Mathematical Models of Learning Efficiency

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ABSTRACT
Put forward four mathematical models of learning efficiency: exam review model, talent cultivation model, career development model and quality education model. Exam review model points out those students of common capacity and quality are likely to get higher score. Talent cultivation model points out that the students very talented in some respects may not get ideal exam result. Career development model gives two elements which decide a personal career achievement: learning rate and learning time. Quality education model describes the expectation of future talent types, and provides a tentative idea of quality education.
Keywords: mathematical model, learning efficiency, exam review model, talent cultivation model, career development model, quality education model

INTRODUCTION

In high school, there are students who study very hard. They spend a lot of time in class, homework, review and examination. Their performance is better and all courses are balanced relatively, but they have no particular strengths. They progress dully and steadily. Neither they have experience of radical change, nor do they have leaps in progress. They generally perform well, but have little potential for development.

There are other students who are the pursuers of learning progress in daily study. Their performance is not outstanding. They may get a very low score in one exam. But the occasional setback is perhaps the turning point they rise up. Reminder and guidance of teachers or parents may uncover their potential box, stimulate their learning enthusiasm and ability, and help them make extraordinary progress in future study.

After years of observation by tracking career development of primary and middle school students, Zhou Wu, a teacher with Tianchang Primary School of Hangzhou, found a phenomenon: the student whose score was the tenth or so in his class during primary or middle school might have faster progress and achievements later. He called it "the tenth phenomenon". Although this statement only stays on the level of experience, even it is slightly rough, it causes extensive concern. Many people reflect that the general trend of the primary and middle school students' future career development is indeed the case, so the expression has a certain value.

Deguo Zhao, as an editor in chief of Chinese Alumnus Network and an expert of Research Group of Top Scorers, introduced that the research group had investigated 1400 top scorers nation

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State of the literature

- There are some studies on student's potentiality development. They focus on "the tenth phenomenon".
- In high school, there are students who study very hard. They generally perform well, but have little potential for development.
- The student whose score was the tenth or so in his class during primary or middle school might have faster progress and achievements later.

Contribution of this paper to the literature

- The article puts forward four models of learning efficiency to explain the "the tenth phenomenon" from different aspects.
- Exam review model explains "the tenth phenomenon" from one aspect and talent cultivation model explains "the tenth phenomenon" from another aspect.
- Career development model points out the fundamental reason of the "the tenth phenomenon" and quality education model is a counterexample of "the tenth phenomenon".

widely from 1952 to 2011. There were a few top scorers in the list of professional elites including academicians, but they were the top scorers in the fifties of the last century. No top scorers after 1977 have been professional elite yet.

Why does the top student in middle school become an average Joe, and why does the tenth student develop better? This paper analyzes and solves this problem by means of mathematics, and explains that "the tenth phenomenon" is reasonable through mathematical models.

EXAM REVIEW MODEL

In modern society, there are all kinds of examinations everywhere. As long as you want to get a high level of academic credential, as long as the job is competitive, the exam will be inevitable. Therefore, to learn examining is one's basic ability of survival and development. This section puts forward the mathematical model to improve efficiency of examination review by the marginal utility analysis method of microeconomics.

When a consumer purchases some kinds of products with a certain amount of money, in order to achieve the maximization of total utility, he must be sure to make the marginal utility of the last dollar is equal to each other, and equal to monetary marginal utility. This is the equal marginal utility law of microeconomics (Li, 2007; Li, 1996).

During exam review, the examinee's time for learning is limited, and his ability is limited too. That is, his vigor is limited. On the other hand, he must pay certain vigor to get one score in each course. It is easy to see that the examinee's exam review is very similar to consumer's purchasing products, thus we can use the equal marginal utility law to establish the model of exam review.

Suppose that there are three courses X, Y, Z in an examination for a certain examinee, and the score of each course is recorded as X, Y or Z. The cost of each course to get one score is $P_x(X, Y, Z), P_Y(X, Y, Z)$, or $P_z(X, Y, Z)$ respectively. These functions are monotonously increasing for each variable. The final score is the total utility TU.

