MODESTUM

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Mathematics teacher argumentation in a didactic perspective

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Abstract

The teacher's argumentation is considered essential to promote the student's mathematical education; the teacher's argumentation is linked to the teacher's didactic-mathematical knowledge and the achievement of instructional objectives. In this sense, a pragmatic proposal is required that links the teacher's didactic-mathematical knowledge with argumentation. Toulmin's (2007) model makes it possible to study the structure of teacher or student argumentation; in contrast, pragma-dialectic considers that in the face of a difference of opinion, the parties involved try to persuade their counterpart while observing argumentative discourse standards. In the context of an argumentative class, both argumentation models do not consider the particularities of the argumentation of a didactic nature that the teacher uses. This document presents a proposal for argumentation suited to mathematics education, based on a mapping of literature and a model of the teacher's knowledge, contrasted in an actual class. The analysis and discussion of data let us conclude that a teacher's argumentation is characterized by epistemic, cognitive, interactional, emotional, and communicative features that reaffirm not only the complexity of educational practices but also the teacher's argumentation for educational purposes and the adequacy of the proposed definition.

Keywords: argumentation, teacher knowledge, communication, didactic facets

INTRODUCTION

Argumentation has achieved status thanks to the impulse of philosophers, scientists, mathematicians, formal logicians, discourse analysts, and communication scholars with which it has become an independent object of study.

In the field of mathematics education, argumentation is essential for several reasons; on the one hand, insofar as some of its characteristics help to understand and explain didactic phenomena (Kazemi et al., 2021; Molina et al., 2019; Solar & Deulofeu, 2016); on the other hand, for its role in the students' construction of knowledge, as it has communicative nuances (Ayalon & Hersckowitz, 2017; Toro & Castro, 2020), rhetorical (Castro et al., 2021) and logical qualities (Conner et al., 2014; Knipping & Reid, 2015; Pedemonte & Balacheff, 2016).

Considering argumentation as an activity of thought "is not only the expression of an individual assessment, but a contribution to a process of communication between people" (van Eemeren & Grootendorst, 2011, p. 62); and because it is closely related to learning (Baker et al., 2019; Forman et al., 1998; Krummheuer, 2011; Staples & Newton, 2016), for example when the validity of a statement is discussed in math class.

While considerable research has focused on argumentation, limited research has focused explicitly on teachers' knowledge and practice in the context of argumentation (Kosko et al., 2014; Mueller et al., 2014). Nonetheless the teacher's support to promote argumentation is key in student's mathematics education (Conner, 2022).

Freinet (1969) supports the thesis that the joint activity of students promotes and supports learning. The results of the research support this hypothesis by Cecchini et al. (1972), who studied the effect of joint work on learning. Socio-constructivist and socio-cognitive theories suggest paying attention to the dialogical dimension of argumentation (Baker, 2009; Muller-Mirza et al., 2009; Rigotti & Greco-Morasso, 2009). In the sociocultural perspective on cognition and learning, argumentation is considered part of a dialogical process

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Contribution to the literature

- The paper defines argumentation in mathematics education.
- The paper links mathematical argumentation and the didactic mathematics knowledge model of teacher knowledge.
- This paper proposes a definition of teacher argumentation in mathematics education, which includes elements of the pragma-dialectical theory and elements of the didactic-mathematical knowledge. The definition is contrasted with the argument of a teacher during mathematics classes to validate and adjust the definition to its daily use in the classroom.

between students and teachers that favors interaction and ideas discussion. Vygotsky (1978) considers that argumentative dialogue leads to the development of higher-level mental processes such as critical reasoning and reflection (McAlister et al., 2004).

Educational practices informed by the latter point of view place importance on students examining alternatives as a condition for developing argumentative reasoning and promoting learning (Kuhn et al., 1997; Terenzini et al., 1995). Simon (2008) reports that introducing argumentative dialogue in science classes improves high school students' understanding and engagement (e.g., Erduran et al., 2022; Evagorou & Osborne, 2013; Osborne, 2010; Sampson & Schleigh, 2013). Through structured group work, and students are encouraged to discuss and evaluate the explanations and accounts on which scientific theory is built. Joiner et al. (2008) report how the structural quality of the argumentation was higher for students from different programs of study who actively participated in discussions. It is agreed with Baker (2016) that the interest in argumentation refers to 'arguing to learn' (Andriessen et al., 2003) instead of 'learning to argue'.

A literature mapping allows for identifying the construct of "collective argumentation" (Conner et al., 2014), which takes up Toulmin's (2007) model and is used as a means to explain the role of the teacher when arguing in mathematics class (Yackel, 2002), as well as to investigate how the teacher can present more sophisticated arguments, in addition to the types of guarantees or support necessary to present them (Yopp, 2012), or to examine the direct contributions of teachers to the arguments, the types of questions they ask and other supportive actions (Conner et al., 2014).

