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Mathematics Education Art and Architecture: Representations of the Elliptic Paraboloid

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ABSTRACT

This paper presents a study, which is an extension of the research done by the first author on the elliptic paraboloid, and which is oriented to mathematics teaching to 17- to 19-year-olds architecture students. The first part presents a study of the graphic representation, modifications and apprehensions of the dynamic graphic register of the elliptic paraboloid when students interact with digital technology. The second part presents a device (sequence of tasks) that allows to the students articulating the apprehensions of the dynamic graphic register, as well as does the connection between mathematics, art and architecture, specifically in architectural constructions where these mathematical notions are present.

Keywords: mathematics, art, architecture, graphic representations, elliptic paraboloid

INTRODUCTION

Different researches carried out in the area of Mathematics Education, such as those of Peñaloza (2016), Salazar and Almouloud (2015), Ingar (2014), Salazar, Gaita and Saravia (2013), López and Anido (2004), among others, address graphic representations through of Dynamic Representation Environments (DRE) as technological means and aspects from the Theory of Registers of Semiotic Representation as a theoretical framework of study, which provides theoretical tools to understand the strategies that students develop when representing surfaces.

Particularly, the research done by Peñaloza (2016) shows that, when students in higher education start studying analytic geometry, they present difficulties of algebraic nature related to elemental arithmetic operations, falling into algebraic and graphic representations of the different mathematical objects they study, particularly the elliptic paraboloid.

At the same time, the designed device shows the need of digital technology in teaching mathematics. The use of technology makes special sense in the area of Mathematics Education.

When students review "quadric surfaces" in reference books, we perceive that symmetry properties of surfaces are not applied in their elements, such as vertex, axis of the surface, planes of symmetry, among others, which we believe it is due to the limitation of the physical environment of graphic representation of such surfaces in order to do transformations and treatments between the elements representing them, which are the conic sections in this case. Therefore, problems presented in reference books consist in asking for the graphic representation of a second-degree equation in three variables through cross sections or relating graphic representations of certain surfaces to their respective equations.

Symmetry is an artistic manifestation whose mathematical basis is established in Euclidean, descriptive and projective geometry, since architecture cannot be understood without symmetries. In that regard, Viollet de Luc states the following:

> *Today, in the language of architects, symmetry does not mean a balance or harmonious relation between the parts of a whole, but a similarity of opposing parts, that is the exact reproduction of whatever there is on the right, on the left side of the axis (quoted from Calcerrada, 20, p. 22).*

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Contribution of this paper to the literature

- The paper offers a theoretical and practical reflection on the different representations in R^2 and R^3 of the elliptic paraboloid as a necessary object of study for teaching mathematics to future architects, and it also brings a different approach to others.
- In theoretical reflection, the dynamic graphic register and its respective apprehensions and modifications when interacting with a DRE also involve aspects of art, specifically in architectural constructions, in which these mathematical notions are present and visible while, in practical reflection, a device is designed favoring the learning of the mathematical object presented due to its organization and analysis.

The study of the paraboloid is present in the syllabi of mathematics classes, which correspond to the curriculum of architecture in different universities, such as Peñaloza (2016) states it in his study on paraboloid visualization, which reinforces its importance in the academic training of future architects. In turn, it is important to evidence the importance of geometry in architecture and art, which we will detail in the following lines.

THE ELLIPTIC PARABOLOID

On the other hand, geometric properties of polyhedral and quadric surfaces have been known by men since the dawn of civilization, and the evidence of this knowledge is in the design of different classic and contemporary architectural constructions, such as coliseums, Egyptian pyramids, church cupolas, amphitheaters, etc. In that regard, Ibáñez (2009) claims that

> *[…] Geometry, and Mathematics in general, have been present in Architecture since the moment that men felt the need to build a home to shelter from the inclemency of nature, rest or keep away from their enemies, whether excavating caves, building huts or setting up tents; and they also felt the need to build special places to burry and venerate their dead or worship gods. […] It seems evident to anyone that, since shape and structure are so important in the design of architectural constructions, Geometry and Mathematics are a fundamental part of Architecture. (p.155)*

With the development and evolution of analytical geometry in the study of the different representations of surfaces that model the desired shape to establish the geometry of the constructions and projects on demand, designers must consider, besides the algebraic or parametric equation representing the surface, the effects produced by loads and efforts (besides the weight itself) that are distributed on the whole surface. According to Aass (1963, p. 1), surfaces with double curvature, such as the hyperbolic paraboloid, have been of great interest to architects and design engineers due to their simple and effective construction for being a ruled surface and for the effort redistribution in different directions, but some studies also prove that the elliptic paraboloid presents properties of high resistance due to the positive Gaussian curvature.

Besides all that, elliptic paraboloids are very attractive from an artistic and architectural point of view since, according to Gaudí, "for an object to be extraordinarily beautiful, it is necessary for its shape not to be superfluous at all". **Figure 1** shows the cupola of Güel Palace in Barcelona, Spain, a work from Spanish architect Antonio Gaudí, which has the shape of a paraboloid of revolution.

