2. $\overline{CM} \cong \overline{DM}$	Given	*Line 2*
3. ∠AMC dan ∠BMD are vertical angles	Definition of vertical angles	*Line 3*
4. ∠AMC ≅ ∠BMD	Theorem of vertical angles	*Line 4*
5. ΔAMC ≅ ΔBMD	Axiom of Side-Angle-Side	*Line 5*
6. ∠MAC ≅ ∠MBD	Definition of congruent triangles	*Line 6*

Figure 7. PMT28's two-column proof for task 5 (prepared by Anwar)

Regarding the understanding of hypothetical syllogism, the percentages of RelHS1 in **Table 5** show that less than 5% of PMTs wrote a conclusion as one of premises (T2a, T3a, and T4a). So, most of PMTs understood hypothetical syllogism and applied this reasoning to connect all components of proof logically.

The percentages of RelHS2 in **Table 5** detail that a few PMTs wrote a premise as a conclusion and/or created an unconnected flow-chart from premises to conclusion (8% in T1c, 4% in T2b, 11% in T3b, and 0% in T4b). So, only a small number of PMTs used circular reasoning to complete the given flow-chart. Circular reasoning is an indication of the lack of understanding of hypothetical syllogism in constructing a proof. For instance, in T3b, PMT38 created two flow-charts, as shown in **Figure 6**. PMT38 wrote an appropriate universal proposition to justify each singular proposition in both flow-charts. However, in the second flow-chart PMT38 wrote "$\overline{AB} \cong \overline{CD}$" as one of premises to deduce an intermediate proposition in which this proposition is a conclusion, as indicated by the red rectangle in **Figure 6**, indicating circular reasoning.

Holistic-Structural Level of Understanding

Table 6 presents the percentages of PMTs who reached the holistic component of understanding. **Table 6** shows that the percentage of PMTs who rewrote the flow-chart proof into a paragraph or two-column proof in task T5 (75%) was slightly higher than the percentage (70%) of PMTs who rewrote the paragraph proof into flow-chart proof in task T6. Although the difference is small and not significant (Wilcoxon signed-rank test: Z=-.688, p=.491), PMTs argued that the flow-chart proof format is easier to understand and makes it easier to write the proof in other formats. For instance, during a whole-class discussion, PMT19 stated about flow-chart proof: "I can see the process easily", "I can see clearly which statements support the conclusion. I mean origin of the conclusion" and "the steps of the proof are easy to be seen".

In contrast to the flow-chart proof, some PMTs had difficulties to identify the connection between statements in two-column proofs. For instance, during whole-class discussion, after PMT11 read the two-column proof of task T5 created by PMT28 as shown in

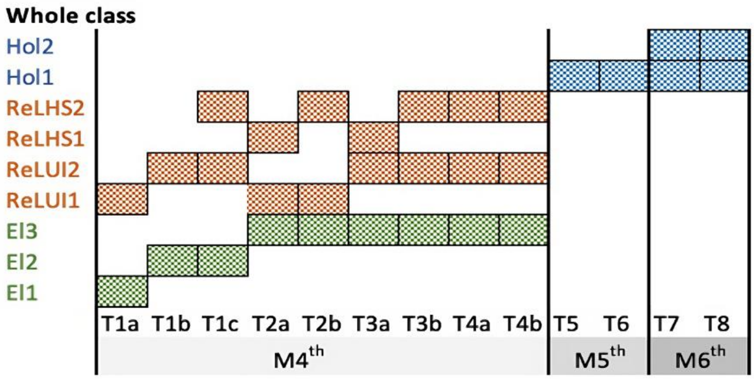
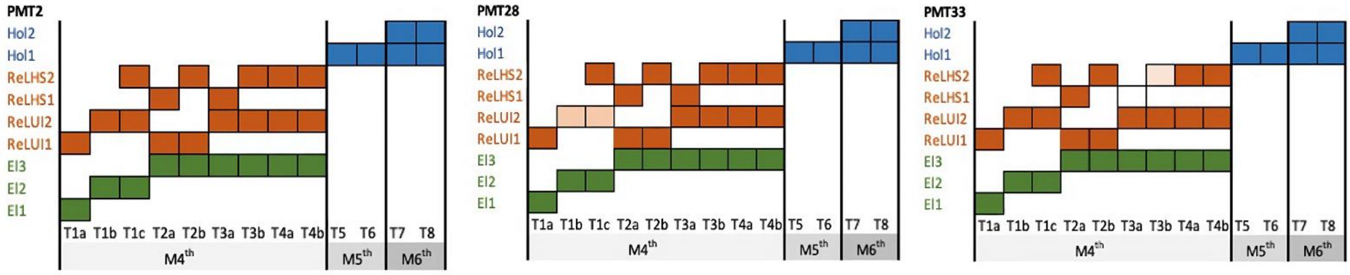


Figure 8. Hypothetical learning trajectory showing tasks targeting components of understandings of geometrical proof (prepared by Anwar)



a. Progression PMT2 b. Progression PMT28 c. Progression PMT33

Figure 9. Learning trajectories of three PMTs (prepared by Anwar)

Figure 7, she stated that statement 5 was deduced from statement 4 and said, "number 4 is derived from number 3". Our interpretation is that PMT11 thought that each statement in the left column is deduced from a previous statement, indicating that the representation of the two-column proof did not lead to PMT11's understanding of the connection between statements in a proof.

Table 6 also shows that the percentages of students at level Hol2 were high in the answers to tasks T7 (91%) and T8 (85%). This indicates that most PMTs were able to construct a flow-chart proof. The percentage of Hol2 in task T7 was higher than that in task T8, because of a difference in the complexity of the problem. In T7, the proof steps were given in the text. In contrast, in T8, PMTs should devise their own proof steps and then write their flow-chart proof. As shown in **Table 6**, the percentages of Hol1 increased from T5 (75%) to T7 (83%) and then remained stable in T8. This indicates that most PMTs were able to write flow-chart proof into other proof formats and that percentage improved over time.

Students' Learning Trajectories

We simplified the information from **Figure 2** and **Table 2** into **Figure 8**, presenting HLT enacted in the intervention and the components of understanding targeted by the tasks. The graph represents the meetings of the intervention (M) and tasks (T). The partial-structural elemental sub-level is shaded in green, the partial-structural relational sub-level in orange, and the holistic structural level in blue. We show how individual students progressed during the intervention meetings and how their progression followed HLT.