Nardi et al. (2012) proposed an approach in which the Toulmin's (2007) model was adapted with the refinement made by Freeman (2005) on collateral classification; and Metaxas et al. (2009, 2016) used the Toulmin's (2007) model and the analysis of argumentation schemes (Krummheuer, 2011; Weinberger & Fischer, 2006; Willard, 1976), to analyze the argumentation structure of mathematics teachers. Nevertheless, more than using or taking up a particular model, some authors are interested in an integral argumentation approach (Demiray et al., 2022) and present types of global argumentation structures used by future mathematics teachers, which they classify as mono structures and hybrid structures (Hunter, 2001; Peldszus & Stede, 2016).

Other contributions deal with how teachers learn to promote discussions during class so that students participate in the argumentation (Kazemi et al., 2021); justify the conclusions; communicate them; and respond to the arguments and refutations of others (Kosko et al., 2014), concerning conditions to promote the development of argumentative competences in the mathematics class (Solar & Deulofeu, 2016), or how the teacher, the class and the mathematical subject shape the opportunities of the students to engage in an argumentative activity (Ayalon & Even, 2016).

The previous contributions emphasize specific aspects of argumentation, but in the literature, no approach studies argumentation in the classroom context as one of the competencies that the mathematics teacher must exhibit. Argumentation, seen as part of classroom practices, is closely related to mathematical knowledge and the teacher's didactic knowledge (Boero et al., 1996; Castro et al., 2021; Pino-Fan et al., 2015, 2022). Various authors have studied the knowledge of the mathematics teacher (Schoenfeld & Kilpatric, 2008; Shulman, 1986; Silverman & Thompson, 2008), Pino-Fan et al. (2015), who identify, describe, and relate various domains of teacher knowledge. Exploring the relationship between some facets of the teacher's knowledge and its use in an environment that promotes argumentation is convenient. The teacher expresses didactic knowledge through locutionary acts, oriented by didactic intentions to promote student understanding with which the teacher's argumentation exceeds the student's acceptance of his statements. Knowing the characteristics of the teaching argumentation of a pragma-didactic nature can help improve students' mathematical training.

The following section presents the theoretical framework, a discussion about the argumentation in mathematics education, the context and the participants, the episode, the discussion, and finally, the conclusions.

THEORETICAL BACKGROUND

One such model is the so-called didactic-mathematical knowledge model (DMK) (Castro et al.,

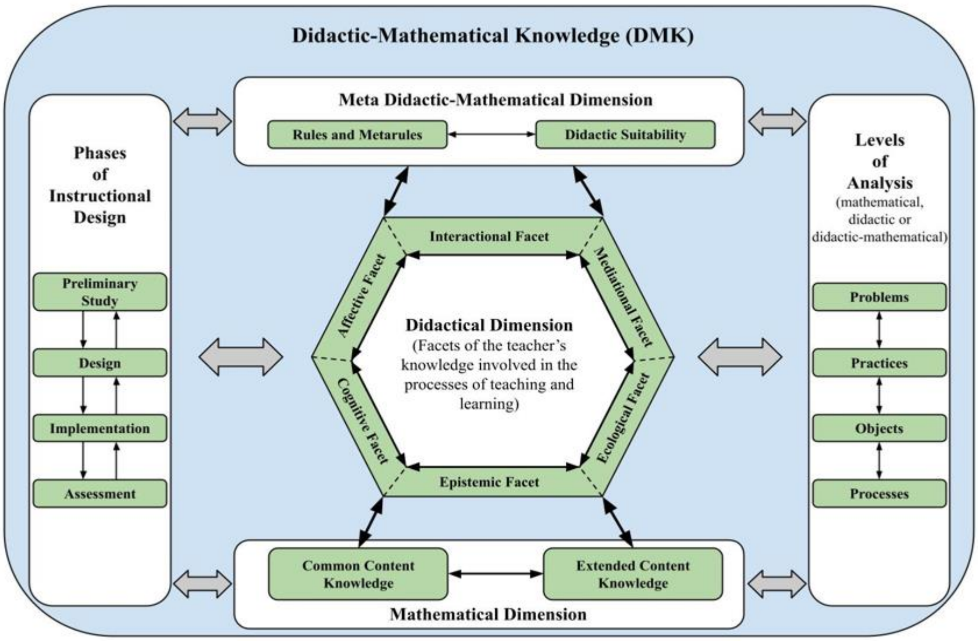

Figure 1. Dimensions & components of didactic-mathematical knowledge (Pino-Fan & Godino, 2015, p. 103)

2018; Pino-Fan & Godino, 2015; Pino-Fan et al., 2018). The international research has led to conceptualizations and models on teachers' knowledge to teach mathematics. Various authors have studied the knowledge of mathematics teachers (Hill et al., 2008; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008; Shulman, 1986), none offer tools to analyze the didactic knowledge. DMK (Pino-Fan et al., 2015, 2018) offers epistemic, interactional features and mediational facets allow studying the didactical dimension for teaching. DMK model interprets and characterizes the teacher's knowledge from three dimensions (**Figure 1**): mathematical dimension, didactical dimension, and meta-didactic mathematical dimension.

DMK's mathematical dimension refers to the knowledge that a teacher needs to teach mathematics or guide classroom mathematical activity, the didactic dimension refers to the set of knowledge required to put the teaching into action, and the meta-didactic dimension refers to the norms–institutional, and cultural–that regulate the teaching and learning process. This model has been widely addressed in Pino-Fan et al. (2015). The paper explores epistemic, interactional, and mediational facets, and propose the framework is useful for investigating and possibly enhancing how teachers support students' reasoning and argumentation as fundamentally mathematical activities. DMK's mathematical dimension refers to the knowledge that a teacher needs to teach mathematic or guide classroom mathematical activity.