Figure 1. Cupola of Güel Palace
Source: https://www.pinterest.co.uk/atasilent/the-gaudí-buildings-in-barcelona/

In that sense, we consider that it is important for a student of architecture to study quadric surfaces, especially the case of the elliptic paraboloid, in his professional training, given the fact that the shapes of artistic, classic and modern architectural constructions are modeled mathematically through the algebraic representation of the elliptic paraboloid, allowing greater accuracy in the shape and design of those constructions and other similar ones since the location of any point of a surface can be determined with respect to a reference system and obtain its technological properties by knowing the equation of such surface in general.

Thus, in this paper we present a theoretical and practical reflection on the different representations of the elliptic paraboloid in the dynamic graphic register (Peñaloza, 2016) and their respective apprehensions and modifications when interacting with a DRE these representations also involve aspects of art, specifically in architectural constructions in which these mathematical notions are present and visible.

We defined the study to the elliptic paraboloid in R^3 with vertex different from the origin of coordinates and main axis parallel to the coordinated axes, and based on aspects of the Theory of Registers of Semiotic Representation from Duval (1995; 1999; 2002), we did a study of its graphic representation, modifications and apprehensions of the dynamic graphic register.

DIFFERENT REPRESENTATIONS OF THE ELLIPTIC PARABOLOID

Given the abstract and intangible nature of mathematical objects, they need a tangible environment in order to do transformations between them and produce new meanings; in that sense,

> *[...] the mathematical activity is necessarily done in a context of representation, since there is no other way to have access to the mathematical object directly from perception, being students capable of recognizing the same mathematical object they know in other contexts and use them (Duval 1995, p. 144).*

In other words, learning in mathematics is related to the processes of semiosis and noesis, where "semiosis" is the apprehension or production of a semiotic representation and "noesis the conceptual apprehension of an object. It is necessary to mention that noesis is inseparable from semiosis. It is necessary for noesis (conceptualization) to occur by means of semiosis (representations) so the apprehension of a mathematical object occurs.

According to Duval (1993, p. 39), "they are productions constituted by the use of signs belonging to a system of representation, which have their own difficulties of meaning and functioning". Besides, the researcher defines the following registers: natural language, algebraic, graphic and figural. In that sense, in **Figure 2** we present the natural language, algebraic and graphic registers of the paraboloid.

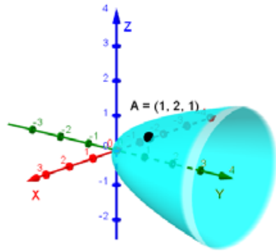
Natural language register	Algebraic register	Graphic register
Paraboloid, whose axis of symmetry is the *Y*-axis, with vertex at the origin of coordinates, opens to the positive direction of the *Y*-axis and passes through point (1, 2, 1).	$y = x^2 + z^2$	

Figure 2. Representations of the paraboloid

It is worth mentioning that, in the natural language register, more information is required in order to be able to describe the elliptic paraboloid, while its representation in the algebraic register describes, in more general terms, the aforementioned object, which can also be represented graphically.

Regarding the graphic register, specifically that of the elliptic paraboloid, we point out that, in the sense of Duval, it is possible to constitute the dynamic graphic register of the elliptic paraboloid; therefore, we based on the researches of Salazar and Almouloud (2015), which configures the dynamic figural register and defines the formation, treatment and dynamic conversion of this register. Regarding the types of transformations (conversion and treatment) of an elliptic paraboloid, we present its transformations in **Figure 3**. We observed that the conversion from the natural language register to the graphic register is not direct, since the graphic representation of a quadric requires its equation.

