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Mathematical Abstraction in the Solving of Ill-Structured Problems by Elementary School Students in Korea

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Ill-structured problems can be regarded as one of the measures that meet recent social needs emphasizing students' abilities to solve real-life problems. This study aimed to analyze the mathematical abstraction process in solving such problems, and to identify the mathematical abstraction level ([I] Recognition of mathematical structure through perceptual abstraction, [II] Application of mathematical structure recognition through internalization, and [III] Formation of new mathematical structure recognition through interiorization) and form of the students, by applying ill-structured problem-solving activities to problem-solving learning approaches for fifth grade elementary school students. The study results showed that 2 out of 6 groups displayed the highest level [III] of mathematical abstraction, while two other groups displayed Level [I], and the other two groups showed Level [II]. Especially, the students in each group showed higher levels and forms of mathematical abstraction (from Level [I] to [II] and [III]) over the course of the problem solving process. These study results showed that the mathematical abstraction levels and forms of the students can be improved by exposure to mathematical abstraction through a problem solving learning approach using ill-structured problems.

Keywords: ill-structured problem; mathematical abstraction; elementary education; problem solving

INTRODUCTION

Ill-structured problems, where frames of reference with respect to real problems are contextualized, require learners to define the problems as well as determine the information and skills needed to solve them. They can be thought of as one of the measures that meet the recent social needs emphasizing students' ability to solve real-life problems encountered in modern society.

The Program for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) show that Korean students have difficulty solving problems that connect mathematical concepts with everyday life. Essentially, according to these organizations, Korean students possess a lower ability than students of other countries to apply mathematics to general

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circumstances. This suggests that it is necessary for them to experience complex real-world problems that are not formalized, i.e., ill-structured problems.

Ill-structured problems can also be considered as a way to improve students' mathematical thinking capacity. According to Ge & Land (2003), such non-routine problems make students associate abstract mathematical knowledge with their everyday lives. Accordingly, they are able to abstract, generalize, and format problems of their daily lives. They learn to reorganize information, the better to focus thinking that leads to new understanding, and they evaluate alternatives in order to find the most appropriate solution. Ill-structured problems with these characteristics can be used for the development of high-level mathematical thinking skills such as abstraction and reasoning.

This study gave an ill-structured mathematics problem to fifth-grade elementary school students, analyzed how they mathematically abstracted it, and looked at their level of mathematical abstraction and its characteristics. The results will contribute to a better understanding of the suggestion that learning how to solve ill-structured problems is important to mathematical education in school, and to the development of students' mathematical proficiency.

State of the literature

- Ill-structured problems are contextualized, require learners to define the problems as well as determine the information and skills needed to solve them.
- There is a problem, clearly understanding it, searching for and selecting relevant information, identifying and justifying various perspectives, organizing information, determining the most appropriate solution, and further debate and articulating of beliefs and values.
- Mathematical abstraction consists of the perceptual level, the internalization level, the interiorization level, and the second level of interiorization.

Contribution of this paper to the literature

- This paper reviews the trends and status of problem solving in ill-structured problems and mathematical abstraction.
- This study analyzed the mathematical abstraction process and identified the mathematical abstraction level by applying ill-structured problem-solving activities to problem-solving learning approaches for fifth grade elementary school students.
- This study showed that the mathematical abstraction levels and forms of the students can be improved by exposure through a problem solving learning approach using ill-structured problems.

BACKGROUND

Ill-structured problems

Ill-structured problems emerge from specific contexts. Their characteristics are as follows: first, aspects of the situation are not concrete; second, the problems are not well-defined; third, they are based on real-life situations and have openness; and finally, complex situations are presented (Chi & Glaser, 1985).

Jonassen (1997), Shin, Jonassen & McGee (2003), Palm (2008), and Torulf (2008) mentioned the properties of ill-structured problems, and Kim, Lee, Hong & Kim (2011) defined authenticity, complexity, and openness as those properties. Authenticity means that everyday life coincides with mathematics homework or problems which depict real life outside the school (Palm, 2008). To have authenticity, the problems should include the context of daily life and be relevant enough to infer an integral part of the real situation. Jonassen (1997) saw the attributes of complexity as follows: they exist uncertainly; and concepts, rules and principles are required to solve the problem, or how they are organized. Also, the relationships between concepts and rules and principles are not fixed. Jonassen also said this about openness: first, multiple evaluation criteria must exist for solving problems; second, a clear means is not presented for appropriate behavior; third, learners must express personal opinions and beliefs about the problems; fourth, it is suggested that learners judge and defend the problems. Shin, Jonassen & McGee (2003) said that the nature of openness allows students to put various interpretations on problem-solving and to justify their interpretations.