Individual Students' Trajectories

All students began at the pre-structural elemental level, see **Table 4**, as we assumed that students already know some geometric terms, definitions and axioms regarding congruent triangles from previous meetings. All students generally progressed in similar ways through the main path of the learning trajectory; individual differences emerged in the timing of development and the degree to which students could engage in constructing all possible flow-chart proofs. In the following paragraphs we present the learning trajectories for three PMTs. These three PMTs succeeded in improving their level of understanding of structure of proof from pre-structural level at the initial stage of intervention to the holistic level of understanding, but they differed in the paths followed to develop their understanding as visualized in **Figure 9**.

The full-colored rectangles indicate that PMT reached a particular component of understanding. In contrast, the light-colored rectangles indicate that PMT only partially achieved to reach the level intended by the task. The non-colored black line rectangles show that PMT failed to reach particular component of understanding.

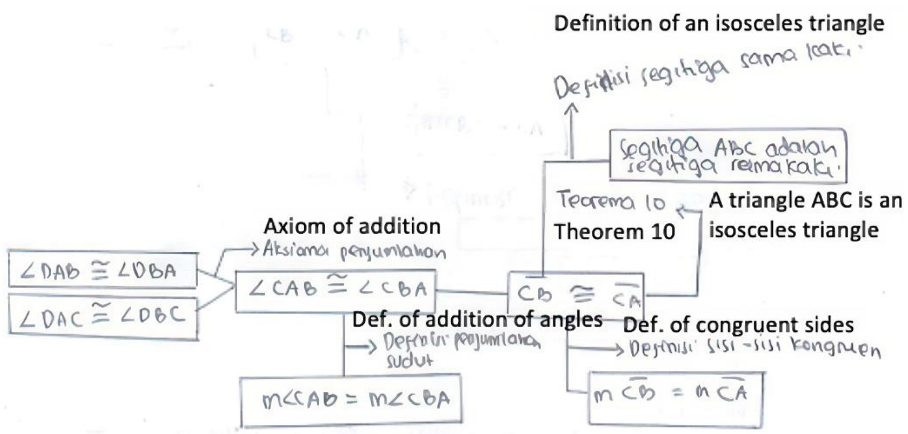


Figure 10. PMT28’s flow-chart proof of task T1c (a scan of the participants’ answers, reprinted with permission)

PMT2

Part a in **Figure 9** presents PMT2’s progression through the learning trajectory. PMT2 is one of PMTs who demonstrated to have reached the highest level after the trajectory, and her progress followed the trajectory targeted by the tasks. During the interview session, PMT2 proved a geometric proposition and used a flow-chart proof format to represent the proof. During the process of constructing the flow-chart, the interviewer posed some questions to investigate PMT2’s strategies. PMT2 answered “firstly, I see the premises written in the statement, see the figure following the statement …”, “... find other conditions that can be concluded from the premises …”, and “then find other conditions [intermediate propositions] such that I can find that segment DB and AE are congruent [conclusion]”. In this case, PMT2 started to construct a proof from the premises, which were written explicitly in the text or extracted from the figure following the statement to be proven, and, next, she found intermediate propositions to come to the desired conclusion by using universal instantiation of universal propositions (i.e., forward thinking). She also argued that presenting geometry proof using flow-chart enabled her to trace previous statements used to deduce the intermediate proposition or conclusion in the proof: “… if I use two-column format it cannot be seen from which statements the congruent triangle is deduced”. The use of flow-chart proof format also enabled PMT2 to find easily a following intermediate proposition to arrive at the desired conclusion “yes, so I can explore the next statement [intermediate proposition]”.

PMT28

PMT28’s progression through the learning trajectory is depicted in part b in **Figure 9**. The light-colored rectangles in T1b (RelUI2) indicate that PMT28’s understanding of universal instantiation in T1b was considered incomplete because one of the universal propositions used to justify the conclusion was incorrect. At the initial meeting, PMT28 used inappropriate universal propositions to create a singular proposition. For instance, in task T1b PMT28 deduced a conclusion “$m\angle CAB = m\angle CBA$” from the intermediate proposition “$\angle CAB \cong \angle CBA$”, but used an incorrect universal proposition (i.e., definition of addition of angles) to justify the conclusion. This indicates that PMT28 lacks understanding of universal instantiation. This is also visible in PMT28’ answer to task T1c, as shown in **Figure 10**, which is visualized by the light-colored rectangles of RelUI2 in task T1c.

However, in task T1c PMT28 connected all statements in the proof logically as indicated by the correct statements in the rectangles. This means that PMT28 did not use circular reasoning. So, PMT28 reached RelHS2, visualized by the full-colored rectangle. The light- and full-colored rectangles of RelUI2 and RelHS2 in task T1c indicate that the understanding of these two components, universal instantiation, and hypothetical syllogism, were developed independently. This also shows that RelHS2 can be reached before RelUI2, and this raises questions on the hierarchical nature of these two levels of understanding.

PMT33

PMT33’s progression through the learning trajectory is depicted in part c in **Figure 9**. For the first tasks T1 and T2, she followed the trajectory as intended by HLT. The white rectangle for task T3a indicates that PMT33 did not reach the component ReLHS1 targeted by this task because she showed circular reasoning. In task T3b, PMT33 chose two premises “ $\overline{OB} \cong \overline{OD}$” and ”$\overline{AB} \cong \overline{DC}$” to conclude the intermediate proposition “$\Delta ABO \cong \Delta CDO$” and then finally drew the conclusion “ $\overline{AB} \cong \overline{CD}$” (see **Figure 11**). This circular reasoning indicated that PMT33 lacked understanding of hypothetical syllogism, which connects singular propositions in the proof. However, PMT33 instantiated appropriate universal propositions (e.g., theorem S-S-S and definition of congruent triangles) to justify singular propositions, which indicated that PMT33 understood universal instantiation. Based on the cases of PMT28 and PMT33,