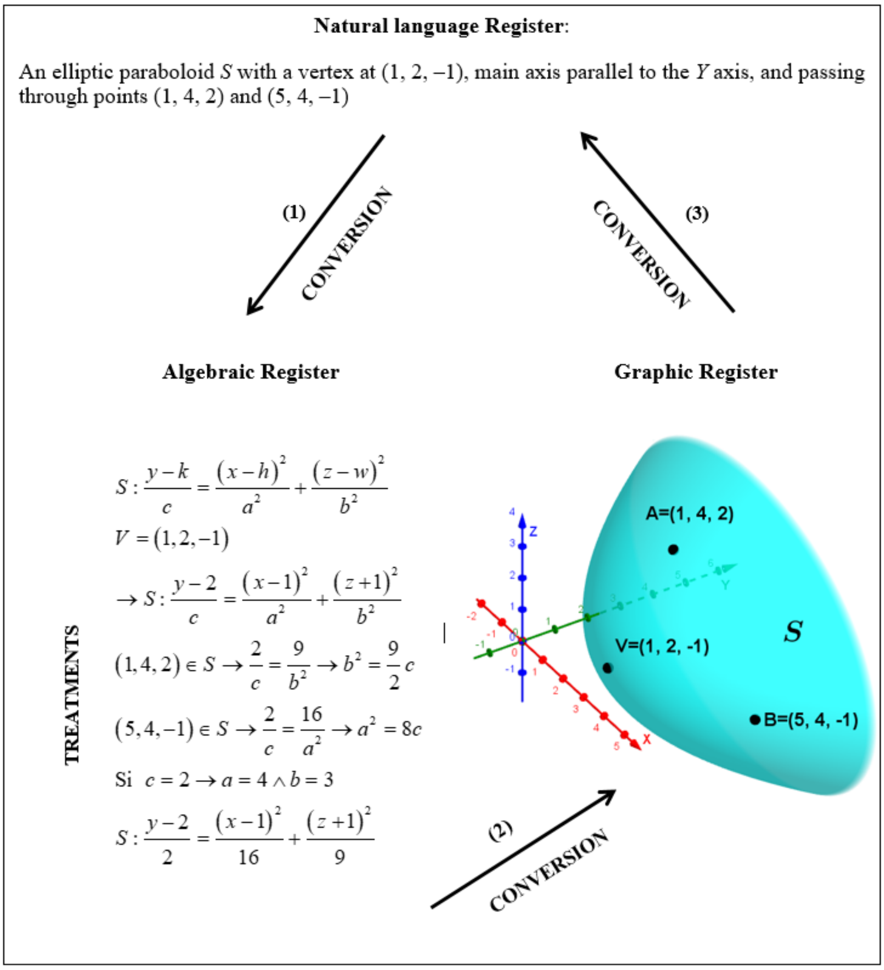


Figure 3. Transformation of a Semiotic Representation into another

Therefore, it is necessary to do the conversion from the natural language register to the algebraic register (1) and, by doing treatments in the algebraic register, the algebraic representation of paraboloid *S* is obtained through its equation, which can be represented graphically, for example, through the DRE *Geogebra 3D*, besides the vertex and the point, doing the conversion from the algebraic register to the graphic register (2). By having the representation of the paraboloid in the graphic register with the vertex and a point, it can be described in natural language, since no information was lost in any of the registers; this way the conversion from the graphic register to the natural language register (3) is done, which implies that conversions (1)→(2)→(3) are done cyclically.

Regarding the apprehensions of the elliptic paraboloid, we understand that if the student is ready to identify the paraboloid in the graphic register – that is, he perceives some constitutive elements but indicated in an explicit way – then we can state that he has developed a ***perceptual apprehension***.

In **Figure 4**, the student can immediately recognize that the object represented in the graphic register corresponds to a paraboloid; however, he could not specify if it is of elliptic or circular type, which implies subsequent greater analysis and thus developing another type of apprehensions.

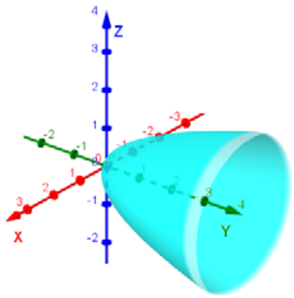
Object represented in the graphic register	**Perceptual apprehension**
	The object is a paraboloid.

Figure 4. Perceptual apprehension of the paraboloid

In case a student has to do a graphic representation of the paraboloid in Geogebra through its Cartesian equation, such student recognizes the need to describe the necessary steps (procedure) to sketch such representation. We can state that the student has developed the ***sequential apprehension*** of the paraboloid. In this type of apprehension, it is extremely convenient to use a sequence of steps of construction through syntax of commands and tools from the software.

If the graphic representation of a paraboloid *S* were done with pencil on paper, the sequential apprehension developed in the student would be equal to him doing cross sections with planes parallel to the coordinated planes and representing each one of those cross sections in the graphic register to represent the surface of the paraboloid, since it does not have trace or sliders.

If the student develops a ***discursive apprehension***, he is able to recognize other mathematical properties essential to the paraboloid, which are not explicitly indicated in its graphic representation, such as the type of paraboloid, the coordinated axis to which the focal axes of the elliptic cross sections are parallel, the vertex of the surface, etc. In this type of apprehension, the shape of the cross sections of planes parallel to coordinated planes with the paraboloid are identified by the student, who recognizes the pertinence of such information as the shape of its equations and main elements (center, vertices, foci, etc.), which are recognized and named. **Figure 5** indicates the discursive apprehension of the paraboloid developed by a student, which allows him to identify if the graphically represented paraboloid is of circular or elliptic type.

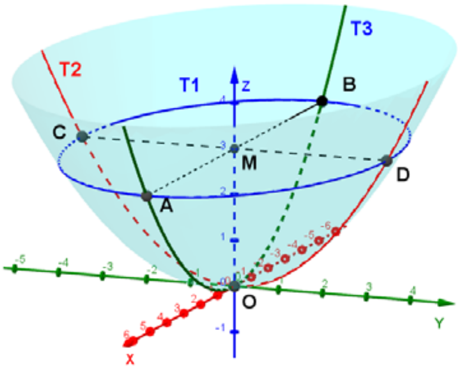
Object represented in the graphic register	Discursive apprehension
	T2 and T3: Parabolas in planes x = 0, y = 0 T2 and T3 intersect in the origin of coordinates (O). AB and CD intersect in M; M belongs to the Z-axis. AM = MB, CM = MD, M: Center of T1 The axis of the surface is the Z-axis. The abscissa of A is more than 4, the ordinate of D is less than 4; therefore, the focal axis of T1 is parallel to the X-axis. The surface is an elliptic paraboloid with vertex at the origin of coordinates, whose axis is the Z-axis and it expands towards the positive semiaxis of the Z-axis.