The total score increment on account of one score increment of a course is called the marginal utility of the course, and the marginal utility of each course is $MU_X = \frac{\partial TU}{\partial X}$, $MU_Y = \frac{\partial TU}{\partial Y}$ or $MU_Z = \frac{\partial TU}{\partial Z}$ respectively. The examinee's vigor is M. The total score increment on account of one unit of devoted

vigor is the marginal utility of his vigor. In order to achieve the highest total score, the examinee needs to maximize the total utility. The mathematical model is as following

$$\max TU = f(X, Y, Z) \tag{1}$$

$$\text{s.t. } p_x X + p_Y Y + p_Z Z = M \tag{2}$$

This is the conditional extremum problem. We can use Lagrange's method of multipliers to solve the problem. Construct the Lagrangian function

$$L = f(X, Y, Z) + \lambda[M - p_X X - p_Y Y - p_Z Z]$$

Solve the following equations

$$\begin{cases} \dfrac{\partial L}{\partial X} = \dfrac{\partial f}{\partial X} - \lambda p_X = \dfrac{\partial TU}{\partial X} - \lambda p_X = MU_X - \lambda p_X = 0 \\[2mm] \dfrac{\partial L}{\partial Y} = \dfrac{\partial f}{\partial Y} - \lambda p_Y = \dfrac{\partial TU}{\partial Y} - \lambda p_Y = MU_Y - \lambda p_Y = 0 \\[2mm] \dfrac{\partial L}{\partial Z} = \dfrac{\partial f}{\partial Z} - \lambda p_Z = \dfrac{\partial TU}{\partial Z} - \lambda p_Z = MU_Z - \lambda p_Z = 0 \\[2mm] \dfrac{\partial L}{\partial \lambda} = M - p_X X - p_Y Y - p_Z Z = 0 \end{cases}$$

and gain the necessary condition under which the function $TU = f(X, Y, Z)$ to get maximum

$$\frac{MU_X}{p_X} = \frac{MU_Y}{p_Y} = \frac{MU_Z}{p_Z} = \lambda \tag{3}$$

Differentiate (1) and gain the following formula according to (3)

$$dTU = \frac{\partial f}{\partial X} dX + \frac{\partial f}{\partial Y} dY + \frac{\partial f}{\partial Z} dZ = MU_X dX + MU_Y dY + MU_Z + dZ = \lambda(p_X dX + p_Y dY + p_Z dZ)$$

Differentiate (2) and gain the following formula

$$dM = p_X dX + p_Y dY + p_Z dZ$$

So we can gain the following formula

$$\frac{dTU}{dM} = \frac{\lambda(p_X dX + p_Y dY + p_Z dZ)}{p_X dX + p_Y dY + p_Z dZ} = \lambda \tag{4}$$

i.e. λ is the marginal utility of vigor $\frac{dTU}{dM}$.

We can get the conclusion from formulae (3) and (4): if an examinee takes part in multi course examination subjected to a certain amount of vigor and wants to obtain the highest total score, he should make the marginal utility of every course be equal to each other, and be equal to the marginal utility of his final vigor.

Comments on the exam review model.

(1) Exam review model has the guiding function for the high school entrance examination, college entrance examination and Graduate Candidate Test.

If the final result is the total score, the model is

$$\max \quad TU = X + Y + Z$$

$$\text{s.t.} \quad p_x X + p_Y Y + p_Z Z = M$$

$$\text{where } MU_X = MU_Y = MU_Z.$$

The necessary condition under which the total score to get maximum is $\frac{1}{p_X} = \frac{1}{p_Y} = \frac{1}{p_Z}$. or $p_X = p_Y = p_Z$. This suggests that if the examinee wants to obtain the highest total score, he should pay the same 'price' for each course to earn the final score. In other words, he should obtain the same score from each course with his final effort, and the score should equal to the score that his final effort should obtain. During the period of exam review, he should pay more vigor on the course which can obtain score with lower cost, increase the cost of the course to obtain score until the cost to obtain one score from every course is equal. Only using this strategy, he can obtain the highest total score.

To be sure, it doesn't mean that the score of every course is same. It just suggests that the examinee should work hard to earn the score which can be earned easily.

For these sort examinations, all courses are equally important, none can be neglected. For every course, there are some basic contents from which the examinee can obtain scores relatively easily. However, when he studies to a certain level, it will be very difficult to get one score. So it is taboo to tend to go overboard on one or some subjects. As is said, four minus one equals zero in the college entrance examination.

(2) The exam review model has the guiding function for pupils, middle school students, the freshmen, the sophomores and so on to study the elementary knowledge.

This model can be used for general education to maximum the total knowledge when learning several courses at the same time. The learner should focus on the basic concept, basic knowledge and basic theory of the course. He could omit special case, counterexample, theoretical derivation and extending contents. He should study widely but not deeply, not tend to go overboard on one or some subjects, so that all courses develop balanced and reach to similar difficulty.