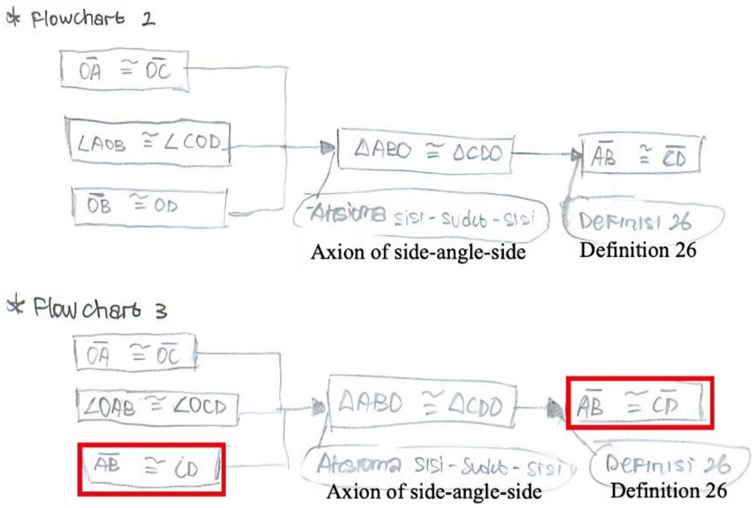


Figure 11. PMT33's flow-chart proof in task T3b (a scan of the participants' answers, reprinted with permission)

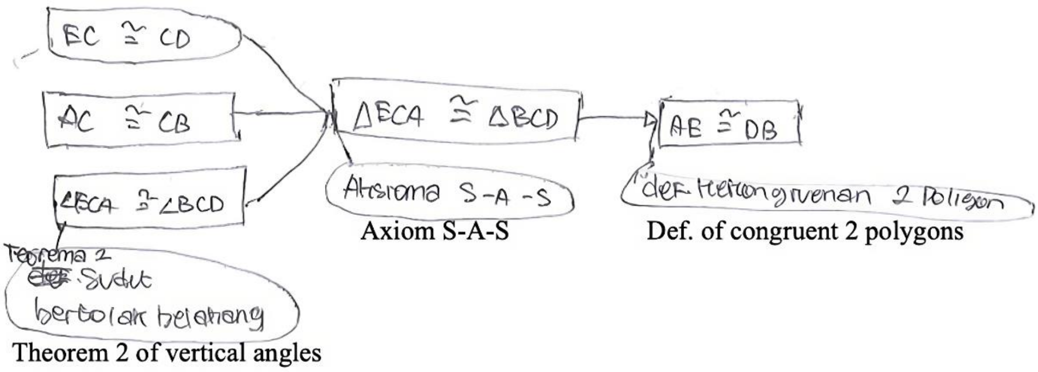


Figure 12. PMT33's flow-chart proof in interview session (a scan of the participants' answers, reprinted with permission)

our interpretation is that these two components of understanding, universal instantiation and hypothetical syllogism, developed independently.

In the interview session, PMT33 constructed the proof of the geometric proposition "In polygon $ADBE$, if $\overline{AC} \cong \overline{CB}, \overline{EC} \cong \overline{CD}$, then $\overline{AE} \cong \overline{DB}$". The point C is a midpoint of segment AB. PMT33 wrote correct premises, intermediate proposition "$\Delta ECA \cong \Delta BCD$", and conclusion "$\overline{AE} \cong \overline{DB}$" (see **Figure 12**). He identified a premise $\angle ECA \cong \angle DCB$, which is not stated explicitly in writing but embedded in **Figure 12**. The student said "it is simple, because these [pointing to angles $\angle ECA$ and $\angle DCB$] are vertical angles, so this [pointing to $\angle ECA$] must be congruent with this [pointing to $\angle DCB$]").

Figure 12 and his verbal explanation indicate that PMT33 chose appropriate universal propositions to justify each singular proposition. PMT33 also explained why he chose "axiom of side-angle-side" as universal proposition: "I have to think how I can prove that these two triangles are congruent, so I have to use side-angle-side, and side-side-side is not possible because this [pointing to $\overline{AE}$ and $\overline{BD}$] must be proven … based on my experience [wrote conclusion as one of premises in task T3b]". This indicates that PMT33 not only understood the universal proposition but also hypothetical syllogism, because all singular propositions are connected logically.

In the interview session, PMT33 explained his thinking to find intermediate propositions to connect premises to the conclusion when he started to construct the proof. He started with choosing the premises ("firstly, I wrote the given statements [premises]"), seeing the conclusion ("see the end [pointing the conclusion $\overline{AE} \cong \overline{BD}$]") and thinking on how to come to this conclusion ("Think how I can arrive to this conclusion"). PMT33 used backward thinking to plan the construction of the proof. This way of reasoning combined with the correct use of premises, intermediate proposition, and universal proposition to justify each singular proposition in the flow-chart, as visualized in **Figure 12**, indicates that PMT33 had full understanding of the structure of proof and ability to construct the proof after the intervention.

DISCUSSION

In this study, we conducted an intervention-based study aimed at enhancing students' understanding of the structure of proof. Our research offers knowledge about the development of students' understanding of the structure of Euclidean proof and how it can be supported, which can be used as a guide for teachers to develop a more detailed discourse (Ivars et al., 2018). Overall, we found that the intervention supported not only PMTs' understanding of the structure of proof but also their performance on proof construction.

Specifically, our quantitative data of pre- and post-test indicated that the intervention indeed supported PMTs' structural understanding of proof. After the intervention 50% of PMTs were at the highest level of structural understanding (as opposed to 0% before intervention), indicating that they see a proof as a whole including the proof elements and their connections. Evidence from students' answers to tasks, in particular the percentages of PMTs who reached specific components of understanding of the structure of proof targeted by each task, also showed that the designed tasks of the intervention (i.e., Tasks T1, T2, T3, T4, T5, and T6) supported PMTs' understanding of geometry proof. An additional 25% of PMTs reached the partial-structural relational level in the post-test. 12 of these 15 PMTs failed to use correct universal propositions to justify all singular propositions (i.e., lack of understanding of universal instantiation) and the others (three of 15 PMTs) lacked understanding of hypothetical syllogism, because they used circular reasoning. Also, during the intervention, we saw that the number of students understanding hypothetical syllogism was higher than the number understanding universal instantiation, showing that the identification of correct universal propositions such as theorems, axioms, or definitions (i.e., understanding universal instantiation) was more difficult than understanding hypothetical syllogism. Our interpretation is that tasks T2b, T3b, and T4b assigning PMTs to construct multiple flow-chart proofs and the whole class discussions about these flow-charts helped PMTs avoid circular reasoning. This interpretation aligns with findings by Inagaki et al. (1998).