Figure 5. Discursive apprehension of an elliptic paraboloid

If a student expresses the need to do transformations in the graphic register of a paraboloid to obtain information that allows him to identify shapes and values of its elements, such as open curves, closed curves, the vertex, distances, etc. and does them, we can state that the student has developed the ***operative apprehension*** of the paraboloid. To do that, according to the need, he must do certain *modifications* in the graphic register. To determine these modifications, we base on Ingar (2014), who did an adaptation of the modifications, in the sense of Duval, to study them in the graphic registers of local maxima and minima of multivariable functions. This way, the modifications in the graphic register of the paraboloid are the following: the ***optical modification*** done by a student can occur when using the *Zoom out* and *Zoom in* tools in the EDR, so that the elements and features of the graphic representation of the paraboloid can be recognized and studied in more detail with close-ups or distancing. For example, this occurs when the values of a, b, and c in the algebraic representation of the paraboloid, whose equation is: $\frac{z-w}{c} = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$, are relatively higher in comparison to the scale of the Cartesian axes, so it is necessary to zoom out the resulting view to study its shape and recognize its main elements. In **Figure 6**, an optical modification has been done, zooming out in order to recognize the shape of the graphic representation of paraboloid S, whose algebraic representation is given by the equation: $S\!: z = \frac{(x-4)^2}{25} + \frac{(y-5)^2}{36}$. In the first case, the surface covers pretty much all the *3D Graphics View*, so it was necessary to do a modification.

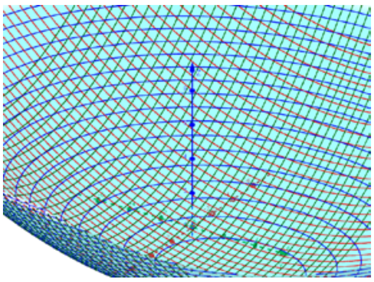
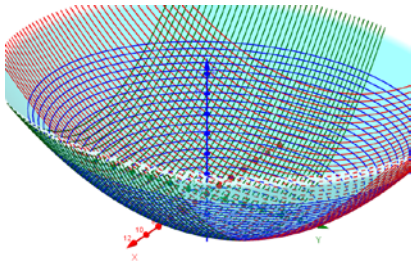
Graphic representation of paraboloid ***S***	Optical modification of the graphic representation of paraboloid ***S*** by using the ***Zoom out*** tool
	

Figure 6. Optical modification in the graphic representation of paraboloid

The ***positional modification*** corresponds to translations and rotations of the graphic representation of the elliptic paraboloid, keeping the shape of such surface and only changing its position with respect to the observer.

Two ***positional modifications*** (see **Figure 7**) have been done to the graphic representation of paraboloid S, whose algebraic equation is: $z = \frac{x^2}{4} + \frac{y^2}{9}$ (translation and rotation).

Graphic representation of	Translation	Rotation

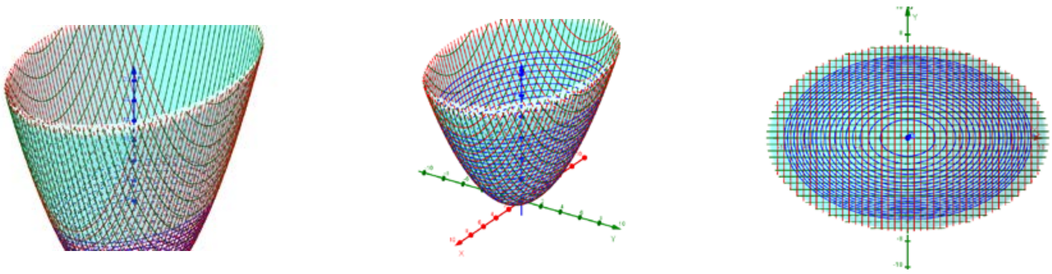


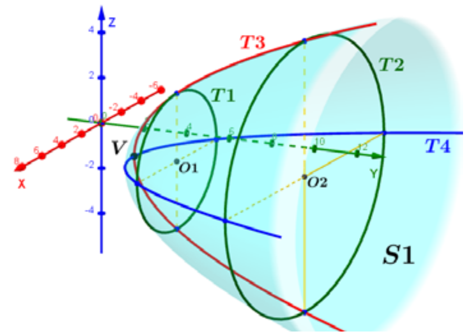
Figure 7. Positional modification

We state in our study that the ***dimensional deconstruction*** of a *primary* paraboloid allows decomposing it in points, lines and planes, which allows graphically representing a *secondary* paraboloid after meeting previously established conditions. The parameters and constants of each one of the conic sections that represent the primary paraboloid can have equal or proportional shapes. Through treatments in the graphic register, this principle allows doing graphic representations without doing rigorous treatments in the algebraic register.