This model recognizes that the teacher must put into play knowledge about the mathematical content, interaction strategies with the student, resources to promote content understanding, the context, where both the instruction and the school are located, and emotional aspects to motivate, in addition to the student's cognition. This knowledge, grouped into facets, takes place in the classroom and is regulated by mathematical norms and meta-norms (Godino et al., 2007; Pino-Fan & Godino, 2015), manifested in the intentional communicative action of the teacher to educate in mathematics.

The argumentation develops in a close and complex relationship with the teacher's knowledge and becomes evident during classroom practices. Various elements used by the teacher and students are part of communication in class, and links have been reported with studies in argumentation (Lin & Tsai, 2016; Planas et al., 2018; Rowland, 2000; Ryve, 2011). These elements can be identified, problematized, and organized with findings made both in argumentation and mathematics education research. At the same time, a proper form of argumentation in mathematics education can be identified, including characteristics of didactic-mathematical knowledge, with formative intentions and characteristics of argumentation recognized in other areas of human activity (Gainsburg et al., 2016). Our intention in this document is to propose a definition of argumentation faithful to the complexity of teaching locutionary acts and their link with argumentation, put into action on mathematical instruction, and highlight the characteristics used in a math class.

ARGUMENTACIÓN IN MATHEMATICS EDUCATION

Different authors have argued that participation in argumentative discourse supports the development of argumentation skills (Billig, 1997; Graff, 2003; Kuhn, 2005). Some researchers have developed research that

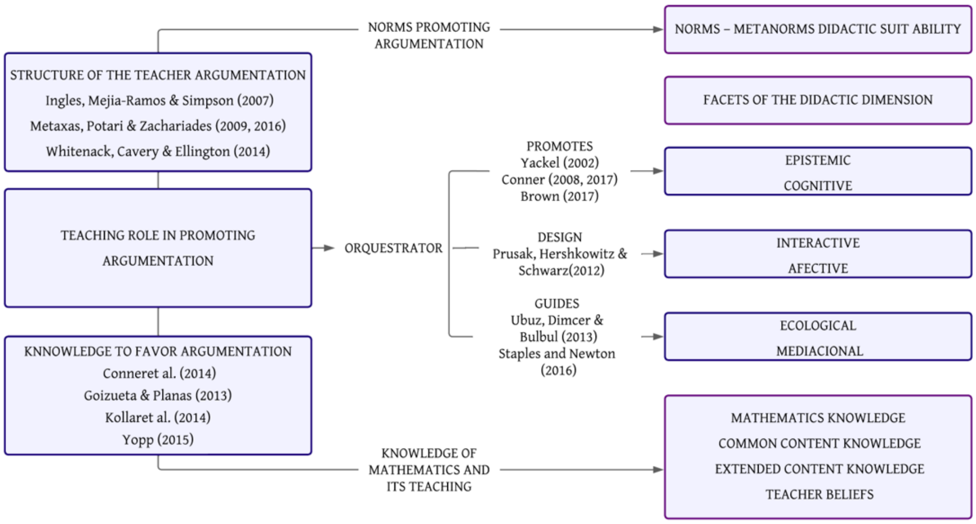

Figure 2. Links between literature on argumentation & DMK (Source: Authors' own elaboration)

employs peer interaction to facilitate the development of argumentative reasoning.

For example, Kuhn (2005) and Kuhn et al. (1997) developed research, where secondary school students were engaged in a series of dyadic discussions on capital punishment. Following these discussions, students' argumentation improved, and there was an increased meta-cognitive awareness of the coexistence of multiple views (Kuhn, 2005; Kuhn et al., 1997). Several other studies have reported similar findings (Anderson et al., 2001; Schwartz et al., 2003). These works support the study of argumentation in mathematics education to offer students opportunities to learn and develop mathematical skills appropriate for their future life.

Some terms associated with the tradition of argumentation must be adjusted to the characteristics of the vast body of knowledge available in mathematics education: models of teacher knowledge, learning difficulties, teaching strategies, social and cultural contexts, cognition, and language. **Figure 2** synthesizes a literature mapping, establishes a link with DMK model (Pino-Fan et al., 2015), and attempts to link two broad research areas.

The argumentation in class is associated with a dialectical perspective (Asterhan & Schwarz, 2016; Ayalon & Hershkowitz, 2018), which is linked with characteristics of DMK (Pino-Fan et al., 2015, 2022) that models the teacher knowledge required to teach effectively. The following definition is rewritten on the proposal of van Eemeren et al. (2014) and fits the context of mathematics education:

> "Argumentation is a communicative and interactional act complex aimed at resolving a difference of opinion with the addressee by putting forward a constellation of propositions the arguer can be held accountable for to make the standpoint at issue acceptable to a rational judge who judges reasonably" (p. 7).