(3) The situation will be different if the examination result is evaluated by the number of passed courses because the marginal utility of one score of every course is not same. For example, one student's current score of the course X is 82, and his current score of the course Y is 59. The utility of the course X increasing 1 score is lower, while the utility of the course Y increasing 1 score is greater. Therefore, even if the cost of the course Y increasing 1 score were ten times that of the course X, he might put energy into the course Y. According to above theory, the necessary condition for the best result is $\frac{MU_X}{p_X} = \frac{MU_Y}{p_Y}$. $MU_X < MU_Y$ leads to $p_X < p_Y$. This situation has the guiding function for college students taking part in the course test, self study examination, certificate examination and so on.

TALENT CULTIVATION MODEL

Exam review model is suitable for reviewing or learning basic knowledge during the exam period. However, a person should not stay in the examination level, nor should he learn basic knowledge forever. He must have his own special skills when he has a certain basic knowledge with some experience of the learning and testing.

We learn from experience that the progress is generally obvious when we begin to study a course; however, the progress seems more and more slowly over time. It is similar to pouring in to the

conical vessel opening up. At the beginning, the water level roses quickly; gradually, the water level roses slowly; later, the water level almost cannot be seen rise when pouring a cup of water to the container.

Assuming A and B are learning three courses X, Y, Z with the same energy meanwhile. The goal of A is to strive for higher total score, and performance of three courses should be balanced according to the exam review model, as shown in **Figure 1**. B focuses on the course Z, the results of the courses as shown in **Figure 2**. Visually, **Figure 1** indicates the exam review model, while **Figure 2** indicates the talent cultivation model.

In **Figure 1**, three containers represent courses X, Y and Z which A is studying. The liquid in each container represents the energy of A spending in the corresponding course. In **Figure 2**, three containers represent courses X, Y and Z which B is studying. The liquid in each container represents the energy of B spending in the corresponding course. Assuming that A and B can get the same amount of knowledge if they spend same energy in every course, so the liquid in one container also represents the amount of knowledge that A or B gains in the corresponding course.

In **Figure 1**, let the height of the liquid surface be h and the radius of the liquid surface of the each container be r. The height h represents the level that A has reached in one course, which is valued by score usually. The amount of knowledge that A gains in the each course is $\frac{1}{3}\pi r^2 h$ respectively, and the total amount of knowledge that A gains in three courses is $\pi r^2 h$. His total score is 3h.
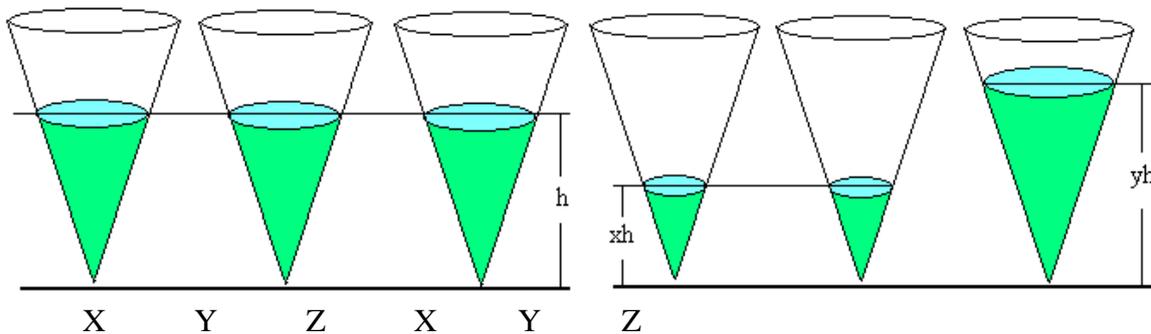

Figure 1. Exam review model   **Figure 2**. Talent cultivation model

In **Figure 2**, let the height of the liquid surface of two former containers be xh and the radius of the liquid surface be xr the height of the liquid surface of the third container be yh and the radius of the liquid surface be yr. The amount of knowledge that B gains in each course is $\frac{1}{3}\pi r^2 h x^3$ or $\frac{1}{3}\pi r^2 h y^3$ respectively, and the total amount of knowledge that B gains in three courses is $\frac{2}{3}\pi r^2 h x^3 + \frac{1}{3}\pi r^2 h y^3$. His total score is (2x+y)h.

