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Modelling the Relationships Between Quantities: Meaning in Literal Expressions

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Algebra is often considered as difficult and mysterious doctrine due to numerous symbols that represent mathematical notions. Results of the research on students' interpretation of literal expressions show that only a small number of students are ready to accept that a letter can represent a variable. The aim of this research with students of the fourth grade of elementary school was to examine the impact of the use of various symbols, which represent the relationships between the quantities, on students' understanding of the meanings of literal expressions. We used experimental method (with one group) and the testing techniques (pre-test and final-test comparisons). The research results show that the iconic representation of the structure of literal expressions, namely, giving meaning to a term through its reliance on certain schemes in the learning process, significantly affects the ability of students in modelling generic quantitative relationships, and in the development of the meaning of letters in the role of algebraic variable.

Keywords: Literal expression, generalization, visualization, modeling, scheme.

INTRODUCTION

Devising new approaches to teaching algebra, which would successfully develop the meaning of algebraic symbols and concepts among students, has been the focal point of international mathematical community in the last few decades. The development of the meaning of symbols, and especially the development of the basis on which the manipulation of symbols was devised and followed by adequate representations is the primary goal of early algebra learning. Algebraic symbols, which function as a highly effective means of thinking for some, are for others a great obstacle in communication. Analyzing the language of algebra, Radford (Radford et al. 2005, p. 685) claims: „Algebraic symbolism does not possess a wide variety of means such as adverbs, adjectives and noun complements which have the main role in written and spoken language. Instead, it offers precision and succinctness which are governed by a few syntactic rules." However, the ability to comprehend how such precision and succinctness function is often a big problem for students of different ages, which was shown in a number of studies aiming at understanding students' mistakes in the interpretation of algebraic symbols (Kuchemann 1981; Stacey nnd Macgregor 1997; Kieran 1996, Radford et al. 2005, etc.). Kuchemann (1981) corroborated in his research that the comprehension of the meaning of literal expressions was difficult even for high school students, as he was exploring the way in which high school students understood algebraic expressions. He concluded that a large number of students had difficulties understanding literal expressions, because they did not accept them as the answer (solution), since they expected the number. He categorised students' answers into six levels of interpretation of literal members: a) The letter is estimated: it is at first given numerical value; b) The letter is disregarded: it is ignored or no meaning is given to its existence; c) The letter is treated as a concrete object: it is regarded as an abbreviation for a concrete object or as an independent concrete object; d) The letter is regarded as a specific unknown: it is regarded as a specific unknown number; e) The letter is estimated as

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State of the literature

- Understanding the term "expression" (both numerical and literal), its meaning and structure, as well as the acceptance of expression as an independent object and the use of its properties constitute fundamental knowledge for further development of algebraic knowledge and skills.
- Some authors see the key difference between arithmetic and algebraic approach in the absence of structural understanding of numerical and literal expressions and regard it as the cause of numerous problems that the students encounter when learning algebra.
- Despite their theoretical orientation, the results of researches agree in the following: algebraic syntax is not transparent.

Contribution of this paper to the literature

- Our aim is not only to comprehend the nature of mistakes and analyze the misinterpretations of algebraic expressions, but also to devise the ways of representing concepts and nature of students' and teachers' activities, which lead to the development of the notion of a variable and literal expression as a mathematical object.
- We aim to address the issue whether fourth-grade students, who are exposed to algebraic content from the first grade (equations, literal expressions), have a developed meaning of literal expressions.
- We aim to address the issue whether the way in which quantitative relations are represented constitute a significant factor in creating the meaning of literal members? What is the nature of iconic means which instigate algebraic thinking and generalizations.

a generic number: it represents or at least could represent several values, not just one; f) The letter is regarded as a variable: it represents a sequence of indefinite values. Although the interpretation that the students use depends on the nature and complexity of the question, only a small proportion of students were able to understand the letter as a general number despite school experience in representing numerical generalizations. Subsequent researches on students' interpretation of literal members confirm the results which show that only a small number of students were able to accept the letter as a variable. It is understood as a concrete object and is given numerical value, or is ignored, etc.