Many scholars (Ge & Land, 2003; Jonassen, 1997; Shin, Jonassen & McGee, 2003, Sinnott, 1989) proposed the processes of solving ill-structured problems, which differ from those of structured problem-solving. Sinnott (1989) proposed a model with the following steps: the process of problem space design, the choice and creation of solutions, monitoring and storage and non-cognitive factors, and think-aloud protocols. Ge & Land (2003) outlined a four-step process: problem representation, developing solutions, developing justification, and monitoring and evaluation. Jonassen (1997) and Shin, Jonassen & McGee (2003) proposed a similar process: problem representation, generating solutions, justification, and monitoring and evaluation. They have these factors in common: recognizing that there is a problem, clearly understanding it, searching for and selecting relevant information, identifying and justifying various perspectives, organizing information, determining the most appropriate solution, and further debate and articulating of beliefs and values. These have all been associated with the process (Hong, 1999).

In several studies such as Palm (2008), Ge & Land (2003), and Shin, Jonassen & McGee (2003), students were given a problem to solve. They took the 'real-world' into account while forming practical solving strategies, organized information and their thoughts clearly to lead to new understanding of the problem, and evaluated and examined the various alternatives to seek the most appropriate solution.

According to the above research, solving problems with characteristics of authenticity, complexity and openness can improve students' high-level mathematical thinking and strengthen their problem-solving skills in real life.

Mathematical abstraction

Abstraction is a process of constructing relationships between objects from a particular point of view (Van Oers & Poland, 2007). This reorganizes previously constructed mathematics vertically to reconstruct a new structure (Hershkowitz, Schwarz & Dreyfus, 2001, p. 202). Abstraction allows the human psyche to select a collection of behaviors or mental items and register them in the memory (Battista, 1999). The concept of mathematical abstraction proposed by Ohlsson & Lehtinen (1997) is that it recognizes commonalities which are isolated from a number of concrete examples and generalizes things. Dienes (1971, 1978) described abstraction as focusing on common properties and proposed that it was one of the keys to learning mathematics and the main characteristics of mathematics. Van Oers & Poland (2007) said that it is the dialectical process between objects, which are specifically given, and their abstract representation. Abstraction could be described by focusing on configuration of the mental model, which could be expressed as an abstract symbolical model.

With respect to abstraction in the study of mathematics, researchers divided procedure into multiple levels or stages. Dreyfus, Hershcowitz & Schwarz (2001) segregated mathematical abstraction procedure into three stages: the need for a new structure, the construction of a new abstract entity, and the consolidation of the abstract entity/structure. Piaget (1972, 2001) proposed empirical abstraction, pseudo-empirical abstraction, and reflective abstraction. Recognizing, building-with and constructing were presented by Hershcowitz, Schwarz & Dreyfus (2001). Spatial structuring, a mental model and a scheme were presented as the procedure of mathematical abstraction by Battista (1999) and Battista & Clements (1996). Battista (1999) also presented Levels [I] to [IV] of mathematical abstraction; the perceptual level, the internalization level, the interiorization level, and the second level of interiorization.

Regarding abstraction in mathematical learning, Battista (2007) suggested a four-step process. Students in Step One show incomplete abstraction for standard unit repetition, which is associated with Level [II] – the internalization level –

advanced by Battista (1999). Steps Two and Three are related to the Battista's Level [III] of mathematical abstraction – the interiorization level. Students in Step Two represent interiorized and adjusted units and students in Step Three represent the interiorized and repeated structure. Students in Step Four symbolize the numbers and use them, which belongs to the second level of interiorization of mathematical abstraction.

Children learn things based on manipulation and observation of specific objects. Therefore, it was thought to be difficult for children to apply abstraction to learning. However, according to several studies such as Kohnstamm (1948), Kohnstamm (1967) and Sheppard (1973), abstraction can be taught to younger children than the age group proposed by Piaget (Van Oers & Poland, 2007). They also argued that younger children were able to think abstractly and that their abstract thinking skills could be developed by learning activities, including appropriate schematizing activities (Davydov, 1990; Egan, 2002; Van Oers & Poland, 2007).

Some studies, such as Kato, Kamii, Ozaki & Nagahiro (2002), Davydov (1990), Van Oers & Poland (2007), can be interpreted in the same context. Kato, Kamii, Ozaki and Nagahiro (2002) individually interviewed sixty Japanese children ranging from ages 3 to 7 years old and analyzed three levels of mathematical abstractions through problem solving in the given tasks. The three tasks given to each child in the interview included: ① Representation-of-Groups-of-Objects Task, ② Conservation-of-Numbers Task, and ③ Writing-of-Numerals Task. In addition, Davydov (1972, 1990) was able to empirically demonstrate that children aged 8 to 10 years old were able to think abstractly whether or not they were taught theoretical models to analyze the world.