Based on the assumptions that A and B are learning three courses X, Y, Z with the same energy meanwhile and that A and B can get the same amount of knowledge if they spend same energy in every course, we can gain

$$\frac{2}{3}\pi r^2 h x^3 + \frac{1}{3}\pi r^2 h y^3 = \pi r^2 h \tag{5}$$

After simplification, we can gain

$$2x^3 + y^3 = 3 \tag{6}$$

Specifically, if $x = 0.7, y \approx 1.32, (2x + y)h \approx 2.72h$. So the total score of A is 3h, and the total score of B is 2.72h. That is, A performs better than B in multi course examination concerning on total score.

Figures and mathematical analysis show that the progress of A getting in every course is very fast, and A has reached considerable level. However, B performs better than A in course Z because the score of B is 1.32h while the score of A is h in course Z. Although the score of B is a few better than A, this gap contains a lot of knowledge, ability, experience, frustration, etc. If A wants to make up this gap, he has to pay the effort that B spending on the course Z. In other words, if the amount of knowledge is not valuated by the height of the knowledge, but by the volume, the amount of knowledge of B in course Z is much more than that of A, and B will go farther than A in course Z.

Look back upon the situation in middle school, we often see above case. The goal of A is through the college entrance examination, and his development of each course is even. Different from A, B tends to go overboard on one or some subjects, and even the bias subjects are not exam courses. The result of exam is obvious that the total score of A is higher than B. However, if they were engaged in the career related to course Z, B would have an advantage over A.

CAREER DEVELOPMENT MODEL

Assumption and Symbol

A person's learning effect depends on his learning efficiency and learning time. Learning efficiency is influenced by many factors, such as IQ, EQ, interest, learning atmosphere, education method, learning method and so on. Learning efficiency is limited by the inherent genetic quality. Meanwhile it is influenced by the impact of environment. Learning time does not imply a person's age or the sum of the time he is used to study, but the effective learning time. For example, he used a day to study the derivative. He did not study the same content until a month later. Now, the content he had learned last time was almost forgotten, so the learning efficiency of two times was just as what one time. If he studied the same content another month later, the content he had learned second time was almost forgotten. According to this analysis, his actual learning time is almost zero. So, we assume that learning time is the effective time for the growth of new knowledge except for the non-learning time and the time for review.

We introduce the following symbols.

e: effect, implying the amount of knowledge that the learner owns.

r: rate, implying the efficiency of the learner learning and accepting new knowledge.

t: time, implying the effective learning time of the learner.

Linear Model

Let one's learning efficiency be r, where r is a constant $r \geq 0$. If he starts to study since $t_0 (t_0 > 0)$, his learning effect is

$$\int_{t_0}^{t} r dt = rt - rt_0 = rt - b \tag{7}$$

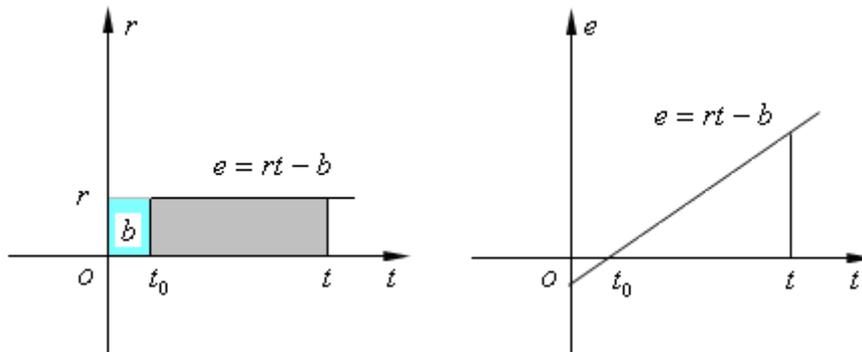That is, the mathematical model of his learning effect is

Figure 3. The learning effect of linear model

$$e = rt - b, (t \geq 0) \qquad (8)$$

The implication of the model is shown in **Figure 3**.

Figure 3 shows that he starts to study since $t_0(t_0 > 0$), his amount of knowledge is 0 at t_0 and his amount of knowledge is $e = rt - b$ at $t_0(t_0 > 0$).

Model (8) indicates that a person's learning effect is mainly determined by the learning efficiency and the learning time. For each person's learning time is always limited, learning efficiency is the most important factor affecting the learning effect. Someone (such as a naughty boy) spends less time for learning in primary school or in middle school, but he still can keep up with other students, so his learning efficiency is relatively higher than others. If he gradually increased learning time later, he would have made considerable progress. This is the key cause of "the tenth phenomenon".