The definition uses concepts proposed and studied by onto-semiotic approach to mathematical cognition and instruction (Godino et al., 2007). According to this approach to mathematical cognition and instruction, understanding a mathematical object occurs in terms of meanings, sequences of meanings, and social or classroom practices in which such meanings are discussed to solve mathematical problems.

Understanding within a school institution is interpreted in terms of the personal-institutional duality. The first refers to students' understanding -the cognitive facet of DMK–and the second refers to the teachers' understanding, contained in the books and regulated by the curricular programs-epistemic facet of DMK. A meaning conflict refers to a difference between personal meanings and institutional meanings. Comprehension refers to configurations of meanings associated with school mathematical objects.

Argumentation is not only a structural entity (Toulmin, 2007) but a complex communicative and interactional act that occurs through language (van Eemeren et al., 2006), which has an educational intention and considers mathematics knowledge specificities and knowledge about student cognition, which is developed in a social and cultural context, which follows interactional strategies, which use specific means to motivate participation under restrictions imposed by mathematical, curricular, and school administrative management regulations. The discussion is not always carried out under similar conditions. It is accepted that, at times, there is no discussion by the participants, only acceptance of authority criteria and replication of ideas. Communicative action (Sfard, 2020) is linked to cognition and is crucial to the understanding, recreation, and use of mathematical knowledge. In this proposal, argumentation is assumed as a process and as a product (Goddu, 2011; Zarefsky, 2014).

This proposal for argumentation in mathematics education, furthermore, requires considering the argumentative process as a complex communicative and interactional act aimed at resolving a conflict of meaning, and the product of the argumentation as a duality, where it is considered as a constellation of propositions enunciated to make acceptable the personal or institutional understanding, the point of view in question, and as the understanding that the student exhibits. The process and product of argumentation have a teaching component and a learning component that are interrelated. However, the perspective of the process is different for the teacher and the student.

van Eemeren et al. (2014), consider the acceptability of the argument of interest when the arguer defends a point of view before a recipient who doubts its acceptability or a different point of view. In mathematics education, acceptability is regulated by the epistemic guarantees of a mathematical nature–epistemic facet–and by previous student knowledge–cognitive facet–on which the construction of school mathematical knowledge is based. The argumentation in mathematics education must assume that it is reasonably judged based on mathematical knowledge in its personal-institutional duality. The teaching objective is not to promote acceptance of a point of view but to convince the students about epistemic acceptability, promote students' understanding, and account for DMK's cognitive facet.

To support or refute the point of view, the constellation of propositions refers to a semiotic configuration (Baker, 2016; Godino et al., 2007) that includes guarantees, which in mathematics refers to theorems or properties, while the backing usually is referred to as the explanations, where the argumentator resource to examples or cases.

In mathematics class, students do not assume a counterpart. However, in case of doubts about a teacher's assertion, indication, or explanation, or doubts about an answer or procedure different from the one presented by the teacher, or different answers to a task in the student's work, the teacher's argumentation is required. It could be affirmed that the teacher's argumentation is not required, only an explanation. However, in this case, the teacher's discursive resources are required to be staged, which is comparable to the presented argumentation definition.

Both students and teachers resort to different types of guarantees–a priori, empirical, institutional, and evaluative (Nardi et al., 2012) located in a mathematical epistemic frame of reference (epistemic framing) given that students' warrants offer evidence of their epistemic framing in their mathematics use.

The warrant can be expressed by a rule that acts as a bridge between the data and the claim; in other words, the warrant is the transition from data to claim. Warrants are used not only to guarantee that the relationship (implication) is valid but also to be taken as knowledge about definitions, theorems, properties, and statements based on the experiences of those participating in the argumentation process. In mathematics teaching, the equivalent of warrants is the theorems or properties, while the backing is usually referred to as the explanations, where students refer to examples or particular cases.

The point of view refers to an affirmation, the statement of a property, or a procedure in the form of a configuration of meanings that constitute the students' knowledge or the institutional knowledge of teachers.

In what follows, the previous definition will be used and adapted according to mathematics education features, and some elements of the proposal on the didactic-mathematical knowledge of the teacher (DMK) will be used. Class segments will be analyzed, and the argumentation's process and product will be considered.

The process includes the interactional and emotional facets, which refer to how and to whom it is said. Since the argumentation takes place in an educational context, with instructional purposes of educating in mathematics, the epistemic facet and the cognitive facet are included, which refers to what it says–includes configurations of meanings of mathematical objects-, why it says it, and why is he saying it? There is a difference between what is said in a daily communicative act and what is said in a communicative act in class when mathematical objects are discussed; in the latter case, the 'that says' refers to mathematical objects, relationships, modes of representation, and modes of use. Nevertheless, while the teacher refers both to mathematical objects of a complex nature and to configurations of meanings that are discussed, he must consider the student, including the cognitive facet. In this research, the mediational and contextual facets were not included.

CONTEXT AND PARTICIPANTS

In order to question the definition of argumentation, an experience was carried out in a mathematics class. The two teachers' classes were observed and recorded on video during September and October 2018 in Emma's and Daniel's classes. The teachers have experience teaching mathematics and hold postgraduate studies in mathematics education. Teachers had the autonomy to prepare their lessons and were provided with general information about the research. In addition, informed consent was obtained from the educational institutions' directors and the students' legal representatives. The observation was non-participant. A video camera focused on the teacher, either when addressing all the students or working with groups of students.