These studies show the following results: students represent the level of mathematical abstraction in the process of solving mathematical problems. Their mathematical abstraction level – their ability to think abstractly – can be taught and developed by allowing them to experience theoretical models, such as schematizing activities, which structure the specific experience. Therefore, it is necessary for mathematics education in school to provide students with an opportunity to experience theoretical models so that they can improve their mathematical abstraction level. One suggestion is that they learn how to solve ill-structured problems.

RESEARCH METHODS

Participants

The objective of the study is to analyze the mathematical abstraction abilities shown by fifth-graders when trying to solve an ill-structured problem. To this end, the study conducted a test on a selected class at an elementary school in Seoul Metropolitan City. The test participants were a group of 20 fifth-graders: 9 boys and 11 girls. They were formed into six groups of three and four members each and participated in the solving of an ill-structured problem.

Research design

Design of an ill-structured problem

As the questions consist of ill-structured problems which do not materialize in the real world based on authenticity, complexity and openness, learners were required to interpret problems on their own, seek solutions and identify a proper solution. In addition, the questions made it necessary for learners to express the problems, present solutions, justify their thinking, and review and evaluate various solutions (Hong, 1999). This plays an important role in enhancing their high-level mathematical thinking capacity and real-life problem-solving ability. These questions, if they are applied to mathematics education at elementary schools, can offer opportunities for students to try to solve a problem in a real-life context and to help improve their problem-solving capacities.

This study referred to the problem-solving process suggested by Jonassen (1997), Ge & Land (2003), Shin, Jonassen & McGee (2003), and Hong (1999), and designed the questions for the second semester of fifth grade to ensure that it can be applied to an elementary school mathematics education. The flowchart for the problem-solving learning process is:

[understanding problem] → [seeking solution] → [applying].

Understanding problem: present problems during class debate and understand them (plenary class debate). Seeking solution: plan for solution seeking, identify an appropriate solution, and induce a mathematical conclusion based on formalization and abstraction (subgroup activities & presentations). Applying: applying the solution to the problem (plenary class debate).

The study focused on analysis of the abstraction process, which appears during the problem-solving process, by applying the above procedure. To design an exercise for the participants, the study confirmed the education goals by establishing the education objectives related to solution-seeking methods among patterns and problem-solving domain, and selected learning experiences which reflected a real-world context. The problem-solving teaching and learning curricula were designed by organizing the selected learning experiences appropriate to the problem-solving learning procedure.

The problem applied in this study was based on related literatures on ill-structured problems, Korea 2007 revised mathematics curricula, NCTM, domestic and international textbooks. We developed a problem that can be given to fifth-graders in Korea to solve in the context of the real world (see Figure 1). It was reviewed and verified in consultation with a mathematics scholar, three elementary school teachers currently enrolled in doctorate and master's degree studies in elementary school mathematics education, and an elementary school teacher with 21 years' experience. It was first given to a class of sixth-graders as part of a preliminary study before being finally modified.

The architectural design and construction company ('A') received an order from a client ("B"), as follows:

I want to build a new house. Here are the design specifications of the home that I want. First, I want the house to be a rectangular shape, 20m in width and 10m in length, with a floor area of 200m^2. The house should have five bedrooms, two bathrooms, a living room, a kitchen and dining area, a utility room and a balcony. The master bedroom is to have an area of 20m^2 with direct access to a bathroom and dressing room. The other rooms have area sizes of 12~16 m^2. The living room and the kitchen and dining area have to be situated at the center. The living room must be 44m^2, and the kitchen and dining area is to be 24m^2. The utility room has to be connected to the kitchen. The master bedroom must be in the farthest corner from the front door. If you send me a couple of drawings which meet these requirements, I will choose one after reviewing them.

Company A notified several teams of Client B's needs and asked them to submit designs. From among the submitted proposals, Company A will choose the design that best meets Client B's needs. Each subject group will become a team of Company A and will try to develop the best design.

Figure 1. Context of ill-structured problem: Architectural drawing

Design of mathematical abstraction process analysis

This study aims to analyze the mathematical abstraction processes demonstrated by the students during problem-solving learning. The study assessed students' learning activities and the observations of researchers. To assess the learning activity of a student, group activity reports and activity results during the study were analyzed. Researchers observed, recorded and analyzed the activities of learners.

Group activity results and observations of their learning activities were independently reviewed, in accordance with the analysis standards for mathematical abstraction levels and forms, by the three elementary school teachers described in Section 3.2.1 and by researchers. They were then analyzed based on the results agreed upon after consultation on the reviewed contents.

After assessing the group activity reports, the activity results and the observations made the group which showed the highest level of mathematical abstraction was analyzed.

This study used modified analysis standards based on mathematical abstraction levels and contents from Battista (2007), Piaget (1972, 2001), Gray & Tall (2007), Hershkowitz, Schwarz & Dreyfus (2001), Ozmantar & Mnaghan (2007), Steffe & Cobb (1988) appropriate to problem-solving learning. The analytical framework for mathematical abstraction levels and forms are as shown in Table 1.