Conventional notation helps abstraction and generalisation. Symbolic mathematical language is precise and concise. However, if symbols are introduced without adequate basis which give meaning for symbol manipulations, students can develop early formalization and for them symbolic language can become semantically empty. A premature use of symbols independently of their meaning always leads to formalism.

Most research studies on understanding and interpretation of literal expressions was conducted with students of final grades of elementary school, often with a focus on the difficulties and misunderstandings. The above studies generally confirm Kuchemann's results, which show that only a small number of students are able to accept a letter in the role of a variable. The main goal of our study was to show that even with younger elementary school students it is possible to develop a structural understanding of the idea of representation with a letter, which is a fundament for further development of algebraic knowledge and skills.

Literature review shows that numerous studies have focused on the relationship of the use of visual representations of algebraic notions. However, there is a lack of studies on the relationship between the different representations and their impact on algebraic thinking. Empirical studies have shown that the transition from the concrete to the abstract representations is not easy. The aim of this work is to conceive and analyse the teaching strategies which are oriented to overcome the "cognitive gap" in the process of developing and linking arithmetic and algebraic generalizations. The research focus is on exploring the relationship of visual and verbal modes of representation of algebraic concepts, and their impact on the way students understand the literal expressions representations of algebraic notions.

THEORETICAL BACKGROUND

When the process-object theories emerged (Freudenthal 1962; Kieran 1996; Kieran 2004; Sfard and Linchevski 1994; Crowley et al. 1994 , etc.), it transpired that the main problem was that students regarded algebraic expressions as evaluation procedures and not as mental entities which can be manipulated. Different understanding and interpretation of numerical and literal expressions in arithmetic and algebraic approach is one of the essential differences between these two approaches, which can cause difficulties in developing algebraic thinking and skills.

After several research projects, in which she aimed to clarify and explore the problem of interpreting mathematical expressions, Kieran (1989, 1992, 1996, 2004) described two concepts of mathematical expressions:

- procedural (pertains to working on concrete numbers, results-based work).
- structural (operations on expressions as mathematical objects).

Kieran states that research showed that students, for instance, did not see the point of expressing the sum 8 + 3, but instead wanted to transform that sum into number 11. This way of thinking leads to the situation where expression x + 3 is meaningless to students. The results of research indicate that students understand the expression x + 3 solely as the addition of number 3 to x, whereas in algebra that expression represents not only the process of adding 3 to x, but also the object x + 3. They think that they should do something with it, but they do not know what. Students are not able to think of operations as focal points. In arithmetic the expression 8 + 3 is interpreted as a problem and is understood as „add number 3 to number 8". In algebra 8 + 3 represents number 11. Thus students see the expression x + 3 as the process of „adding number 3", and not as a solution per se. Two interpretations of the sum 8 + 3, arithmetic and algebraic, correspond to the concepts procedurally and structurally. Although proclaimed goals of school algebra are structural in nature, Kieran stresses that most algebra textbooks use procedural approach for the introduction of algebraic content.

Different understanding of mathematical expressions is also presented by Sfard (1991), who states that abstract mathematical concepts can be understood in two fundamentally different ways: operational (as processes) and structural (as objects). She claims that for the majority, operational concept represents the first step of adopting new mathematical concepts. The shift from grasping „the process" to grasping „the object" is accomplished neither quickly, nor without great difficulty. As they are fully developed, these aforementioned approaches play a significant role in mathematical education. We can notice that in the context of school algebra, the term procedural, which is used by Kieran, has the same meaning as the term operational, which is used by Sfard. Exploring the ways of shifting from arithmetic to algebraic thinking, Kieran (Kieran 1992, p. 392) states that "the study of school algebra can be interpreted as a series of process-object adjustments which students have to make in structural aspect of algebra". Most authors see the division between arithmetic and algebra as an ontogenetic gap caused by operational-structural dualism of the mathematical concept.