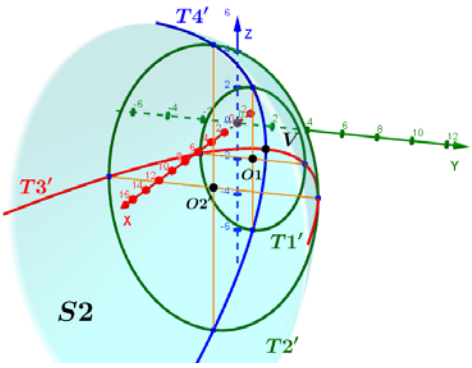
If the parameters (in absolute value) and other constants of the conic sections (obtained by cross sections) are equal, the graphic representation of the secondary paraboloid can be interpreted as a translation and/or rotation of the primary paraboloid.

For example, in **Figure 8** we show the graphic representations of the elements of the graphic representation of paraboloid S_1, whose algebraic representation is $S_1\colon \frac{y-2}{2} = \frac{(x-1)^2}{16} + \frac{(z+1)^2}{9}$, to plot surface S_2, whose algebraic representation is $S_2\colon \frac{x-1}{2} = \frac{(y-2)^2}{9} + \frac{(z+1)^2}{16}$, by using the graphic representations of the elements of the S_1 graph, since the graphic representations of both surfaces have *the same shape* and thus the same represented elements, but located in different positions in the space.

Decomposition of the graph of surface S1 to plot surface S2



Primary Paraboloid
Elements of S1 represented by:
V: Vertex, Axis of S1 parallel to the Y-axis
T1: Closed curve (Ellipse) in a plane parallel to plane XZ, center: O1, focal axis parallel to the X-axis
T2: Closed curve (Ellipse) in a plane parallel to plane XZ, center O2, focal axis parallel to the X-axis
T3: Open curve (Parabola) in a plane parallel to plane YZ, and it extends towards the positive semiaxis of the Y-axis
T4: Open curve (Parabola) in a plane parallel to plane XY, and it extends towards the positive semiaxis of the Y-axis.



Secondary Paraboloid
By translating curves T2 and T4 from S1 to their new positions in the graphic register, the S2 graph has been composed from the decomposition of the S1 graph.
Elements from S2 represented by:
V: Vertex, Axis of S2 parallel to the X-axis
T1′: Closed curve (Ellipse) in a plane parallel to plane YZ, center O1′, focal axis parallel to the Z- axis
T2′: Closed curve (Ellipse) in a plane parallel to plane YZ, center O2′, focal axis parallel to the Z-axis
T3′: Open curve (Parabola) in a plane parallel to plane XY, and it extends towards the positive semiaxis of the X-axis
T4′: Open curve (Parabola) in a plane parallel to plane XZ, and it extends towards the positive semiaxis of the X-axis.

Figure 8. Dimensional deconstruction of the paraboloid

Having identified the apprehensions and modifications of the elliptic paraboloid in the graphic register, as well as its dimensional deconstruction, which is a cognitive activity that allows articulating different dimensions, we think that, based on these theoretical aspects, it is possible to develop a training device (sequence of activities) for mathematics teachers that favors the developments of the process of visualization.

THE DEVICE

In that sense, with the development of the theoretical basis presented in the previous items, we think it is pertinent to design and analyze a device on the object of study. **Table 1** details the organization of the device, which consists of a sequence of three activities.

Table 1. Activities from the experimental part of the present research

Activity	Type of Surface or Curve	Description
1	Elliptic ellipsoid / Plane / Ellipse	Use the different tools from the EDR to plot an ellipsoid, planes parallel to the coordinated planes, intersection of the ellipsoid with one of the planes, use sliders, associate the slider to the equation of a plane, show/hide objects.
2	Ellipse / Elliptic ellipsoid	Create a macro or new tool to plot ellipses to plot ellipses since their vertices and ends of the minor axis are previously inserted in the 3D Graphics View, as well as to represent an ellipsoid (3D) through those closed curves (2D).
3	Elliptic Paraboloid	Given the graphic representation of an elliptic paraboloid through its vertex and four cross sections of such surface with planes parallel to the coordinated planes, represent another paraboloid with the same shape as the first one, using the same elements but in another position in R^3.

After designing it, in this paper we present the a priori analysis of activity 3, called "Visualize a quadric", which consists of five items.

1) Open the ***Actividad_3.ggb*** file in which the curves of intersection T_1, T_2, T_3, T_4 of a quadric surface S_1 is shown with planes parallel to the coordinated planes, as well as its vertex V. Identify the shape of each curve and explain the tools used for **just one** of those curves.
2) With curves T_1, T_2, T_3 and T_4 that represent surface S_1, represent another surface S_2, which has to have **<u>the same shape</u>** as surface S_1, with vertex in point V as well. Surface S_2 opens towards the positive X-semiaxis, and the focal axis of its closed curves will be parallel to the Z-axis, getting the new T'_1, T'_2, T'_3 y T'_4 curves.
   Suggestion: First, plot the closed curves; you can use the *Ellipse through 4 points* macro.
3) With the information provided in the algebraic and graphics view, obtain the Cartesian equations of a closed curve (T'_1 or T'_2) and an open curve (T'_3 or T'_4), expressed in **canonical shapes**. Then, obtain the equation of the surface.
4) Represent surface S_2. To do that, in the *Input Bar* insert the equation in three variables, obtained in the item (3). Verify that the represented surface passes through curves T'_1, T'_2, T'_3, T'_4 and vertex V.
5) Explain how the equation of surface S_1 can be obtained **directly** from the equation of surface S_2 obtained in step (3). Write it down and represent surface S_1, making sure it passes through curves T_1, T_2, T_3, T_4 and vertex V.