Nonlinear Model

The linear model assumes that a person's learning efficiency r is a constant, but it does not conform to the actual. In fact, his learning efficiency will gradually increase with the increase of knowledge, the accumulation of experience, the summary of the failure lessons, and the improvement of learning methods. On the other hand, his learning efficiency will gradually reduce with the growth of age, memory declining, daily affairs increasing, energy and time being difficult to be focused on learning, and continuous uninterrupted learning opportunities reducing. So model (8) should be modified as

$$e = [r_1(t) - r_2(t)]t - b \qquad (9)$$

where $r_1(t)$ is the rate that the learning efficiency increases with time, $r_2(t)$ is the rate that the learning efficiency decreases with time, and they are all the functions of time t satisfying $r_1(t) \geq 0, r_2(t) \geq 0$

If $r_1(t) > r_2(t), e$ is the monotonously increasing function of time t. This indicates that the knowledge he is getting is more than the knowledge he has forgotten, and his total knowledge is increasing with time. Oppositely, if $r_1(t) < r_2(t), e$ is the monotonously decreasing function of time t. This indicates that the knowledge he is getting is less than the knowledge he has forgotten, and his total knowledge is decreasing with time.
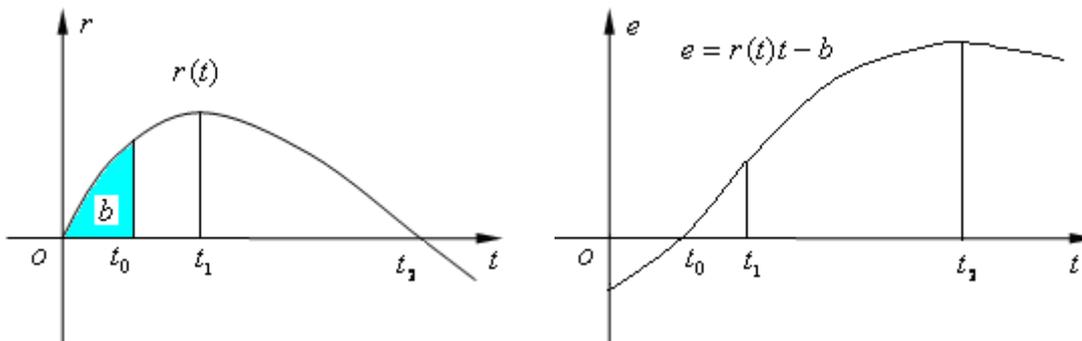
Model (9) is able to be simplified as

Figure 4. The learning effect of nonlinear model

$$e = r(t)t - b, (t \geq 0) \tag{10}$$

where b is a constant satisfying $b > 0$, $r(t)$ is the function of time t which indicates the rate of a person's learning efficiency changing with time.

The implication of the model is shown in **Figure 4.** He starts to study since $t_0 (t_0 > 0)$. When $t_0 \leq t \leq t_1, r(t) > 0, r(t)$ is the monotonously increasing function of time t. Now, his knowledge is increasing, furthermore, his knowledge is increasing more and more quickly. When $t_1 \leq t \leq t_2, r(t) > 0, r(t)$ is the monotonously decreasing function of time t. Now, his knowledge is increasing, but his knowledge is increasing more and more slowly. When $t > t_2, r(t) < 0, r(t)$ is the monotonously decreasing function of time t. Now, his knowledge is decreasing and the forgetting progress is dominant.

It has realistic foundation that learning efficiency $r(t)$ is the function of time t. For example, when one learns the field theory, he will have deeper understanding of specific fields learned previously such as gravity field, electric field and magnetic field, at same time, he will have a global understanding of general fields such as number field and vector field. Now, he will feel like increasing a lot of knowledge in a short while just like seeing the light in the darkroom suddenly. At this moment, his learning efficiency is very high, and the learning effect is very obvious. For another example, some people may not study consciously after retire, and the new knowledge increased is less than the knowledge forgotten, so his learning efficiency $r(t)$ is negative. His amount of knowledge is getting smaller and smaller, and he appears to have little knowledge gradually.