In this paper, Emma's class is reported. Emma's class is a tenth-grade class of 32 female students (15- and 16-

year-old). The math class has four hours a week, two hours corresponding to trigonometry, one hour for geometry, and another for statistics.

Emma followed the institutional curriculum plan consistent with the Colombian curricular standards. Therefore, the presented episode corresponds to a trigonometry lesson, where Emma and her students discuss the value of trigonometric ratios in angles developed during two lessons. In the first lesson, Emma and her students find the value of sine, cosine, and tangent, and in the second lesson, the students work autonomously based on Emma's indications. The episode being discussed takes place during the second lesson.

The episode presents the turns, participants, and intervention; the teacher's reactions to the students' interventions are identified. A placeholder with lowercase letters and the reaction code are included at each reaction's end. Finally, the episode is analyzed regarding our definition of argumentation and DMK facets (Pino-Fan et al., 2015).

EPISODE

This episode occurs at the lesson's beginning when Emma revisits some procedures discussed in the previous lesson. The analysis episode are presented first. A capital E names the students, and a number is added when only one student intervenes. Position markers are used, which are identified with the letters a, b, and c and account for specific didactic actions. In addition, codes are used: Exp for Explain; Ret for Retake; Apre-tra to Aprove and to translate; and Ave for find out.

1. Emma: [....] So we have the cotangent of 30°, the root of 3. The secant of 30° is two roots of 3 over three, and the cosecant of 30° to a … how much is it? B.

2. E: 2 […].

6. E1: Teacher, if he gave me ½ is it wrong?

7. Emma: Here she gave you a half [points to the answer that is projected]aAve, who else did she give ½ to? b Ave.

8. E: [Some students raise their hands].

9. Emma: Okay, so let's check what happened aAve. Remember that these trigonometric ratios are coming out of the triangle we built Find out guide and interpret bExp. What triangle did we make? cAve.

10. E: Equilateral.

11. Emma: The equilateral triangle is the one with all sides equal Apr-Exp. So, we had an equilateral triangle, and from that triangle, we took a triangle, half of the triangle, that triangle has an angle of 90°, 60°, and an angle of 30° [he says this while working on the board] bExp How long was the hypotenuse of that triangle? cAve.

12. E: 1 unit.

13. Emma: 1 unit aApr, how long was this side of the triangle? [Points to one of the legs] bAve.

14. E: Half unit.

15. Emma: Half unit aApr. And the other leg? bAve.

16. E: the root of three divided by two.

17. Emma: [Nods]aApr. So let us go back to cosecant bRet: What is the cosecant of any angle? key cAve.

18. E2: Hypotenuse on opposite leg.

19. Emma: Hypotenuse divided by the leg opposite Apr-Tra. So, I'm going to do the cosecant of 30°... This is going to be equal to bRet ... How long is the hypotenuse? cAve.

20. E3: 1 unit.

21. Emma: It is worth 1 unit to aApr. This is the 30° angle bExp. What is the opposite leg?... The one that measures root of 3 over 2 or ½? cAve.

22: E: ½.

23. Emma: ½, that is, it would be divided ½aApr-Tra. What was wrong with those who gave you ½? Well, different difficulties can arise; here we have a division between fractional numbers bRet. How are fractional numbers divided? What would we have to do? cAve.

24. E1: Internal and external law.

25. Emma: Apply the law of internal by external, true aApr-Tra. However, we need to put a number bExp. Where do we put the 1 that we need? cAve.

26. E: [Some students say above and another numerator].

27. Emma: So, let's remember that we also mentioned that last class. The fractional that they are already giving us is the one at the bottom, that is, the one we need to complete is the one at the top aRel. In other words, this would be over 1. Here we would have internals times externals [points to the board], that is, the cosecant of 30° is

Table 1. Characteristics of interactional facet

Characteristics	Actions
Affirmations for:	Find the answer to the task [1a] [1a].
	Examine a student's statement [7a] [1a].
	Explore the student's claim [9a] [1a].
	Expand on a student's statement [9b, 25b] [1a].
	Approve and extend a student's statement [11a] [1a].
	Extend a student's statement with the help of visual supports [11b, 21b, 27b, 29b] [1a].
	Approve a student's statement [13a, 15a, 21a, 29a] [1a].
	Take back a student's statement [17b, 19b] [1a].
	Propose an affirmation to resume previous participation and answer the initial question. [31c] [1a].
	Approve and translate the statement made by a student [19a, 23a, 25a, 31a] [1a].
	Restate the statement made by a student [27a] [1a].
	Disapprove of a statement made by a student [29a] [1a].
	Expand on a student's statement and present a solution to the task [31b] [1a].
Questions to:	Check student response [1b] [1a].
	Explore other (possible) solutions to the task [7b] [1a].
	Find out the procedure addressed in previous lessons [9c, 11c, 13b, 15b, 19c, 23c, 25c, 27c, 29c] [1a].
	Check concept understanding covered in previous lessons [17c, 21c] [1a].
	Retake student intervention and present possible answers [23b] [1a].
Gestures	Approve student' statement [17a] [1a].

going to be equal to bExp… When I multiplied the externals, where is the product of that multiplication? Where is the result? Is it in the numerator or in the denominator? cAve.