In traditional teaching, which is faced with students with little understanding of algebraic syntax, the only solution is often found in so-called "fruit salad" algebra (Crowley et al. 1994) or „mnemonic symbols" (McNeil and Weinberg 2010). The symbol "3a + 4b" may signify „three apples and four bananas". Some students, who accept this deception, are capable of interpreting the expression "3a + 4b + 2a" as „three apples and four bananas and two apples", which is "five apples and four bananas", or "5a + 4b". Thus it seems that they are capable of simplifying expressions. But, in that way, students develop the notion of letters represented by the object. Research has shown that many students regard letters as the representation of objects, rather than as numbers in equivalence relation. Such interpretation of letters excludes the interpretation of equivalence not only as structural mathematical object, but also as procedural interpretation. However, studies (Knuth et al. 2005, Asquith et al. 2007) show that students, who are under the age of 15, are capable of interpreting a letter in literal expressions as a variable. These results are, according to authors, explained by the curriculum that provides opportunities for students to work with literal symbols in ways which support the interpretation that a single literal symbol can stand for multiple values. Thus, materials based on this approach seem to help children develop a conceptual understanding of variables (Blanton and Kaput, 2005; Brizuela and Schliemann, 2004; Carraher et al., 2006; Kaput, 2000).

Recognizing the origin of misunderstanding algebraic notation is necessary for improving algebra teaching. The issues how to bring the language of symbols closer to students and how to develop their meaning stir a strong debate.

The relation between the use of visual abilities and students' mathematical abilities is an interesting field of research, although it seems that no consensus has been reached in this area. Anderson et al. (2008) found that the "spatial imagery" is an important factor in solving geometric problems. However, Tolar and collaborators (Tolar et al, 2009) concluded that "spatial imagery" related to the manipulation with symbols, has only a moderate effect on the achievements in algebra. They established that spatial visualization has effect on solving algebraic problems involving other aspects of algebraic thinking (generalization, modelling). In contrast to that, many studies many studies stress the importance of visualization in the problem-solving process (Andrade 2011; Radforad 2003; Kieran 1996; Abdullah et al. 2014). Numerous authors consider the use of different representations for illustrating problem situations as the important component of algebraic thinking (Kieran 1996; Duval 1999; Rivera 2010; Kabaca 2013). Radford (2003) stressed the importance of developing the meaning of algebraic expressions by using different semiotic means of objectification. Tall sees the entire school mathematics as „[...] a combination of visual representations, including geometry images and graphs, together with symbolic calculations and manipulations" (Tall 2008, p. 5); Mason et al. (2007) maintain that the use of several representations for the same concept can help students enhance their learning, as it helps them realise and express the generalized nature of the concept. The problems that students face in grasping and using algebraic symbols partly occur due to the drawbacks of deep mathematical structures which provide meaning to symbols, while visualization is a significant means of exploring mathematical problems and providing meaning to mathematical concepts and their correlations.

METHODOLOGICAL FRAMEWORK OF RESEARCH

The aim of our research was to examine the influence of using different representations to show the relations between quantities on understanding the meaning of literal expressions by fourth-grade students in primary schools (aged 9-10). This goal was operationalized through the following research tasks:

1. Explore to what extent fourth-grade students in primary school were capable of applying and comprehending the steps of modelling and iconic representation of general quantitative relations, prior to and after the application of the Model;
2. Explore whether fourth-grade students in primary school, prior to and after the application of the Model, understood literal expressions only procedurally, or whether they had a developed notion of expressions as independent objects with their own meaning.

The sample comprises 58 fourth-grade students (aged 9-10) from one primary school in Belgrade - 36 girls and 26 boys.