The first level of mathematical abstraction, Level [I], is mathematical structure recognition through perceptual abstraction, in which a learner recognizes the need for a mathematical structure to solve a given problem and understands the problem by applying it to mathematics. At this level, students recognize the need for

Table 1. Analysis standards for mathematical abstraction levels and forms

Mathematical Abstraction Level	**Analysis Contents**	**Mathematical Abstraction Forms**
[I] Recognition of Mathematical Structure through Perceptual Abstraction	・Does a student recognize the need for a mathematical structure to solve the given problem situation, and does he or she apply it to mathematics? ・Is a student aware of the previously learned mathematical structure (including mathematical knowledge, concepts and principles)? Can he or she recognize and identify common attributes related to the problem? ・Does a student recognize mathematical attributes involved in the problem based his or her own experiences and intuition and by utilizing physical objects?	-Perceptual abstraction -Recognition of need for mathematical structure -Recognition of mathematical structure and attributes
[II] Application of Mathematical Structure Through Internalization	・Does a student simplify the problem into a concise form? Does he or she express it using a mathematical relation and structure by formalizing informal concepts? ・Does a student actively introduce, utilize and apply the previously learned mathematical structure (including mathematical knowledge, concepts and principles) to solve the problem?	- Internalization (Simplification, Formalization) - Formal expression of ill-structured concept - Application and Utilization of Mathematical Structure
[III] Construction of New Mathematical Structure Through Interiorization	・Can a student solve problem by generalizing a mathematical concept included in the problem? ・Does a student form new mathematical knowledge and structure when solving the problem? Can he or she generalize it to a problem in a different real-life context? ・Does a student develop a new structure based on the previously learned mathematical structure (including mathematical knowledge, concepts and principles)?	- Interiorization - Generalization of Mathematical Structure - Vertical Reconstruction of Mathematical Structure

perceptual abstraction and express it as a form of mathematical abstraction such as a mathematical structure or attribute recognition.

Level [II] of mathematical abstraction is the application of a mathematical structure through internalization, in which a learner expresses various ill-structured parts of the given problem in a certain form, or simplifies it to a concise mathematical relation or mathematical structure. The mathematical abstraction form which a student of this level uses includes internalization, including simplification and formalization of ill-structured concepts, as well as application and utilization of mathematical structure.

The last level of mathematical abstraction, Level [III], is the development of a new mathematical structure through interiorization. A learner forms new mathematical knowledge and structure by solving problems, and generalizes it to a problem with a different real-life context. The student of this level shows mathematical abstraction forms such as interiorization, generalization of a mathematical structure, and vertical reconstruction of a new mathematical structure.

RESULTS

Mathematical abstraction level and form which appear in solving an ill-structured problem

The study assessed the mathematical abstraction levels and forms shown by students, in accordance with the above analysis standards. During the lesson, they were told that their assignment was to draw on a B3 sheet an ‘architectural drawing’ that would meet all design requirements.

Students collected information about the areas, locations and shapes of each room of the house. They decided on an appropriate scale for the sheet of paper, and completed the architectural drawing. After that, they discussed their work with other groups during a presentation and looked for real-life cases to which they could apply their methods. Through this process, the six groups of students showed their mathematical abstraction levels and forms (see Table 2) in solving the ill-structured problem ‘architectural design.’

According to the analysis results on mathematical abstraction forms that each group of students displayed, Groups <1> and <3> showed Level [I], Groups <2> and

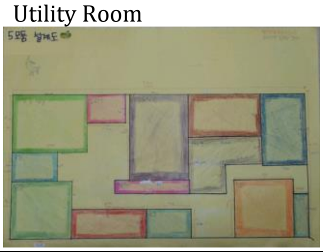
Table 2. Mathematical abstraction levels and form in the process of solving an ill-structured problem

Mathematical Abstraction		Group					
Level	**Form**	**<1>**	**<2>**	**<3>**	**<4>**	**<5>**	**<6>**
[1] Recognition of Mathematical Structure through Perceptual Abstraction	ⓐ Perceptual abstraction	○		○			○
	ⓑ Recognition of need for mathematical structure	○		○			
	ⓒ Recognition of mathematical structure and attributes	○		○	○	○	○
[2] Application of Mathematical Structure Through Internalization	ⓐ Internalization (Simplification, Formalization)		○		○	○	
	ⓑ Expressing informal concept formally					○	○
	ⓒ Application and Utilization of Mathematical Structure		○			○	○
[3] Construction of New Mathematical Structure Through Interiorization	ⓐ Interiorization				○		
	ⓑ Generalization of Mathematical Structure				○	○	
	ⓒ Vertical Reconstruction of Mathematical Structure					○	

<6> showed Level [II], and Groups <4> and <5> showed Level [III]. Table 3 shows the examples of students' mathematical abstraction response from each level/form.