28. E: [Some say above and another numerator].

29. Emma: In the numerator aApr-Des. So 2 x 1... 2, divided by 1 x 1 … 1 bExp, and anything divided by 1?cAve.

30. E: Anything.

31. Emma: The same anything aApr-Tra. So, I would keep the cosecant of 30°, it is going to be equal to 2 bExp. Suddenly, the error was, where the 1 was placed or how the multiplication of internals by externals was done, ready cRet.

The argument begins in [6] with a question posed by a student who obtains a different solution. Emma notices in [7] that several students must correct the solution. However, she does not make it explicit and decides to check the answer [9a]. Next, it presents a constellation of propositions (van Eemeren et al., 2014) [9b-c, 11, 13, 15, 17, 19, 21, 23a], or configuration of objects and meanings (Godino et al., 2007) discussed in previous lessons to show that the answer is not correct; these actions are consistent with the reports by Giannakoulias et al. (2010) on teaching arguments to convince students about the validity of statements. The data and guarantees offered are related to the epistemic framing linked to the epistemic facet of DMK. The teacher mediates between opposing positions and tries to create a consensus to motivate the participation of the students (Hershkowitz et al., 2014). The teaching intention is to convince the students that the answer is not ½. The teacher's sensitivity to favor participation, interaction, and in some cases, the discussion is highlighted, which allows us to intuit certain aspects that one of the teacher's roles is associated with "guiding and interpreting" class discussions (Ayalon & Even, 2016; Azmon et al., 2011). In [23a], there is a primary closure; the final closure is presented in [31b], where Emma ensures all students are convinced that the cosecant of 30° is 2. The fact that the teacher seeks to favor participation and discussion indicates that he could eventually favor collective argumentation and 'pay attention to learning' (Conner, 2008).

Other interventions [23b-c, 25, 27, 29, 31a-c] refer to students' configurations of procedures that cause meaning conflicts; the teacher manifests knowledge about conflicts of meaning associated with the topic–cognitive facet–and about the interaction with the students–interactional and emotive facet–, uses a calm tone of voice, which encourages the participation of the students, which is evident in subsequent interventions [24, 26]. Throughout the episode, it is possible to appreciate several patterns in the teacher-student interaction. First, it is appreciated that the teacher plays a crucial role as a guide-support and guide-questioner and acts as a "manager" of opportunities to engage the students (Bleiler et al., 2014; Staples & Newton, 2016). From **Table 1**, it can be identified that the interaction pattern between the teacher and the students refers to teacher-student-teacher.

The teaching argument arises when there is a difference of opinion–a conflict of meaning–manifested by the student with a different answer; the teacher decides how to indicate the error. How to convince the students that the answer needs to be corrected? How to use prior knowledge? and to do so, it must use epistemic guarantees–epistemic facet–known to the students–

Table 2. Characteristics of interactional facet

Characteristics	Actions
Participation	Involve students in answering the task [1]. Engage students in answering questions posed by the teacher or by a student [9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29].
Means and norms	Use the means available in the class (board, computer) to support the justification [11, 13, 27].
Convince	Convince students of the answer to a question posed by a student [9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31].
Discuss	Noticing mistakes made by some students and discussing them with the other students [7a-b].

Table 3. Characteristics of epistemic facet

Category	Actions
Mathematical object treatment	Propose properties of the mathematical object associated with the solution of the task [1]. State properties of the mathematical object associated with the answer to a question [9b, 11, 13, 15, 17, 19, 21].
Go back to other lessons	Review mathematical objects studied to answer a question. Review solution procedures already studied [23c, 25, 27, 29].
Treatment of meaning conflicts	Identify conflicts of meaning in response to the task [7, 9a]. Make sure that the conflict of meaning has been solved [21b, 31c].
Justify or refute	Present justification about using concepts associated with a question's solution [23b, 27a, 31b-c].

cognitive facet–, it must refute and involve the students–interactional facet–, it must motivate participation and the manifestation of disagreement–emotional facet–. The teaching action encourages the student to recognize the error and convinces her that the answer is not ½ but 2. The student accepts the epistemic guarantees used by the teacher.

Next, the communicative characteristics of the interactional facet are stated.

Teaching actions entail specific intentions: expand, approve, disapprove, explore, or resume student statements to answer questions asked and present a conflict of meaning in the solution of the task, verify answers, explore other possible ways of solution, investigate student procedures, or test understanding of concepts. The teacher invites student participation and tries to convince and involve the students in the discussion. The question posed in [1b] reveals the verification reaction and the use of gestures to approve the affirmation of a student–interactional facet–. **Table 2** shows the characteristics of the interactional facet.

The interventions and questions of the students encourage the teacher to use her didactic-mathematical knowledge (Pino-Fan et al., 2015), shown in **Table 3**.