The research consisted of three phases:

a) Initial Testing: All students in the fourth grade had to solve a test of knowledge, which was designed to show whether they have developed understanding of numerical and literal expressions as structural expressions (allowing manipulation with them as with objects), or only as a processes. According to the existing curriculum of the Republic of Serbia, students in grade one encounter with the notion of the term of "unknown number" (in equations), and in second grade they should know how to read and write summation, difference, product and quotient using the letters. In third grade, they are expected to know to determine the value of an literal expression from a given value of a letter, to solve simple forms of equations and inequalities, and in the fourth of all these understandings are more profound. Since our study includes only students in fourth grade (at the end of school year), we assumed that these students have acquired and are able to understand the concepts of numerical and literal expressions.

b) Application of the Learning Model**:** (explained in detail below). Teachers who usually teach in these classes have organized a class, based on detailed scenarios prepared by the researcher, and previously discussed with the teacher.

c) Final test: This test was designed to show the impact of the Learning Model on students' achievements tested with a parallel version of the initial test.

The Learning Model

Existing literature has stressed the significance of using and connecting several different representations in the process of acquiring mathematical concepts (Dreyfus 1991; Duval 1999; Drouhard and Teppo 2004; Mason et al. 2007,etc.). Many authors (Kieran 1996; Lins 2001; Cooper and Warren 2011, etc.) connect algebraic thinking with the ability of iconic representation of quantitative relations. Radford (Radford 2011, p.319) introduces the phrase „contextual algebraic thinking", which emphasizes the fact that the meaning of algebraic symbols is deeply connected to other modes of representing algebraic ideas. In the process of developing algebraic abstractions and generalizations it is important to use different representations, which are developed from representing situations in the real world in abstract diagrams. Existing literature indicates several models of learning aiming at developing algebraic concepts (Lins 2001; Cooper and Warren 2011). Cooper and Warren use a variety of models (e.g., balance, line, function machine) and representations (e.g. natural language, figures/diagrams, symbols and, in later years, graphs) when introducing algebraic generalizations. In that process, the order of using representations is highly important, as the subsequent choice should compensate for the limitations of the previous one. (Cooper and Warren, 2011). Many authors stress the importance of hierarchy in the level of meaning of used representations in the process of developing adequate mental images (Hiebert and Carpenter 1992; Sfard 2000; Smith 2006; Cooper and Warren 2011). Representations that retain many of the original details of experience are called perception-based representations (Van Oers 2001). "Expressive" and communicative representations assume pointing to what is important, and they are more abstract than reality display (Terwel et al. 2009). Recent studies (e.g. Cai 2004; Koedinger et al. 2008) show that the abstract representations are more effective than the concrete ones in solving complex problems. Empirical studies have shown that the transition from the concrete to the abstract representations is not easy. Cognitive psychologists (Goldstone and Son 2005) have proposed a method called "concreteness fading" to address the issue. The authors defined the method as "a process of successive reduction of specificity" which aims at grasping a relatively idealized and de-contextualised representation that is still clearly associated with the physical situation which the model represents (Goldstone & Son, 2005).

Learning through images is clear, i.e. we understand it as immediate comprehension. In that sense, the images we use as the bearers of meaning of concepts have to contain as little noise as possible, i.e. they have to project the meaning in the most direct way.

All our concrete understanding of a number is preceded by the notion of a set of objects that we perceive. The way of grouping the elements is irrelevant for abstract understanding of numbers, but that noise is significant and meaningful in the construction of arithmetic expressions. We will highlight two types of meaningful noises in arithmetic. The first type is decadal grouping of elements. Decadal grouping of elements is the noise for abstract understanding of a number, but not for its decadal record. The second type pertains to schemes to which we react by adding, multiplying or creating other arithmetic expressions. Schemes are ways of grouping the elements which serve as a basis for creating and transforming arithmetic and algebraic expressions and provide their intuitive meaning to these expressions. Schemes are first perceived in real situations and are later represented in a paradigmatic picture which is not changed. These paradigmatic pictures become iconic signs to which students get accustomed and accept them as a way of representing information. Scheme is a meaningful iconic sign and all expressions which have realistic meaning in a set of integers have to adhere to a certain scheme. We maintain that the activity of connecting expressions to certain schemes is significant "firstly, as it provides meaning to expressions and secondly, it contributes to understanding the expression as 'an independent object'" Content development within the Model can be schematically represented in the way presented in Figure 1.