The intervention also supported PMTs' performance of proof construction. 50% of PMTs were at the holistic level in the post-test being able to construct a complete proof, consisting of all premises, intermediate propositions and conclusion, with correct universal propositions, and all statements connected logically from premises to conclusion. This PMTs' ability to construct correct proofs aligned with Miyazaki et al. (2017) who hypothesized that students who reach the holistic level are able not only to reconstruct previously taught proofs, but also to plan and construct their own proofs. A notable finding was that the post-test percentage was lower (50%) than PMTs' performance on task T8 (85%), which was a similar proof construction problem. Some issues might explain this difference. Firstly, PMTs answered task T8 immediately after they discussed task T7 (i.e., proof construction problem with proof planning) about how to construct a proof. This fresh experience might have helped them to solve task T8 as opposed to the post-test problem a week after the last meeting. This issue raises questions about the retention of students' performance. With our present data we cannot draw conclusions on the sustainability of students' performance, and further research is needed to gain more insight. Secondly, the limited time allotted to students to complete the post-test might also contribute to the lower student's performance. For instance, after the post-test, PMT51 said that the time was not enough to complete the proof.

While related studies (e.g., Miyazaki et al., 2017) proposed a theoretical framework of understanding of the structure of proof, we reported in this paper that students progressed through different routes along the levels of understanding. We identified two ways of progression in reaching the holistic level of understanding. One is that students start from the pre-structural and go to the partial-structural elemental sub-level, then reach the holistic level via firstly understanding universal instantiation then hypothetical syllogism. A second path is similar, but the difference is that students reach the holistic level via firstly understanding hypothetical syllogism and then universal instantiation. Besides the two different routes along the levels of understanding, the progression of PMTs' understanding of the structure of proof occurred along the three levels of understanding, following the framework by Miyazaki et al. (2017).

Reflecting on our study we became aware of some limitations. The first one is that our learning trajectory was designed for a specific group of students whose initial understanding of geometric terms, symbols, its definitions and properties (e.g., axioms and basic theorem) was limited to the concepts of congruent triangles. The proof in this context was limited to a relatively simple, direct proof of a geometric proposition in the form of an implication or *if, then* statement, which is high school level content in other countries. Some other characteristics of our intervention could have affected our conclusions, particularly the use of GeoGebra Geometry application in the first three meetings for supporting students' understanding of geometric terms, definitions, basic axioms, and theorems of congruency. This might have influenced PMTs' understanding of how to instantiate the definitions, axioms, and theorems, for instance, in order to deduce a conclusion from given assumption(s). Additionally, conjecturing through constructing and dragging by using GeoGebra might have affected students' understanding of a conditional statement, particularly the logical status of premises and conclusions (Anwar et al., 2022; Baccaglini-Frank & Mariotti, 2010). Further, we are aware that the statements in this paper were translations of students' utterances from Bahasa Indonesia into English, and those translations might not fully represent the meaning of the Indonesian words to the reader.

CONCLUDING REMARKS

Our findings showed that the use of the flow-chart proof format in our intervention enabled PMTs to see the components of proof and their relational connections. We introduced flow-chart proofs involving one or two singular propositions before students worked with more singular propositions. The positions and representations of premises and singular propositions in the flow-chart (i.e., rectangle, rounded rectangle, and connecting arrows) helped students understand the structure of proof, as suggested by Cirillo and Herbst (2011), elaborated by Miyazaki et al. (2017) and confirmed by our findings.

The tasks of reading and constructing flow-chart proofs and rewriting flow-chart proofs in other formats supported PMTs in constructing more formal proof formats. Therefore, we suggest that a flow-chart proof can be introduced to develop the use of other, more formal formats like paragraph and two-column proof, which are more commonly used in presenting a proof, specifically in Indonesian geometry classes. In addition, findings of our previous study (Anwar et al., 2021) bore out that the use of flow-chart proof together with paragraph and two-column proof improves students' RCGP.

Working with tasks designed to encourage PMTs to choose premises, intermediate propositions and conclusions in order to complete and create flow-charts, stimulated their ability to think forward and backward in constructing geometry proof, particularly proof in the form of an *if*, *then* statement. This finding justified Miyazaki et al.'s (2015) claim that flow-chart proof with open problems enhances students' ability to think forward and backward interactively. This forward and backward thinking helped PMTs to plan for a proof that precedes its construction, particularly determine the intermediate propositions by connecting premises to conclusion. This was particularly true for our students who did not have experience in proof construction, although the instructional emphasis on understanding the structure of proof certainly played a role as well.

Our learning trajectory offers insights that might inform curriculum design and instructional development for effective teaching of deductive proof, particularly in an early stage of learning deductive proof in geometry. Specifically, the levels of understanding of the structure of proof and their progression with special attention to the two independent aspects of relational understanding (i.e., universal instantiation and hypothetical syllogism) may help in this respect. Also, our data showed various instances of students' circular reasoning. The manifestation of circular reasoning in class provides an opportunity for teachers to consider with their students what a proof is, what role premises, intermediate propositions and conclusion play, and how the hypothetical syllogism connects them. Last but not least, the introduction of flow-chart proof format, particularly in the context of a direct proof of an *if*, *then* proposition, may help students' understanding of the structural relationships of proof, as a preparation to learn other, more formal proof formats (i.e., paragraph and two-column proof).

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: This study was supported by the Islamic Development Bank (IsDB) Project 4 in 1, Ministry of Education and Culture (MoEC), through Award IDN-1008.

Ethical statement: Authors stated that the study was approved by the Research Ethics Committee (CETO) of the Faculty of Arts, University of Groningen on 24 September 2019 with the approval code 65186319.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Antonini, S., & Mariotti, M. A. (2010). Abduction and the explanation of anomalies: The case of proof by contradiction. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6th Conference of European Research in Mathematics Education* (pp. 322-331). PME.