Figure 9. Visualize a quadric

In item 1 of this activity, participants are expected to open the file in which the primary paraboloid S_1 is represented by open and closed curves. The whole representation is set with the first trihedron oriented towards the observer (see **Figure 10**), so we think that identifying the shape of the closed curves has a high level of difficulty.

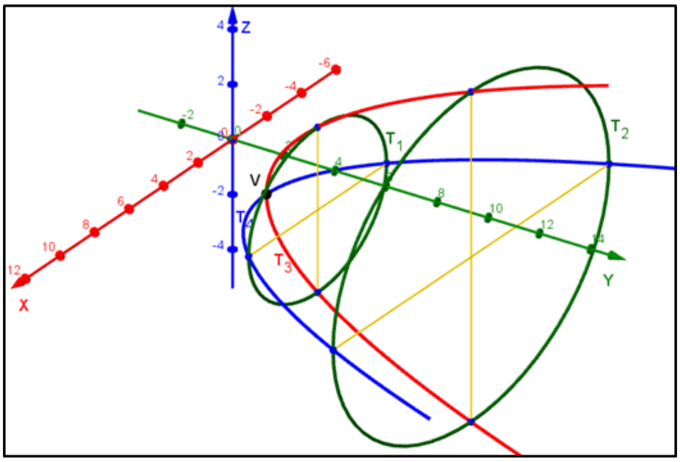


Figure 10. Primary paraboloid S_1

Also, it is expected that, through optical and positional modifications of translation and rotation, it is possible to get frontal views of the plane curves of paraboloid S_1, which allows identifying them, as shown in **Figure 11**.

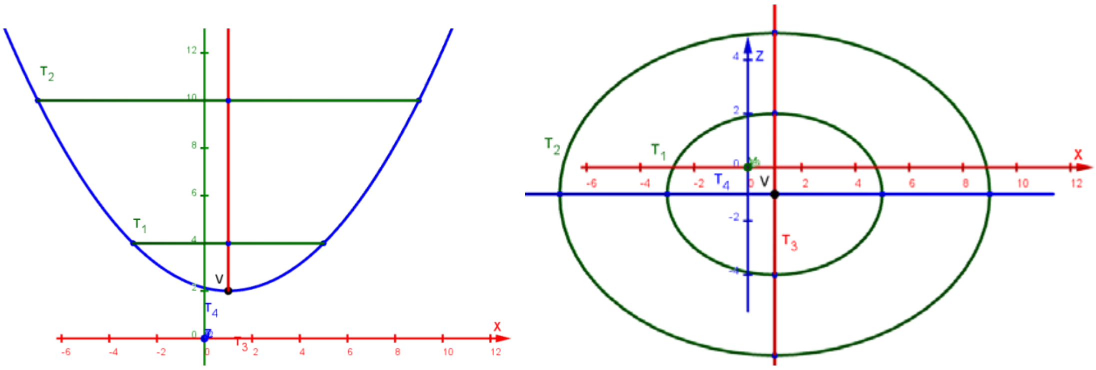


Figure 11. Frontal views of planes *XY* and *XZ* of primary paraboloid S_1

In item 2 of the activity, in terms of Duval, the aim is to analyze the dimensional deconstruction of paraboloid S_1 to represent paraboloid S_2 and, according to the suggestion, closed curves have to be represented first. **Figure 12(a)** shows the translation of center J of curve T_2 towards point J', both at a distance of eight units with respect to vertex V. Then, with the lengths of the *EF*- and *GH-axes* of ellipse T_2, constants a and b of such ellipse are obtained, which have to be the same for ellipse T'_2, representing its vertices and ends of the minor axis. Finally, T'_2 is represented with the *Ellipse through 4 points* tool.

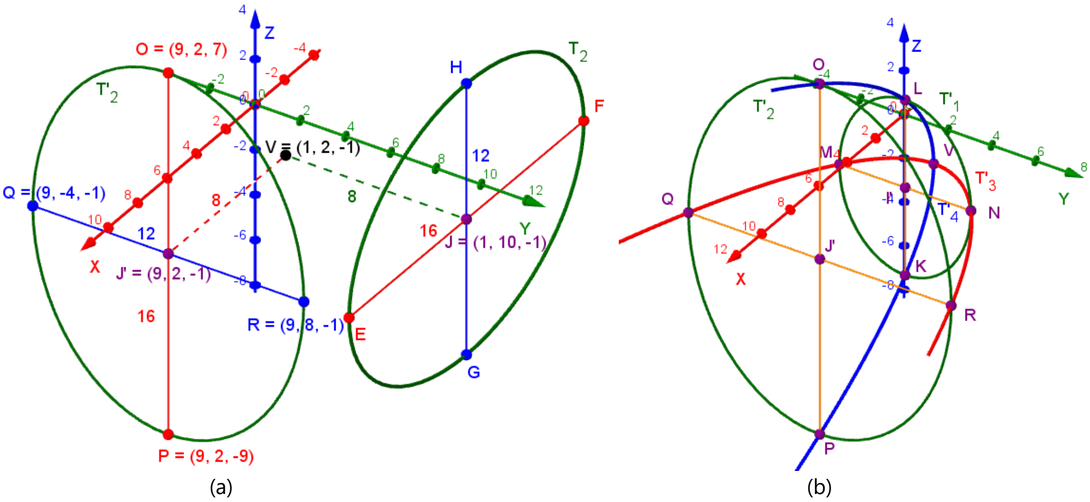


Figure 12. Representation of paraboloid S_2 through dimensional deconstruction of S_1