In the process presented by the diagram, the arrow indicates that by abstracting realistic situations (which are verbally expressed) we come to the iconic representation through schemes. This type of verbal representation is not the expression of generalization in a natural language (in the sense of Hewitt 2001 or Radford 2002). We consider the textual problems as concrete representations (Gerofsky 2009), because they develop presentations that can serve as a cornerstone in the construction of abstract concepts.

Instead, it is the mode of devising schemes, which are regarded as the form of generalization. Different situations taken from the realistic environment are reduced through abstraction to the same images (schemes) which represent corresponding mathematical relations. We devised highly abstract images, where all concrete situations can be represented by the same scheme. The level of abstraction of the applied representation corresponds to students' level of abstraction and generalization. We regard the use of such iconic images as a shift from the usual „from concrete to abstract". We encourage students to express generalizations in a natural language (Radford 2002) by asking them to describe schematically represented relations. We represent schemes with rectangular frames with a stated number of elements (regardless of their

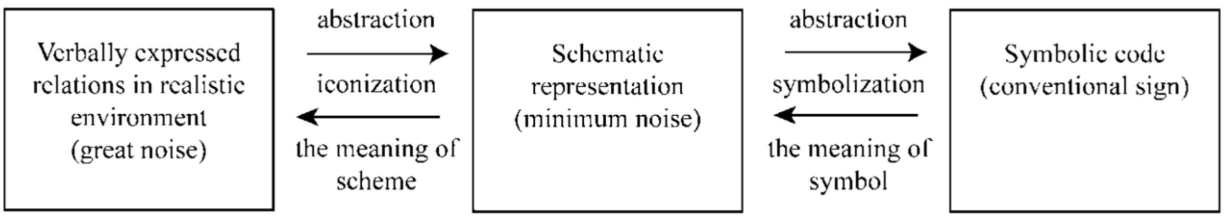

Figure 1. Diagram of the development of the learning model

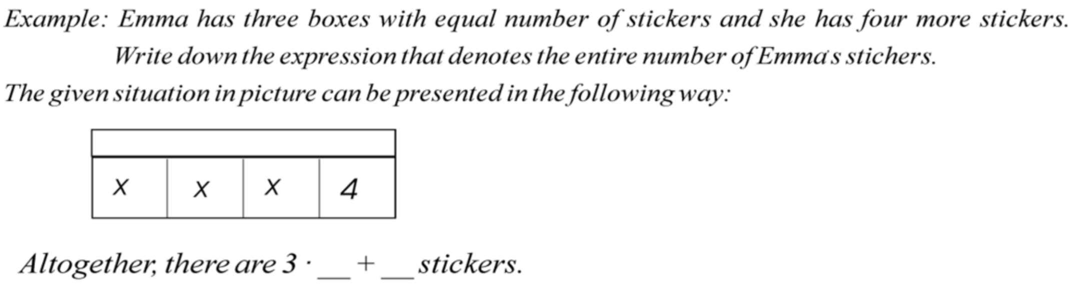

Figure 2. The examples of shematic representation of the meaning of literal expression