The students of Groups <1> and <3> recognized that they needed a mathematical structure (a scale) to depict the floor plan of the house on a piece of paper. This appeared in the [1]-ⓑ form of mathematical abstraction. They perceived the correlation between the actual and relative lengths as well as the width and length of a plane figure in the process of scaling the house plan and its internal spaces down to B3 size. And when they tried to solve the problems related to this, the mathematical abstraction in the form of [1]-ⓒappeared. However, while these two

Table 3. Examples of students' mathematical abstraction response

Level	Form ⓐ	Form ⓑ	Form ⓒ
[1]	Calculate and write down on an A4 size paper the area, length, and distance between rooms that a client wants. If the figures are not correct, do it over on a separate A4 size paper. After completing this, write down the actual area and length. Measure the length on the map and mark it in red and blue.	Write down the area, length, and width of the room, which the client wants. Calculate the length on the map and the actual length. Addition, subtraction, multiplication, division, ratio and proportion are used to display. Display them by drawing in accordance with the blueprint. Objects on the map are displayed by measurements with a ruler.	J: Hey, should we make 5cm equal 1m? What do you guys think we should do? Hanbin: 4cm equals 4m. J: Then, we have too much room here. I: Yeah, he's right. Hanbin: No. No. I: Then, it's too short. -- Omitted-- J: Why don't we make it bigger than the one we just drew? Hanbin: Make it larger. We have a big paper so we can roughly match it.  (We recognized that a blueprint must be drawn after downsizing it to fit on a given paper size. Here, we need to use ratio and proportion concepts that we have learned before.)
[2]	Draw a rectangle, scaled to the size of the paper and include the remaining downsized objects . Record the actual and reduced lengths on the blueprint. Reduce the remaining objects down to the size of the two edges of the rectangle drawn on the blueprint.	2×2.5=5 2.5×2.5=6.25 3×1.5=4.5 5×2.5=12.5 4×2.5=10 2.5×4=10 3×2.5=7.5 2×2.5=5 3×2.5=7.5 2×2.5=5 4×2.5=10 5×2.5=12.5 5.5×2.5=13.75 8×2.5=20 4×2.5=10 5×2.5=12.5 4×2.5=10 6×2.5=15	-Veranda: 2m (width), 2.5m (depth) → 5cm, 6.26cm -Room1: 3m, 5m → 7.5cm, 12.5cm -Room2: 4m, 4m → 10cm, 10cm -Room3: 3m, 5m → 7.5cm, 12.5cm -Bed Room4: 4m, 4m → 10cm, 10cm -Dress Room: 5m, 2m → 12.5cm, 5cm
[3]		A planar figure was created using downsized proportions.	* Sequence Biggest room → Entrance → Veranda → Bathroom → Bedroom → Bedroom → Dress room → Bathroom → Bedroom → Bedroom → Kitchen & Dining Room → Utility Room 

groups of students recognized the need for mathematical knowledge and structure such as a scale, they failed to properly utilize or apply a mathematical structure or relation to problem-solving and ended up showing a form [1]-ⓐof mathematical abstraction.

The two groups failed to scale down the actual widths and lengths to the relative lengths on paper. They showed a form of perceptual abstraction as they roughly estimated the width and length of each internal space by measuring the lines already drawn with a ruler. The students thought they had to measure the dimensions of the sheet of paper and reduce the house size to fit. However, they failed to generalize it to a mathematical relation between the length and width of a B3 sheet and the floor area of the house. That is to say, they recognized the scale-down attribute but only to the extent that they simply shortened the lengths. It can be said that the students showed Level 1 mathematical abstraction. They recognized the mathematical attributes involved by utilizing their own insight and physical objects.

The students of Groups <2> and <6> solved the problem by introducing previously gained mathematical knowledge, concepts and principles. They showed Level 2 mathematical abstraction as they finished their task. The students of these two groups applied the mathematical structure such as the ratio and rate to calculate the relative lengths on an architectural drawing and draw them on the B3 sheet of paper ([2]-ⓒ). However, Group <6> drew only some parts correctly, so they incompletely solved the problem. Group <2> calculated the scale-down ratio to ensure the house plan would fit on the sheet, and estimated the relative widths and lengths of internal spaces of the house according to the scale-down ratio. But they finally did not generalize this mathematical structure to problem-solving. They failed to make an accurate assessment on a problem-solving method and could not find the most appropriate solution to the problem.

The students of Group <2> showed Level 2 mathematical abstraction in the form of [2]-ⓐand [2]-ⓒ, from understanding problems and seeking solutions to looking for other real-life cases to which they can apply them. In contrast, the students of Group <6> showed the [1]-ⓒform of mathematical abstraction in the first problem-solving process, and the [1]-ⓐ, [2]-ⓑor [2]-ⓒform in the process of calculating the relative lengths of the house and internal spaces to make the house fit on a B3 sheet.