Anwar, L., Mali, A., & Goedhart, M. (2022). Formulating a conjecture through an identification of robust invariants with a dynamic geometry system. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2144517

Anwar, L., Mali, A., & Goedhart, M. J. (2021). The effect of proof format on reading comprehension of geometry proof: The case of Indonesian prospective mathematics teachers. *EURASIA Journal of Mathematics, Science and Technology Education*, 17(4), 1-15. https://doi.org/10.29333/EJMSTE/10782

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225-253. https://doi.org/10.1007/s10758-010-9169-3

Bakker, A. (2018). *Design research in education: A practical guide for early career researchers*. Routledge. https://doi.org/10.4324/9780203701010

Cirillo, M., & Herbst, P. G. (2012). Moving toward more authentic proof practices in geometry. *The Mathematics Educator, 21*(2), 11-33.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Examining the role of logic in teaching proof. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: New ICMI study series* (pp. 369-389). Springer. https://doi.org/10.1007/978-94-007-2129-6_16

Heinze, A., Cheng, Y. H., Ufer, S., Lin, F. L., & Reiss, K. M. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: Teaching experiments in Taiwan and Germany. *ZDM-International Journal on Mathematics Education (A), 40*(3), 443-453. https://doi.org/10.1007/s11858-008-0092-1

Inagaki, K., Hatano, G., & Morita, E. (1998). Construction of mathematical knowledge through whole-class discussion. *Learning and Instruction, 8*(6), 503-526. https://doi.org/10.1016/S0959-4752(98)00032-2

Ivars, P., Fernández, C., Llinares, S., & Choy, B. H. (2018). Enhancing noticing: Using a hypothetical learning trajectory to improve pre-service primary teachers' professional discourse. *EURASIA Journal of Mathematics, Science and Technology Education, 14*(11), em1599. https://doi.org/10.29333/ejmste/93421

Mchugh, M. L. (2012). Interrater reliability: The kappa statistic. *Biochemia Medica, 22*(3), 276-282. https://doi.org/10.11613/BM.2012.031

Mckee, K., Savic, M., Selden, J., & Selden, A. (2010). Making actions in the proving process explicit, visible, and "reflectable." In S. Brown (Ed.), *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*.

Mckenney, S., & Reeves, T. C. (2012). *Conducting educational design research*. Routledge. https://doi.org/10.4324/9780203818183

Mills, G. E. (2014). *Action research: A guide for the teacher researcher*. Prentice Hall Columbus.

Miyazaki, M., Fujita, T., & Jones, K. (2012). Introducing the structure of proof in lower secondary school geometry: A learning progression based on flow-chart proving. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 2858-2867). ICMI.

Miyazaki, M., Fujita, T., & Jones, K. (2015). Flow-chart proofs with open problems as scaffolds for learning about geometrical proofs. *ZDM-Mathematics Education, 47*(7), 1211-1224. https://doi.org/10.1007/s11858-015-0712-5

Miyazaki, M., Fujita, T., & Jones, K. (2017). Students' understanding of the structure of deductive proof. *Educational Studies in Mathematics, 94*(2), 223-239. https://doi.org/10.1007/s10649-016-9720-9

Plomp, T. (2013). Educational design research: An introduction. In T. Plomp & N. Nieveen (Eds.), *Educational design research* (p. 204). SLO. https://doi.org/10.1007/978-1-4614-3185-5_11

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM-Mathematics Education, 47*(6), 877-891. https://doi.org/10.1007/s11858-015-0722-3

Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: New ICMI study series* (pp. 391-420). Springer. https://doi.org/10.1007/978-94-007-2129-6_17

Selden, A., & Selden, J. (2017). A comparison of proof comprehension, proof construction, proof validation and proof evaluation. In R. Göller, R. Biehler, R. Hochmuth, & H. G. Rück (Eds.), *Proceedings of the Conference on Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 339-345). KHDM.

Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS, 28*(1), 31-45. https://doi.org/10.1080/10511970.2017.1355858

Stavrou, G. S. (2014). Common errors and misconceptions in mathematical proving by education undergraduates. *IUMPST: The Journal, 1*, 1-8.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237-266). National Council of Teachers of Mathematics.

van Engen, H. (1970). Strategies of proof in secondary mathematics. *The Mathematics Teacher, 63*(8), 637-645. https://doi.org/10.5951/MT.63.8.0637

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119. https://doi.org/10.1023/A:1015535614355

Weber, K. (2004). A framework for describing the processes that undergraduates use to construct proofs. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Annual Meeting of the International Group for the Psychology of Mathematics Education* (pp. 425-432). PME.

Yang, K. L., & Lin, F. L. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics, 67*(1), 59-76. https://doi.org/10.1007/s10649-007-9080-6

APPENDIX A

Task 1

In **Figure A1**, we know $\angle DAB \cong \angle DBA$ and $\angle DAC \cong \angle DBC$, find all possible conclusions that can be derived from the given statements. Complete the following flow-chart or construct your own flow-charts to visualize the connection between the premise(s)/given statement(s) and conclusion.

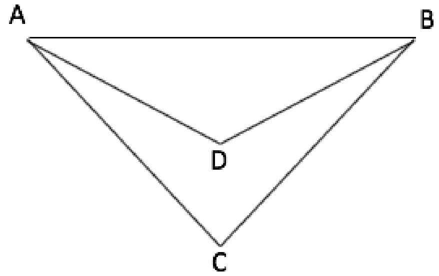
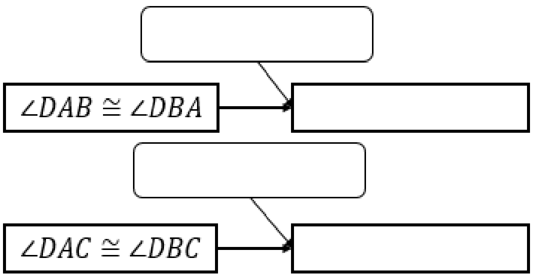
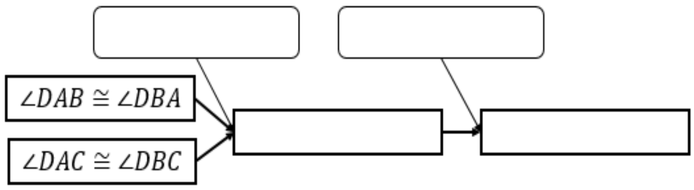


Figure A1. Example-A1 (prepared by Anwar)

Flow-chart version 1:



Flow-chart version 2:



Note. Rectangle: A conclusion derived from previous statement. Rounded rectangle: A reason(s) justifying the conclusion (definition or axioms).

Task 2

In **Figure A2**, we know $\overline{AO} \cong \overline{BO}$. We want to make ΔACO and ΔBDO congruent. Which angles and sides should be congruent and what condition (axiom/theorem) of congruent triangles should be used? Complete the flow-chart!