Table 1. Textual and Symbolic Representation of Literal Expressions – Differences Between Achievements on Initial and Final tests (Wilcoxon Signed Ranks Test)

	z2 - i4	z3 - i7
z value	-6.226	-5.825
Asymp. Sig. (2-tailed)	.000	.000

Table 2. Textual and Symbolic Representation of Literal Expressions – Differences Between Achievements on Initial and Final Tests (Ranks)

	z2 - i4	z3 - i7
Negative Ranks	0	0
Positive Ranks	48	42
Ties	9	15
Total	57	57

nature and position). The third component is the symbolic environment. We wanted to avoid stating generalizations in our model with the use of "quasi variable" (Fuji and Stephens 2001) and instead to create mental images which correspond to generalizations expressed by algebraic symbols. Space holders indicate empty fields (where a number or a letter should be filled in) and do not have numerical value. They are not equalled with literal signs, i.e. they do not have the role of a variable. The first component will be taken as an example for a concept, the second as the bearer of meaning for the concept and the third as a symbol for the concept.

When creating lessons, with which we wanted to develop the meaning of algebraic concepts for students, we devised examples in which textual tasks (which represent verbal description of real-life situations) were paradigmatically represented (in the form of schemes), so that students would get acquainted with such a form of iconic symbols and that the expressions, used by students, would acquire meaning through the link with the corresponding scheme. The meaning of literal equations is linked with their spatial codification in the form of schemes (Example in Figure 2).

In order to encourage deeper understanding, we went (in terms of requirements) in different directions of abstraction. Thus, after several similar examples, we asked the students to write down the expression, based on the picture, and then to think of an example (word problem) which corresponded to the picture (or expression). Those signs take the form of graphic schemes and are the bearers of meaning of the expression. It is our intention *to explore how and to what extent they contribute to the understanding of corresponding mathematical content.*

RESEARCH RESULTS AND THE DISCUSSION OF THE RESULTS

By comparing students' success rate on tasks which pertain to literal expressions and examine the ability of iconic representation of literal expressions after the application of the Model, we concluded that differences in students' achievement on initial and final tests were statistically significant at level .01. (Tables 1 & 2). The tables show ordinal numbers of the tasks from the tests which are marked in the following way: i1 (initial test, 1. task), z1a (final test, 1. task).

Students were therefore significantly more successful in understanding the structure of literal expressions and representation of that structure as a scheme after the application of the Model. By analyzing students' visual models from the initial test we wanted to examine the nature of visual representations with which students showed the meaning of expressions. Our analysis of the ways in which students modelled problem situations in order to express the structure of the problem indicated that schemes which our students constructed were non-algebraic in nature. We put the systematic mistakes which the students made during iconic representation of the structure of numerical expressions into four groups:

a) quantitative relations are not generalized;

b) iconic and syntactic signs are mixed;

c) the structure of expressions is ignored when devising the picture and

d) the structure and meaning of the expressions are misunderstood.

Due to the significance of the ability of iconic representation of quantitative relations on algebra teaching, we shall explain here some of aforementioned types of mistakes and illustrate them with adequate examples.

Let's analyze the iconic models derived from the example: *Sanja has 36 stickers more than Emma. Emma has x stickers. Present given relations with a picture.*

Iconic model (Fig. 3) shows that the student understood the mathematical structure of the given situation, but that *quantitative relations were not generalized* (he drew 36 stickers) and represents the example of *mixing iconic and syntactic signs*. This model contains a lot of perceptive details (Van Oers 2001) and does not express the meaning of the concept (Terwel et al. 2009).

Duval (1999) stresses that it is impossible to comprehend mathematics without considering the difference between the object and its representation. In order to separate the object from its representation, student has to be able to represent mathematical concept in at least two semiotic systems. We believe that two different modes of representation were mixed in this approach: real objects or images and symbols. Variable was represented symbolically, and the number of stickers was represented by drawing each sticker individually. The images that show the number of stickers correspond to the real situation which they represent and are not the result of abstraction of a real situation. Cooper and Warren (2011) used in their model a wide range of models and graphs in representing algebraic generalizations, but such approaches are hierarchically structured.