The above four groups all showed either Level 1 or Level 2 mathematical abstraction in the process of solving an ill-structured problem. Groups <4> and <5> showed Level 3 mathematical abstraction. The students collected information, decided on a scale-down ratio to make the drawing, and calculated the relative lengths of each internal space of the house to be drawn on paper.

These two groups of students first understood and identified the problem through mathematical abstraction in the form of [1]-ⓒ. After that, as they simplified and formalized it so as to set up a problem-solving plan by scaling the actual dimensions down to the size of a B3 sheet, they showed the [2]-ⓐ form of mathematical abstraction.

After that, the students of Group <4> decided to use a scale of 1:100 through the interiorized concept of rate and ratio; that is, to scale 1m on the house floor plan as 10cm on the drawing. They calculated the relative dimensions of the internal spaces of the house and drew them, which indicated the [3]-ⓐform of Level 3 mathematical abstraction, or "interiorization."

Group <5> not only expressed a certain space in a rectangular shape in the process of drawing various internal spaces of the house, but also separated a space which was impossible to form a rectangular shape into several plane figures by vertically mathematizing a mathematical structure about the shape and area size of a plane figure involved in the given task.

During the presentations on their work by each group, Groups <4> and <5> could assess the other groups by generalizing the mathematical structure of ratio and rate to each group's problem-solving process. As they found and pointed out an error in the problem-solving method of Group <3>, they showed the [3]-ⓑ form of mathematical abstraction.

If all mathematical abstraction forms are considered, the students demonstrate mathematical abstraction as shown in Figure 2.

First, the students recognized the need for a mathematical structure to process the information, including the actual dimensions of the house and the relative dimensions on the drawing, in the process of *understanding problem*. They understood the problem by perceiving its mathematical attributes. After that, they applied a mathematical structure such as ratio and proportion in the process of *seeking solution* and found a solution by calculating the relative, scaled dimensions on the drawing. Finally, the students carried out *applying* by expanding and generalizing their own solution to the other groups' problems.

Mathematical abstraction examples: Group <5>'s activities

The students of Group <5> showed Level [I] mathematical abstraction, given that they understood the problem in the problem-understanding process of the given 'architectural drawing' task, perceived the mathematical knowledge and concept involved in the problem, and they applied mathematics to it. After that, they showed Level [II] or [III] mathematical abstraction in the process of collecting information and identifying a proper solution. As they summarized their work and evaluated other groups' work, they showed Level [III] mathematical abstraction in the process of drawing a mathematical conclusion such as mathematical concepts and principle involved in the problem-solving process.

Level [I]

The students of Group <5> showed the first level forms of mathematical abstraction in the process of understanding the architectural drawing task:

> The total floor size of 200m2 with five bedrooms, one dressing room, two bathrooms, one living room, kitchen and dining room, one utility room, one balcony: the master bedroom with direct access to bathroom and dressing room of 12~16m2, living room of 44m2, kitchen & dining area of 24m2, a balcony connected to the living room: lastly the master bedroom must be located in the farthest corner from the front door.

The students recognized the mathematical knowledge and concepts involved in the ill-structured problem and they applied mathematics to it. They used mathematical knowledge and concepts such as the elements of a rectangle (width

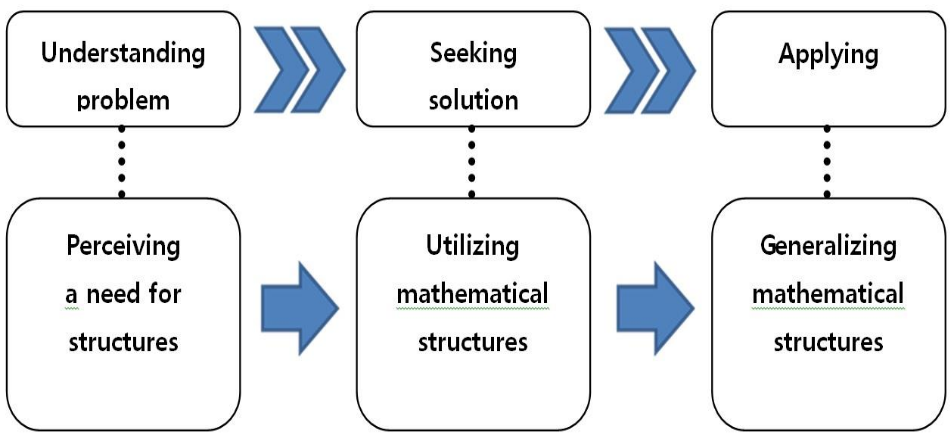


Figure 2. Mathematical abstraction in the process of solving an ill-structured problem

and length), the area of a plane figure and size units, and applied them to the shape of the house and the specified location and area of its rooms. They applied mathematics to the problem and showed the [1]-ⓒ form of mathematical abstraction.