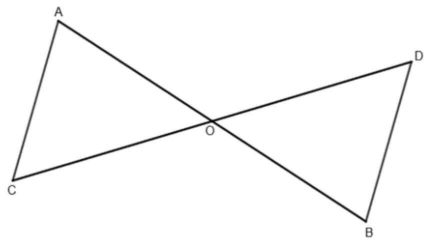
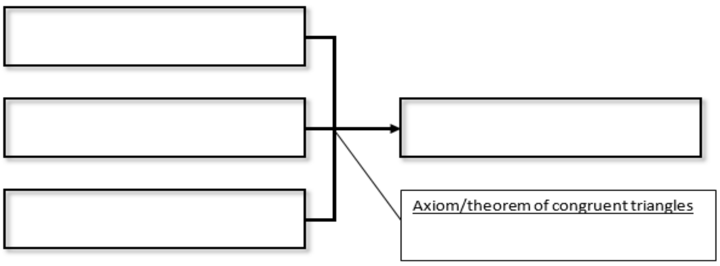


Figure A2. Example-A2 (prepared by Anwar)



Note. You may find/create more than one complete flow-charts.

Task 3

In **Figure A3**, it is already shown that $\overline{OA} \cong \overline{OC}$. Starting from that, you will show $\overline{AB} \cong \overline{CD}$ by showing two triangles in the diagram are congruent. What else do you need to add to draw the conclusion? What type of condition of congruence do you use in there?

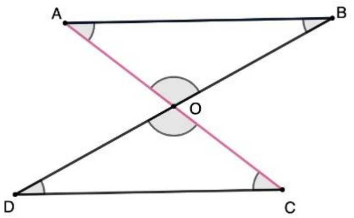
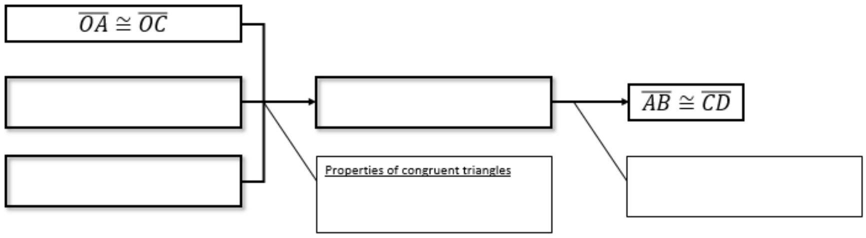


Figure A3. Example-A3 (prepared by Anwar)

Complete the following flow-chart!



Note. You may find/create more than one complete flow-charts.

Task 4

In **Figure A4**, we would like to prove $\overline{BE} \cong \overline{CD}$ by using congruent triangles. What do we need to show this, and what conditions of congruent triangles can be used? Complete the flow-chart!

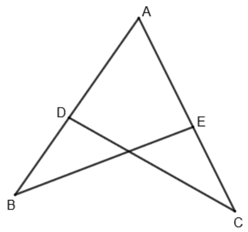
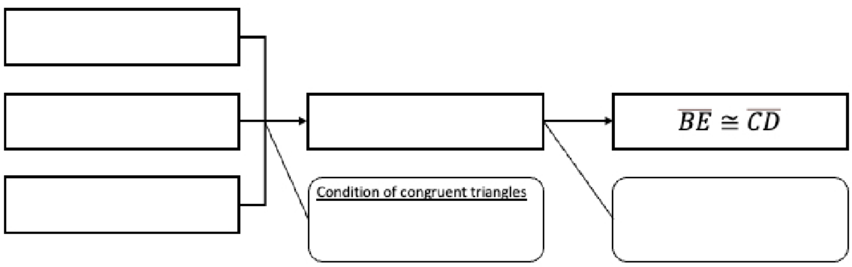


Figure A4. Example-A4 (prepared by Anwar)



Note. You may find/create more than one complete flow-charts.

Task 5

In **Figure A5**, $\overline{AB}$ and $\overline{CD}$ intersect at the point $M, \overline{AM} \cong \overline{BM}$ and $\overline{CM} \cong \overline{DM}$; then, must $\angle MAC$ and $\angle MBD$ be congruent?

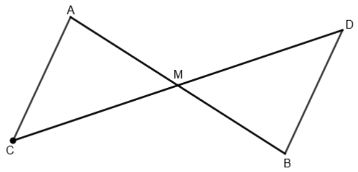


Figure A5. Example-A5 (prepared by Anwar)

To this problem, Jaka gives the following flow-chart proofs:

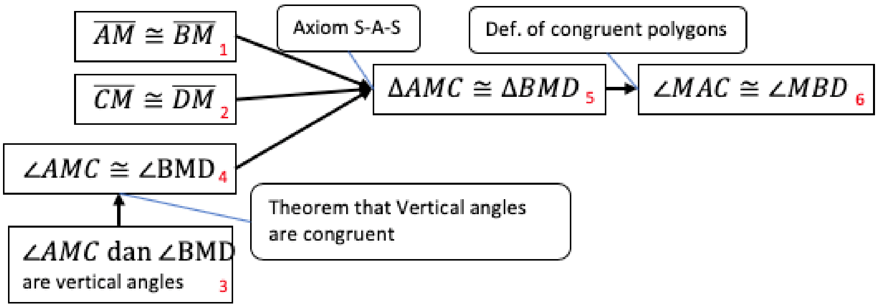


Figure A6. Flow-chart proof (prepared by Anwar)

Answer the following on the basis of this question and the proof process.