Nonlinear model has important guiding function to one's career planning and even can improve his life quality. At first, continuous and effective learning time is the main factor affecting the learning effect. If someone wants to succeed in one field, he must engage in the field effectively and continuously without distractions. Secondly, learning efficiency $r(t)$ is an important factor affecting the learning effect, and it may enlarge with the increase of knowledge, the accumulation experience, and the improvement of learning methods in a certain period. So we need to sum up learning methods constantly, pay attention to draw some successful experience of famous scholars, and gradually improve our learning efficiency. Thirdly, one's learning efficiency will gradually decline and even become less than 0 to a certain time. The forgetting progress will occupy a dominant position, and his knowledge will be reducing. To slow down the process and keep our spirit alive, we need to study new knowledge consciously when we are no longer young.

QUALITY EDUCATION MODEL

Quality education has always been a hot topic in the educational circles and the society. It focuses on thick foundation and wide caliber. The students studying natural science and technology should strengthen the cultivation of humanities, and the students studying social science and humanities should have the basic knowledge of natural science and technology. The goal is to cultivate students' adapting ability to a wide range of job, integration ability of knowledge and innovation ability based on the comprehensive use of knowledge (Zhu Jin, 2000).

Talent cultivation model emphasizes that a student should have the ability to accumulate and develop in one field. This model makes him have great development potential in this field. If he is engaged in the career of this field, he will have advantage. However, if he engages in the career of other fields, his advantage will not exist. Maybe he will not be as good as the students who develop balanced among all courses. His adapting ability is not up to the standard of quality education, and excessive emphasis on professional also makes him unable to integrate knowledge of several branches, not to mention the comprehensive use of knowledge and innovation. Therefore, talent cultivation model is not quality education.

Exam review model emphasizes the relatively balanced development of each course. Although it seems like quality education in some respects, but it is not up to the standard of quality education. The examination course number is limited, the content is less and the course is independent each other. The only goal of examination is total score, so as to the learner focuses on remembering formula, doing exercises and taking part in simulation test. He hardly pays attention to the origin, the development process and the frustration experience of the knowledge. He neither thinks the knowledge system, the relation of the knowledge with other disciplines, the status and role of the knowledge in thinking development process and so on. The result is that the learner has a smattering of courses. He will not discuss and research the knowledge after the examination any longer. Exam review model trains a group of students with high score and low ability, so it is not quality education.

What is the real quality education? There is no universally accepted definition currently. The majority of discussion is focuses on describing the talent type in future. The author is not intelligent enough to give a complete model of quality education yet. I only discuss simply here to draw forth more valuable findings.

Different from the exam oriented education, we hope students read more widely. It is something like filling in containers when a student learning and practicing n related courses. He can get the basic knowledge, basic method and thinking style by spending less energy, so that the n containers are fitted with a certain number of things and reach a considerable height. Then, he tries to explore the similarities and differences of these courses, seek the relationship among the courses, and dump the things of n containers into a larger container so that they are fully integrated. Let the existing knowledge react. By this way, it is possible to generate new knowledge and form his body of knowledge. Let this process repeat indefinitely, enrich the knowledge of each course continuously, increase the new course, even dabble interdisciplinary course, and integrate endlessly. In the process of this move in circles, he will reach a higher level per cycle. Thus and thus, his comprehensive quality will be improved in the long run.

CONCLUSION

Exam review model indicates how to distribute energy and review during the exam review stage to obtain the highest total score. Specifically, in the review period of high school entrance examination or college entrance examination, we must make the scoring difficulty of each course be balanced. This also indicates that the student who follows teacher's rhythm and learns step by step may get better result although his ability is not better than his classmates. This model explains "the tenth phenomenon" from one aspect.

Talent cultivation model shows that there are some students whose exam results are not very prominent in middle school, but perhaps they are developing their own expertise and there target is not just to pass the exam. Although their performance are common in middle school, but they may likely have better development in future. This model explains "the tenth phenomenon" from another aspect.

Career development model points out that one's achievement is the monotonically increasing function of his learning efficiency and learning time. In middle school, the learning efficiency of some students is higher, but they spend less time to study, so they have a common performance. However, they will get a good result with the increase of age, learning motivation and learning time. This is the fundamental reason of the "the tenth phenomenon".

Quality education requires students to have a wider range of knowledge, to study deeper, to pay attention to the connection among courses, and to integrate the acquired knowledge. We often see such a student who not only has a high score, but also has prominent performance on each course. He has better comprehensive quality in one word. We cannot simply say that he is imbecility with high score. We have to think that he is a counterexample of "the tenth phenomenon".

The article puts forward four models of learning efficiency to explain the "the tenth phenomenon" from different aspects, and points out the matters which the learners should pay attention to. The learner can select the appropriate model to guide his learning based on his own actual case.

ACKNOWLEDGEMENTS

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