Level [II]

The students of Group <5> generally showed Level [II] mathematical abstraction in the overall activity of calculating a suitable scale and making an architectural drawing based on the scale. First, they simplified the problem into a concise form by writing down reduce the rectangle and rooms based on the scale-down ratio on the drawing paper. They expressed the problem as a mathematical relation and structure by recording the lengths are scaled down in proportion to the reduced lengths of the two opposite sides of the rectangle. These activities showed the [2]-ⓐ form of mathematical abstraction.

In addition, the students of Group <5> estimated the reduced lengths to be drawn by multiplying the actual dimensions of each room of the house by 2.5, based on the previously calculated dimensions of the entire house. They expressed the relation between the actual and reduced dimensions of each room in a multiplication form such as '2 × 2.5 = 5, 2.5 × 2.5 = 6.25, … '. This showed the [2]-ⓑform of mathematical abstraction.

Meanwhile, the students first had to calculate the width and length of the house to decide on a suitable scale for their drawing. In the process of calculating the relative width and length of the house, they applied the ratio of the actual width to the length of the entire house to find out possible combinations of widths and lengths which could be scaled down to fit onto the drawing paper, and finally they found the best possible relative width and length. In such an activity (see Table 4), the students applied a mathematical structure like the rate and ratio of the dimensions of a rectangle to the problem, which belongs to the [2]-ⓒ form of mathematical abstraction.

In the meantime, after measuring the length of the house on the drawing, the students decided to represent 1m in actual length as 25cm on the drawing. Based on this decision, they recorded 'as 1m is scaled down to 25cm, multiply the actual lengths with 2.5.' They multiplied the dimensions of internal spaces, including bedrooms and the balcony, by 2.5, divided them by 40 (the scale ratio of 1m to 25cm), and finally replace the unit of 'm' with 'cm.' These activities showed that the students of Group <5> estimated the dimensions of each internal space of the house by utilizing the ratio of the actual length to the drawing length, which belongs to Level [II] of mathematical abstraction.

Level [III]

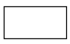
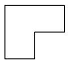
The students of Group <5> showed Level [II] mathematical abstraction forms when calculating a scale to ensure the house would fit onto a B3 sheet, and when they were depicting an architectural drawing based on the scale-down ratio.

Table 4. Group <5> students' activity and teacher's observation

Students' activity	Teacher's observation
To make an architectural drawing fit on a B3 paper, the students tried both 20cm and 40cm but decided that 50cm was the most suitable for the width. They drew a rectangle 50cm in width × 25cm in length.	When calculating the scale-down ratio, the students did not measure the width and length of a B3 sheet, but randomly set a certain number for the width and then estimated the relative scaled-down length in consideration of the relation between the length and width. They compared the lengths and decided whether it would be possible to reduce them by comparing a 50cm ruler to the width of length side of the B3 sheet. After a number of experiments with random numbers, they decided to try to scale down it to 50cm in width and 25cm in length.

However, as they checked and reviewed their problem-solving processes from time to time during their activities, they realized they had omitted the utility room. They had to modify the drawing after deciding the location and shape of the utility room, which showed Level [III] mathematical abstraction forms. The students revealed a vertical reconstruction of a new mathematical structure and a generalization of a mathematical structure among Level [III] mathematical abstraction forms.

When the students of Group <5> added the utility room to the drawing, they went beyond simply expressing the utility room as a rectangle and laying out it into several separate plane figures of the same area size. When they added the utility room to the drawing, they came to know how to express new shapes of plane figures of the same area separately on the drawing, based on previously learned mathematical structures including figures, lengths and plane figures. This can be interpreted as a vertical reconstruction of a new mathematical structure, which belongs to the [3]-Ⓒform of mathematical abstraction.

When adding the utility room to the drawing, the students worked out that it would be 20m2 in consideration of the area of the living room. They tried to divide the 20m2 area into two rectangles, but they could not find space for the hallway. And they tried to divide it into three rectangles but then realized that they could not divide it into the three rectangles of equal area. It was recorded that they divided the utility room area into four rectangles of the same area and decided to attach four rectangles of 5m2 to one another. As a result, there is a utility room with two separate figures (: 5m^2 & : 15m^2) on their architectural drawing. Such activities of Group <5> are the prime example of the [3]-Ⓒform of mathematical abstraction.

The students of Group <5> delivered a presentation on their problem-solving processes and activity results. And while they monitored other groups' presentations, Group <5> asked questions and offered their opinions about errors and mistakes. Hyunsun from Group <5> asked questions about the problem-solving process of Hanbin from Group <3> and pointed out an error when Group <3> said that they made the house fit onto their B3 sheet by reducing the actual dimensions to the drawing dimensions without calculating the scale-down ratio.