With a similar procedure, ellipse T'_1 is represented and, with the *Conic through 5 Points* tool, taking the vertices and ends of the minor axis of each ellipse and vertex V of the surface as points, parabolas T'_3 and T'_4 are represented, as shown in **Figure 12(b)**.

In order to get the equation of paraboloid S_2, according to item 3, it is necessary to obtain the equation of an open curve and a closed curve, for example T'_4 and T'_2. Therefore, with the points, planes of cross sections and focal axes of the cross sections, we can do such task.

According to the information of **Figure 12(a)**, the equation of ellipse T'_2 is the following:

$$T'_2:\begin{cases}\dfrac{(z+1)^2}{64}+\dfrac{(y-2)^2}{36}=1\\ x=9\end{cases}$$

Similarly, the canonical equation of parabola T'_4 is:

$$T'_4:\begin{cases}(z+1)^2=8(x-1)\\ y=2\end{cases}$$

That is, the cross sections of surface S_2 with planes $x = 9$ and $y = 2$ are the following:

$$x=9;\frac{(z+1)^2}{64}+\frac{(y-2)^2}{36}=1$$

$$y=2;\frac{(z+1)^2}{64}=\frac{(x-1)}{8}$$

From both equations, we infer that the equation of second degree in three variables that generates these cross sections is the following: $S_2:\frac{(z+1)^2}{64}+\frac{(y-2)^2}{36}=\frac{(x-1)}{8}$

Simplifying: $S_2:\frac{(x-1)}{2}=\frac{(z+1)^2}{16}+\frac{(y-2)^2}{9}$

Regarding item 4, it is done directly in the DRE, getting a graphic representation that is similar to the *secondary* paraboloid of **Figure 8** and, in the case of item 5, the activity consists in exchanging the terms of the equation of surface $S_2:\frac{(x-1)}{2}=\frac{(z+1)^2}{16}+\frac{(y-2)^2}{9}$ so that, since both surfaces have the same vertex in common, the axis of surface S_1 is parallel to the Y-axis, and the focal axis of the closed curves is parallel to the X-axis, according to **Figure 8**. Therefore, the equation of surface S_1 would be as follows:

$$S_1:\frac{(y-2)}{2}=\frac{(x-1)^2}{16}+\frac{(z+1)^2}{9}$$

whose graphic representation is that of the *primary* paraboloid shown.

The dimensional deconstruction of the paraboloid through a DRE might develop students' decision making in the design of architectural shapes due to its symmetric, acoustic and technological properties in general because of the optical properties of the parabola, addressing the effects of a source located in its focus, changing the orientation of its focal axis, extending such property from a 2D object to a 3D one. In the case of the paraboloid, it would be equivalent to doing treatments in the algebraic register of its representation, hence the importance of the relation between the abstract mathematical thinking and the design of tangible objects in reality.

ELLIPTIC PARABOLOID: ART AND ARCHITECTURE

The acoustic properties of the paraboloid, both elliptic and of revolution have been useful for the design of cupolas, amphitheaters, shell structures, among others. One of these is the stage of the Hollywood Bowl amphitheater in California, designed by architects Fank Gehry and Lloyd Wright (built in 1922), inspired by the natural amphitheaters of ancient Greece and Roma, whose interior shape is that of a semi-paraboloid of revolution, formed by structural semicircular rings; the connection between mathematics and art is evident in this construction (see **Figure 13**).

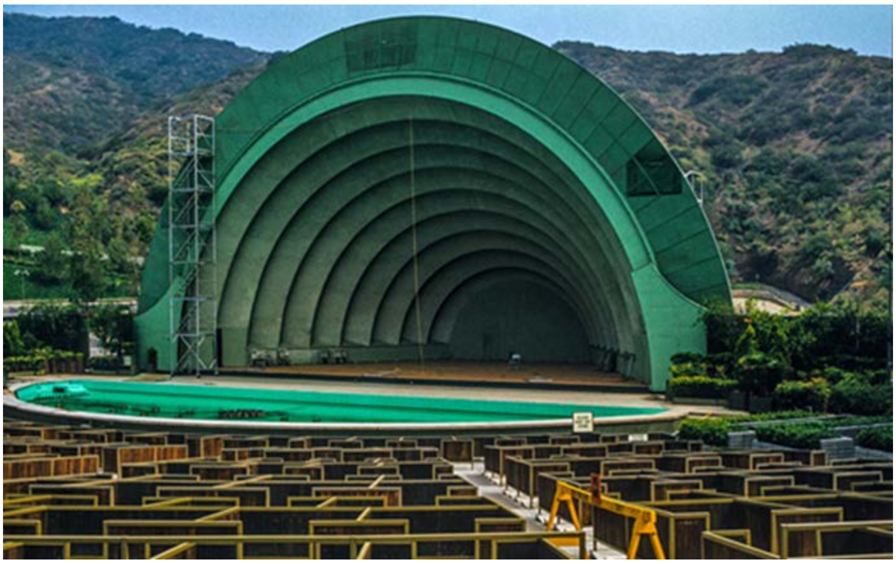


Figure 13. View of the Hollywood Bowl before its remodeling in 2003
Source: http://davelandweb.com/hollywood/images/