1. Label $\angle AMC$ of this figure as 1 and $\angle MAC$ of this figure as 2.
2. Do you agree that $\angle AMC \cong \angle BMD$? Explain why or why not?
3. If ΔAMC and ΔBMD are congruent, what is the corresponding angle of$\angle MAC$?
4. Besides the known conditions ($\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM} \cong \overline{BM}, \overline{CM} \cong \overline{DM}$), which conditions can be directly applied?
5. If someone suggests that the proof process of flow-chart 1, 2, 4, 3, 5, 6 is correct after box 3 and 4 are interchanged, would you agree with his or her opinion?
6. If someone suggests that the proof process of line 1, 2, 3, 5, 4, 6 is correct after box 4 and 5 are interchanged, would you agree with his or her opinion?
7. Which properties (axioms, theorems) are applied in this proof?
8. On the basis of the question (must $\angle MAC$ and $\angle MBD$ be congruent?) and the Jaka's proof,
   a. What conditions are necessarily used?
   b. What is derived from this proof?
   c. Which axioms/theorems/definitions are applied in this proof?
9. From this proof process or flow-chart proof, it firstly derives an *important result* from $\overline{AM} \cong \overline{BM}, \overline{CM} \cong \overline{DM}$ and other conditions, and then derives a *conclusion*.
   a. What is this *important result*?
   b. What is this *conclusion*?
10. Which statements can be validated from this proof?
11. Do you agree that this proof process is correct? Why or why not?
12. Statement A: if $\overline{AB}$ and $\overline{CD}$ intersect at the point $M, \overline{AM} \cong \overline{BM}, \overline{CM} \cong \overline{DM}$, then $\angle MAC \cong \angle MBD$.
    a. Do you agree that this proof process can prove that statement A is always correct?
    b. Do you agree that this proof process can prove that statement A is sometimes correct and sometimes incorrect?
13. Write the paragraph or two-column proof by reference to the flow-chart proof.

Answer the following questions on the basis of what you know.

14. If a quadrilateral $PURV$ has two diagonals $\overline{PR}$ and $\overline{UV}$, and Q is the midpoint of both $\overline{PR}$ and $\overline{UV}$, then is $\angle RPV \cong \angle PRU$ correct?
15. If $\overline{PR}$ and $\overline{UV}$ intersect at a point Q, and Q is the midpoint of both $\overline{PR}$ and $\overline{UV}$, which conclusions can be derived?
16. If $\overline{XY} \cong \overline{YZ}, \overline{MY} \cong \overline{YN}$ and $\angle XYM \cong \angle ZYN$, then are $\angle YZM$ and $\angle YMN$ congruent?

Task 6

As shown in **Figure A7**, L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{AB}$ at D, and intersects $\overline{BC}$ at M; and $\overline{DA} \cong \overline{DB}$; must $\angle DCA$ and $\angle DAC$ be equal?

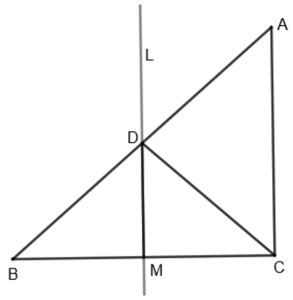


Figure A7. Example-A7 (prepared by Anwar)

To this problem, Iwan gives the following two forms of proof:

A paragraph proof

As shown in **Figure A7**,

Since L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{BC}$ at M (Line 1)

$m\angle BMD = m\angle CMD = 90$ and $\overline{BM} \cong \overline{CM}$ (Line 2)

And $\overline{DM} \cong \overline{DM}$ (axiom of reflective of congruency) (Line 3)

$\therefore \Delta BMD \cong \Delta CMD$ (axiom S-A-S) (Line 4)

$\therefore \overline{DB} \cong \overline{DC}$ (def. of congruent polygons, corresponding sides) (Line 5)

And $\overline{DA} \cong \overline{DB}$ (Line 6)

From Line 5 and Line 6 → $\overline{DA} \cong \overline{DC}$ (Line 7)

Because $\overline{DA} \cong \overline{DC}, \angle DCA \cong \angle DAC$ (Line 8)

Answer the following in the basis of this question and the proof process:

1. Do you agree that $\overline{BM} \cong \overline{CM}$? Explain why or why not?
2. Label $\angle BMD$ in this figure as 1 and $\angle CMD$ as 2.
3. If ΔBMD and ΔCMD are congruent, what is the corresponding side of $\overline{DB}$?
4. Besides the known conditions (the perpendicular bisector of $\overline{BC}$, intersects $\overline{AB}$ at D, and intersects $\overline{BC}$ at M; and $\overline{DA} \cong \overline{DB}$), which conditions can be directly applied without any explanation?
5. If someone suggests that the proof process of line 1, 2, 4, 3, 5, 6, 7 and 8 is correct after line 3 and 4 are interchanged, would you agree with his or her opinion?
6. If someone suggests that the proof process of line 6, 1, 2, 4, 3, 5, 7 and 8 is correct after line 6 has been changed, would you agree with his or her opinion?
7. Which properties (Axioms, theorems) apply in this proof?
8. On the basis of the question and the proof,
   a. Which premises are necessarily used?
   b. What final conclusion is derived from these premises?
9. Which statements can be validated from this proof?
10. In this proof process, an ***important result*** is first derived from the condition is that L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{BC}$ at M and other conditions.

a. What is this ***important result***?

b. According to this important result (10-a) and $\overline{DA} \cong \overline{DB}$, one ***reason/condition*** can be derived to confirm $\angle DCA \cong \angle DAC$. What is this a ***reason/condition***?

11. Choose the correct statements.
12. Do you agree that this proof process is correct?
13. Statement A: if, L, the perpendicular bisector of $\overline{BC}$, intersects $\overline{AB}$ at D, and intersects $\overline{BC}$ at M; and $\overline{DA} \cong \overline{DB}$; then $\angle DCA$ and $\angle DAC$ must be equal.
    a. Do you agree that this proof process can prove that Statement A is always correct?
    b. Do you agree that this proof process can prove that Statement A is sometimes correct and sometimes incorrect?
14. Refine the paragraph proof by placing them into a flow-chart proof format.

Answer the following questions on the basis of what you know.

15. There is a circle with center point P, radius $\overline{PS}$ and $\overline{PQ}$. If T is the midpoint of $\overline{PQ}$, $\overline{ST} \perp \overline{PQ}$, and S is the midpoint of $\overline{PR}$, is ΔRSQ an isosceles triangle?
16. There are three points P, Q and R. If S is the midpoint of $\overline{PQ}$ and $\overline{ST} \perp \overline{PQ}$, what conclusions can be derived?
17. If D is the midpoint of $\overline{AE}$, and $\overline{BD}$ and $\overline{AE}$ are perpendicular to each other, and $\overline{AB} \cong \overline{BC}$, then $m\angle AEC = 90$. Is this correct?

Task 7

Budi is trying to solve the following problem.

Problem

In **Figure A8**, given points A, B, C and D are on OX and OY of ΔXOY, so that $\overline{OA} \cong \overline{OB}$ and $\overline{OC} \cong \overline{OD}$. When A and D, and B and C are connected, prove $\overline{AD} \cong \overline{BC}$.