In relation to the characteristics of the epistemic dimension, actions related to the treatment of the mathematical object, retaking other lessons, treatment of errors or justifying are distinguished. These actions show the knowledge and professional experience of the teacher. It is conjectured that without the mathematical knowledge and without the experience, the episode would have concluded in [7], when the error is specified, and the correct answer is indicated. Concluding the interaction at this point could be seen as wasting a good opportunity to close the argument, since it is at this point that Emma's argument begins to have other types of interpretations, not only in relation to the analysis of class discourse, but with the implications for learning. From this turn of speech, Emma tries to:

(1) present a justification that indicates a conflict of procedural meaning,

(2) explain and try to persuade the students of the correct answer when she insists on emphasizing the care with the treatment of the associated mathematical objects,

(3) link the task with other lesson tasks, and

(4) favor the participation of the students during the validation; the presentation of justifications so that they assume a position and present their arguments.

The teaching work in the classroom is carried out through spoken language that includes the form, identity, acts, and intention of what is said (Gee, 2008). Since the argumentation is expressed orally and by a group of participants (Knipping & Reid, 2015), some elements of speech acts are considered. Ruthven and Hoffman (2016) take up some aspects of IRF initiation-response-feedback to analyze the reactions of teachers and students in class, which refer to the teacher's actions when faced with a question.

Table 4 presents the different reactions of the teacher (Ruthven & Hoffman, 2016) to the participation of the students; a code has been included to analyze the data.

The teaching reactions are linked to the didactic dimension of the teacher's didactic-mathematical knowledge (DMK). For example, when the teacher examines a student's participation, he must interact (interactional facet) to encourage students to participate–but it does so about the student's previous knowledge and interprets the possible conflicts of meaning–epistemic facet and cognitive facet–. Teacher reactions

Table 4. Types of teacher reactions to a student's intervention (adapted from Ruthven & Hoffman, 2016)

Type of reaction	Associate DMK facets	Characterization	Code
Approve	Epistemic	Explicitly indicate approval of the student's intervention.	Apr
Disapprove	Epistemic	Explicitly indicate disapproval of the student's participation.	Des
Repeat	Interactional Emotional	Repeat (essential part of) student's participation in the exact words.	Rep
Reassert Stakeout	Interactional Emotional	Reframe (key part of) student engagement in different words.	Rel
Translate	Interactional Emotional	Translate (essential part of) student's entry into an equivalent form or idea.	Tra
Redirect	Interactional Emotional Epistemic	Redirect the knowledge shown in the student's participation.	Red
Figure out	Interactional Emotional Cognitive	Find out (probe, explore, & examine) participation of the student.	Ave
Expand	Epistemic Cognitive	Expand (enlarge) student participation or draw on it.	Exp
Retake Resume	Interactional	Return to (restate, refer to) the previous question or participation.	Ret
Inquire	Interactional	Pose a question	Inq
Transfer	Interactional Emotional	Transfer consideration of the student's participation to another student or the class.	Trn

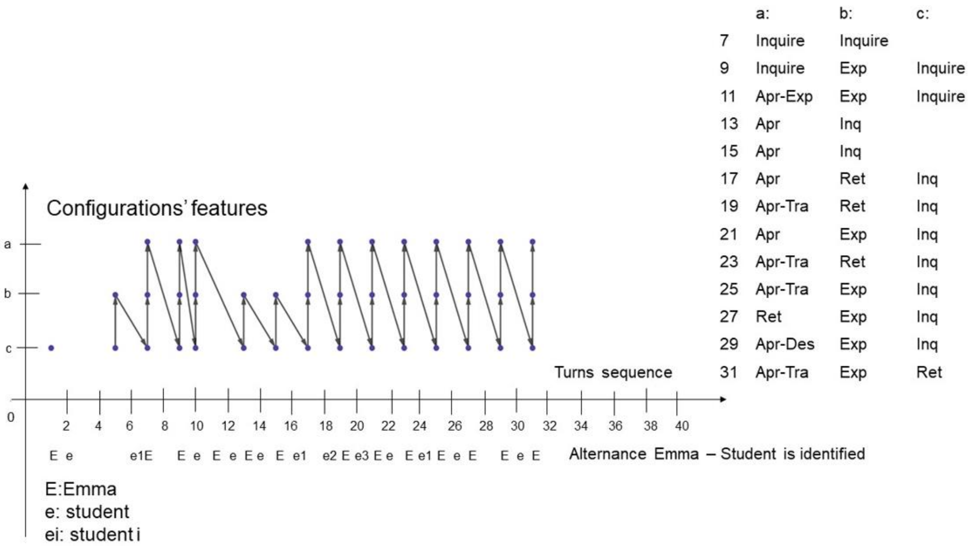

Figure 3. Pattern of interactions (Source: Authors' own elaboration)

promote the student's argumentation understanding through a constellation of meanings about mathematical objects put into play by the teacher. According to Conner and Singletary (2021), the teacher's role in supporting collective argumentation is well established.

DISCUSSION

The episode shows that the teacher guides the discussion (Hershkowitz et al., 2014), which promotes student participation. A question activated the teaching argument to convince students (van Eemeren et al., 2014), not just persuade them or answer their questions. From **Table 1**, **Table 2**, **Table 3**, and **Table 4**, he obtains graph 1, which shows the interactional sequence between teacher E and the students, which is supported by the argumentative teaching pattern that begins in a (explain), continues with b (explain) and ends at c (explain), then starts again at "a" in the next interaction. Although the "explain" pattern is found in the three moments of the teaching activities, it has a different intention and is given in response to different motives (Vygotsky, 1986). The interaction ends in c), whose intention is to "find out," and given the student's reaction, the teacher restarts her subsequent intervention with "a" (**Figure 3**).