After additional explanation about the error in Group <3> by a teacher, Hyunsun from Group <5> asked how Group <3> was able to calculate the drawing dimensions of each internal space of the house without a scale-down ratio, pointed out that the process of scaling down the actual dimensions of various internal spaces was missing, and suggested that the dimensions on the architectural drawing of Group <3> might be inaccurate.

These activities of Group <5> showed the [3]-ⓑform of mathematical abstraction by evaluating the validity of a problem-solving method, by generalizing the scale-down ratio they had calculated to solve their own group task, as well as mathematical knowledge and structures about the relation between an actual length and a relative length on the drawing, and by identifying an error.

> Hyunsun (Group <5>): "I wonder how you could make a drawing without a scale-down ratio."
> Hanbin (Group 3>): "After calculating the actual length and width, we tried to draw them on the B3 sheet."
> Hyunsun: "But I still don't understand…"
> Teacher: (The teacher confirmed with Group <3> whether he or she understood correctly.) "Your guys calculated the actual width and lengths based on what was suggested in the problem and then drew them on a B3 sheet without a certain scale-down ratio. You randomly depicted a small space as small and a big space as big. Group <3>! Have I understood correctly?"

Hanbin: "Right."
Teacher: "Now, do other students know how Group <3> was able to draw their architectural drawing?"
Hyunsun: "Well, without a scale-down ratio, how were you able to calculate the relative area of each internal space on the drawing?"
Hanbin: "After measuring what was drawn on a paper, we estimated them with a calculator."
Hyunsun: "I still don't get it..."

Group <5> also showed the [3]-ⓑform of mathematical abstraction in the process of selecting the most appropriate method among all groups' problem-solving processes. This group of students selected the method of Group <5> as the best of all the solutions to the problem, during the presentation delivered by all groups. The first reason they chose Group <4> was because Group <4> replaced 1m with 1cm, which they said was the easiest and simplest method of reducing actual lengths to the relative lengths on the drawing. The second reason was that Group <4> could calculate accurate relative lengths to be drawn because they used a scale-down ratio.

However, Group <5> complained that the drawing depicted by Group <4> was a little inconvenient because it was too small. When they said "Some of us said we outperformed all the other groups, but we chose another group," they showed that they generalized the mathematical knowledge and structures they already used in their own problem-solving process to other groups, in order to find the best solution. Such activities of Group <5> is a good example of the [3]-ⓑ form of mathematical abstraction.

CONCLUSIONS

This study aimed to analyze the mathematical abstraction processes of elementary school students, which appear in the course of solving an ill-structured problem, and to find the mathematical abstraction levels and forms of the students, by giving ill-structured problem-solving activities to the fifth grade of elementary school. The study results showed that among the six groups, Groups <4> and <5> showed the highest Level [III] of mathematical abstraction. Groups <1> and <3> showed Level [I] and Groups <2> and <6> showed Level [II]. This means that it was not easy for the students to interiorize, which is the highest level and form of mathematical abstraction, when solving an ill-structured problem. This is because it is hard for elementary school students, who build thinking skills by handling concrete objects, to assess a given problem in a symbolical and abstract manner.

If all the mathematical abstraction aspects that the students showed during this study are taken into account, their mathematical abstraction proceeded like the following flowchart (Figure 2) as they tried to solve the ill-structured 'architectural drawing' problem. First, the students recognized the need for a mathematical structure to convert the information on the actual house dimensions to its dimensions on the drawing.

After that, the students utilized mathematical structures such as ratio or proportion in their solution-seeking processes, to calculate the relative lengths to ensure the whole building would fit on the B3 sheet they were given. In the final process of application, the students evaluated each group's problem-solving process by reference to their own. The students of Group <5> showed higher levels and forms of mathematical abstraction (from Level [I] to [II] and [III]) over the course of their problem solving process. This indicated that the activity of solving the ill-structured problem gave an opportunity for the students to enhance their mathematical abstraction capacity.

These study results showed that the mathematical abstraction levels and forms of students can be improved by giving them an ill-structured problem to solve, such as the problem given by this study. This is in the same context as those by Van Oers & Poland (2007) and Davydov (1990), who said that young children can improve their abstract thinking capacity by experiencing a theoretical model which helps structure concrete experiences such as schematizing activities. English (2003) said that students can improve their ability to formalize and generalize ill-structured understanding and presumptions by finding out a mathematical structure in a problem with a real-life context, and by looking for an appropriate solution and applying it. They can also share them with other members of a group by making notes of their learning, thinking and problem-solving processes by using mathematical symbols.

For elementary school students who are gradually developing from a thinking process based on handling concrete objects to a formal and abstract thinking process, it is very important that they have opportunities to experience mathematical abstraction. This study, that gave elementary school students such an opportunity, has many implications for children's mathematical abstraction development. The analysis results of the mathematical abstraction levels and forms that the students showed during this study are also meaningful in regard to mathematics teaching and learning, and for the development of advanced mathematical thinking ability in schools.

AUTHORS' NOTE

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