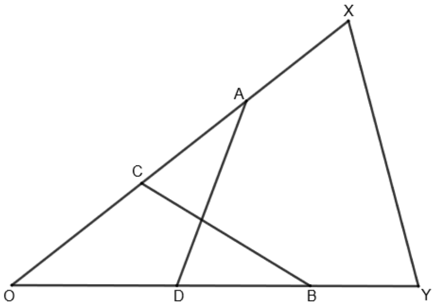


Figure A8. Example-A8 (prepared by Anwar)

Budi described his plan to prove it as follows.

Budi's memo

1. To prove $\overline{AD} \cong \overline{BC}$, it is enough to show $\Delta AOD \cong \Delta BOC$.
2. I see ΔAOD and ΔBOC of **Figure A9**. More clearly, I can divide it into two parts and show what is assumed, as follows.

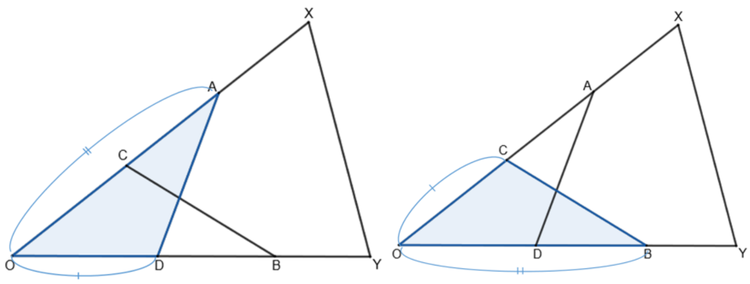


Figure A9. Example-A9 (prepared by Anwar)

3. Based on #2, I think I can prove $\Delta AOD \cong \Delta BOC$.

1. Which property (axioms) should be used to say 'in order to prove $\overline{AD} \cong \overline{BC}$, it is enough to show $\Delta AOD \cong \Delta BOC$' as seen in #1 of Budi's memo? Choose from a)-d).
   a. In a congruent polygon, corresponding sides and angles are equal.
   b. In a congruent polygon, two corresponding sides and corresponding included angles are equal.
   c. In a congruent polygon, two corresponding angles and corresponding included sides are equal.

In a congruent polygon, corresponding sides are congruent.

2. Prove $\overline{AD} \cong \overline{BC}$ of the problem by making the flow-chart proof.
3. Write the proof into two modes of proof representation (a paragraph proof and two-column proof) by reference to the flow-chart proof.

Task 8

In **Figure A10**, if $\angle PMN \cong \angle PNM$ and $\angle QMN \cong \angle QNM$, then is $\overline{PQ}$ a bisector of an angle $\angle MPN$? If YES, write a flow-chart proof of the statement! Then, write the paragraph proof and two-column proof by reference to the flow-chart proof.

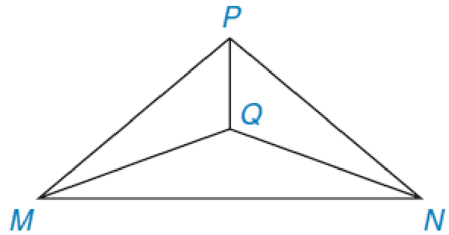


Figure A10. Example-A10 (prepared by Anwar)

APPENDIX B

Interview Task

Problem

Please construct a geometric figure by following the step-by-step construction below:

1. Draw a line segment AB
2. Construct a midpoint C of a line segment AB
3. Draw a line DC
4. Construct a circle with Center C and radius CD
5. Construct intersect point between the circle (#4) and line CD
6. Draw a line segment ED
7. Construct a polygon ADBE

Please construct a conjecture based on the constructed geometric figure. Write the conjecture in the form of an implication (if, then statement) including the geometric figure, then prove the statement (proposition).

Post-Test

Given a statement: "In **Figure B1**, If $\overleftrightarrow{AD}$ is a bisector of a line segment $\overline{BC}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$, and $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$, then $\overleftrightarrow{BC}$ is a bisector of $\overline{AD}$." Is the statement TRUE? If it is TRUE, Prove it!

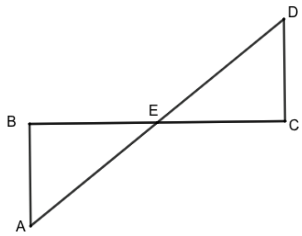


Figure B1. Example-B1 (prepared by Anwar)

Table B1. Rubric of students' level of understanding of structure of proof for ***pre-test***, ***tasks,*** and ***post-test***

Levels		Description
Pre-structural (0)		• No answer/proof or no flow-chart proof • No mention: premises, intermediate conclusion, conclusion & reason (universal propositions) • Most of premises or intermediate conclusions or conclusion are missing or incorrectly stated or inadequate, for instance incorrect symbols
Partial-structural elemental (1)		• Most of premises or intermediate conclusions or conclusion are correctly stated or adequate including the symbols • Reference to prior theorems and/or axioms and /or definitions as the reason to justify is generally lacking or stated inaccurately or use of inappropriate reason (theorems and/or axioms and /or definitions) • All statements (singular propositions) in the proof are not connected logically from premises to the stated conclusion or acceptance of circular reasoning (e.g. use a conclusion as one of premises)
Partial-structural relational (2)	Universal instantiation	• All of the premises or intermediate conclusions or conclusion are correctly stated or adequate including correct symbols • Reference to correct or appropriate prior theorems and/or axioms and /or definitions or use of appropriate reason (theorems and/or axioms and /or definitions) • All statements (singular propositions) in the proof are not connected logically from premises to the stated conclusion or acceptance of circular reasoning (e.g. use a conclusion as one of premises)
	Hypothetical syllogism	• All of the premises or intermediate conclusions or conclusion are correctly stated or adequate, including the symbols • Reference to prior theorems and/or axioms and /or definitions is generally lacking or stated inaccurately • All statements (singular propositions) in the proof are connected logically from premises to the stated conclusion or no acceptance of circular reasoning (e.g. no use of a conclusion as one of premises)
Holistic-structural (3)		• All of the premises or intermediate conclusions or conclusion are correctly stated or adequate, including the symbols • Reference to correct or appropriate prior theorems and/or axioms and /or definitions or use of appropriate reason (theorems and/or axioms and /or definitions) • All statements (singular propositions) in the proof are connected logically from premises to the stated conclusion or no acceptance of circular reasoning (e.g. no use of a conclusion as one of premises)