The argumentation duality "process–product" is considered in terms of actions to achieve students' understanding. The product refers to students' understanding that students show once the presentation of a mathematical topic has finished. The interaction highlights the teaching actions that are in debate with the student's answers. The teacher inquires, approves solutions' relevance, expands her explanation or students' response, and based on the answers obtained, she extends her explanations. Each new interaction starts after asking. The teaching action regulates their explanations based on the responses of the students.

The students recognize the result of the argumentative process in terms of understanding; the process is developed according to the teacher's management, but the result is appreciated in the student's understanding. The process-product duality could be considered for both the teacher and the student; however, sufficient data robustness to validate the hypothesis is not available.

The argumentation includes disagreement and criticism (Asterhan & Schwarz, 2016), 'disagreement' when the students find a different answer than the one presented by the teacher, and 'criticism' by the teacher when pointing out the conflict of meaning, retaking the student solution procedure and identify knowledge that explains possible conflicts of meaning. Disagreement and criticism occur in an environment of respect and openness, where the students seem comfortable with their class participation. Without the participation of the students, an excellent argumentative process is not achieved with which the emotive facet is essential to promote argumentation in the classroom. Argumentative practice occurs when both teacher and students participate, which requires conditions to promote it. It does not occur spontaneously. Therefore, there needs to be more than the meeting between teacher and student to promote.

The teacher's questioning strategies are also highlighted. Although the questions generally appealed to memory, which according to some authors, might not be argumentatively effective (Kosko et al., 2014), the teacher's argumentation could be recognized. Furthermore, the questions evoke previous lessons, where the value of other trigonometric ratios had been discussed, so the questions are intended to evoke solution procedures carried out before, to be used to answer a wrong question from a student, and therefore they become a favorable condition for argumentation in the class.

The data in **Table 1**, **Table 2**, **Table 3**, and **Table 4** show the various teaching actions that promote the students' participation and are in some of the facets of the didactic dimension of didactic-mathematical knowledge (DMK). The teaching objective is to achieve students' understanding as the goal of their argumentative work.

CONCLUSIONS

The research shows that the mathematics teacher's argumentation characteristics constitute a complex formed by two articulated dimensions: interactional and epistemic, whose objective is to educate students in mathematics. Furthermore, the argumentation process refers to a configuration of actions that correspond with facets of didactic-mathematical knowledge–DMK, which are linked to argumentation characteristics and occur continuously, not spontaneously, and in response to questions or actions by the students. The interactional facet is recognized by teachers' reactions for identifying their actions in each characteristic and as valuable the contribution of this research by expanding said typology by including other reactions. It should be pointed out how the different communicative actions favor observing the participation that promotes understanding with which the teaching and learning participation is recognized. For example, Emma seemed interested in her students discussing the proposed tasks rather than accepting her statements.

Additionally, it is appreciated how the teacher engages students, how she presents justifications to convince students that answers are correct, how she uses students' questions to discuss and promote their participation, and how she seems to be interested not only in correcting an answer but participating and understanding some of the meanings associated with mathematical objects when solving a task.

Regarding the characteristics of the epistemic dimension, some notable aspects are identified:

The link of this dimension with professional knowledge, Emma's actions indicate her experience to the degree to which the research was done, since it was identified how she raises justifications for specific procedures, insisting on when, how, and why they should be done.

The way to deal with errors, which is used to promote task discussion, is recognized by students as an opportunity for learning.

The treatment of mathematical objects required appropriately naming the objects of each task and constantly insisting on understanding the associated semiotic configurations.

The definition of argumentation proposed, and the validation carried out in Emma's classes shows that it is appropriate to describe an argumentation typical of school mathematics because they allowed selecting episodes of class lessons that correspond to the argumentation of the teacher, which is not an easy task since during one hour of class different events take place, sometimes facilitated by the teacher and others that are appropriate to the requirements of the context. This form of argument requires conditions to promote it, firstly, the teaching disposition and secondly, the orchestrated use of the different facets of the didactic dimension of DMK.

Argumentation is a way of promoting students' mathematical training. However, argumentation in mathematics education differs from other forms of argumentation–political, legal, scientific–in terms of objectives, participants, epistemic and cognitive conditions, and emotional to get the parties involved. Therefore, the document recognizes the relevance of the definition of argumentation that could be named as a 'pragma-didactic' definition that recognizes that argumentation in mathematics education not only requires both conditions to activate it but also a teaching disposition that recognizes value of the argumentation in the classroom as a condition to promote mathematical education and student understanding.

The "pragma-didactic" definition of argumentation requires not only validation in environments, where the teacher promotes student collective argumentation but also determining how the teacher regulates argumentation based on mathematical didactic knowledge. The pragma-didactic definition of argumentation seems adequate, but longitudinal studies are required to validate and refine it. The relationships between DMK model and argumentation seem promising, but relationships of incidence and priority between both research domains have yet to be established.

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