Initially, concerts at the Hollywood Bowl were without sound system, applying the acoustic property of the paraboloid of revolution; interpreters stood approximately on the focus of the parabola of the base. **Figure 14** shows a modeling of the stage of the amphitheater in a DRE through paraboloid $S\!: y = \frac{x^2}{6} + \frac{z^2}{6}$, in which cross sections were made with planes parallel to plane $y = 0$ (coaxial circumferences) and a cross section with plane $z = 0$, getting parabolic curve C, whose focus in point A. Vector u, which goes from A to point P that belongs to curve C, is reflected through the curve in vector v, which is parallel to the Y-axis (axis of the S surface).

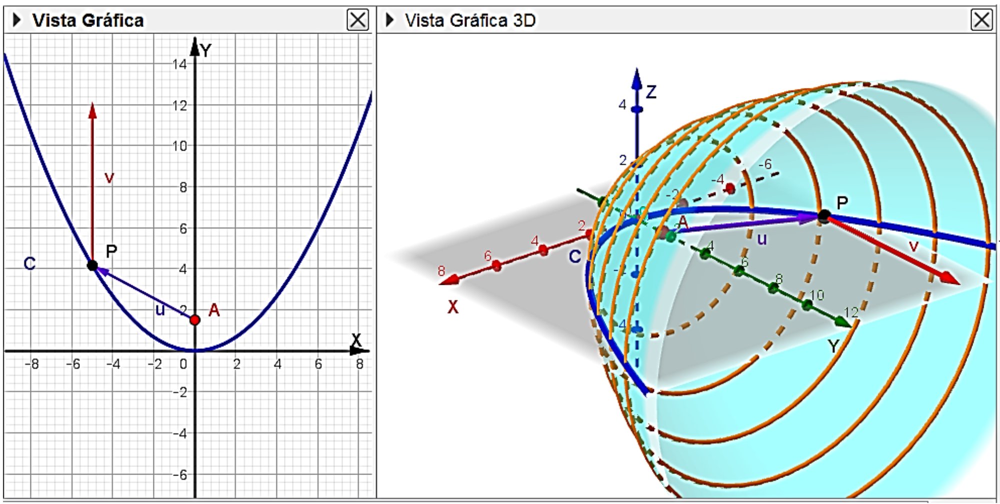


Figure 14. Modeling of the Hollywood Bowl in a DRE

In the 3D Graphics View the coaxial curves represent the structural semi-circumferences of the Hollywood Bowl, plane XY the stage, vector u the acoustic wave of the source located in A, and vector v the reflected wave oriented towards the spectators, thus creating a solid beam of acoustic waves in the shape of a semi-paraboloid. The optical property of the paraboloid can be seen in greater detail in the Graphics View.

This way, we see that the paraboloid, as well as the properties of its elements that allow representing it graphically and thus modeling different applications in architectural and artistic constructions, is a mathematical object that students of architecture should study carefully in order to take advantage of its structural and artistic qualities, revealing the essence of every successful designer or on the way to become one.

FINAL CONSIDERATIONS

In the present paper, we have evidenced that not all the apprehensions of the dynamic graphic register occur sequentially or in strict order; some can be developed with more emphasis and importance than others and, in general, there can be more than one solution criterion, depending on the educational level of the subject or subjects involved, as well as on their accumulated experience in putting their knowledge into practice.

Regarding the operative apprehension, unlike the others, it requires the use of a DRE in order to do modifications in the dynamic graphic register, without modifying the represented elements nor their algebraic representations. Such elements constitute the *visual variables* of the paraboloid in the dynamic graphic register because they allow representing it, and their identification as well as the elements to represent them are important in order to do conversions between representations of the algebraic and dynamic graphic registers and vice versa.

Also, this is in order to identify the algebraic representation of a paraboloid S when this one is translated and/or rotated in a new position S', such as it was observed in the case of **Figure 8**, where the algebraic representation of S' is different from S, which corresponds to a *dimensional deconstruction of the shape* where all apprehensions intervene, which requires a deeper study of the visual variables of the elliptic paraboloid.

We also emphasize that the DRE allows doing treatments, modifications and the development of apprehensions, as well as establishing conjectures and the construction of meanings, which we believe would be better assimilated in an environment of dynamic representations.

Finally, the results of the present research show the pertinence of the DRE in teaching and learning mathematics, particularly the Elliptic Paraboloid, as well as the need to use these advances in mathematics classes of higher education, particularly in the mathematics teaching of future architects. In regards to the connection between mathematics, art and architecture, it is evident that this one exists, and it is present in different constructions projected in architecture.

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