The following iconic model (Fig. 4) shows lack of understanding of mathematical structure and students' inability to transform the given situation into a mathematical expression. These pictures show disregard and misunderstanding of the mathematical structure of the expression. Students understood the requirement as the illustration of the context of the task. This tendency is in accordance with the results of numerous studies which show that students' self-constructed graphic representations have not always the appreciated effects on the learning process (DiSessa 2002; Terwel et al. 2009). The same tendency occurred when students devised iconic models based on symbolically represented expressions.

From the aforementioned, we can conclude that the comprehension of literal expressions and schematic representation of that structure was a difficult task for fourth-grade students. However, the reason for such results may lie in the fact that the students did not have the experience in modeling and iconic representation of quantitative relations. This claim is based on the results of the initial test, when we examined the process of iconic representation of quantitative relations in numerical expressions. In the following task:

Three boxes contain 41 beads each and the fourth one contains 205 beads. Write down the expression which shows total number of beads and represent with an image the situation which corresponds to this expression.

Students devised images which represented each unit of counting despite the fact that they included a large number of figures (Fig. 5). Although they recognized the structure of multiplicative-additive scheme, images correspond to the real situation which they represent and are not the result of abstraction of a real situation. This type of image, which the students used to represent quantitative relations, we cannot regard as wrong, but that it is non-algebraic in nature (Cai, 2004; Koedinger et al., 2008).

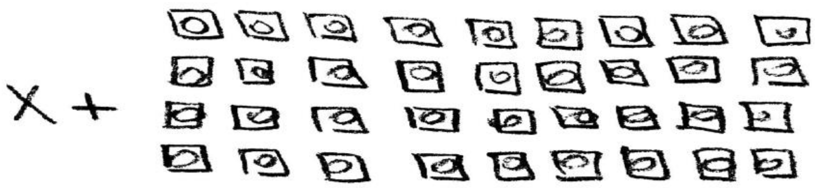

Figure 3. Quantitative relations are not generalized; iconic and syntactic signs are mixed

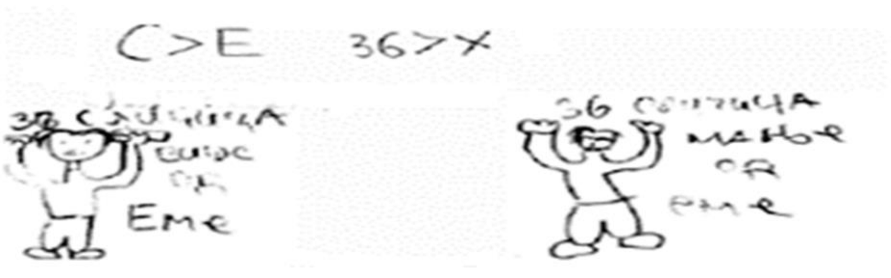

Figure 4. Disregard and misunderstanding of the mathematical structure of the expression

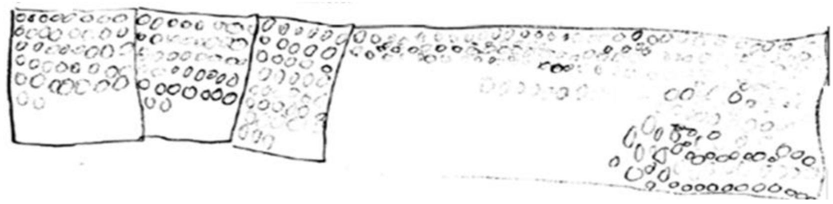

Figure 5. Quantitative relations are not generalized: the representation of quantitative relations in numerical expressions

Table 3. Structural Understanding of Literal Expressions – Differences in Achievement on Initial and Final Tests (Wilcoxon Signed Ranks Test)

	i5 - z1	i6 - z1	z2 - i4	z4vg - i9
Z value	-5,291(a)	-5,680(a)	-6,226(b)	-4,490(b)
Asymp. Sig. (2-tailed)	,000	,000	,000	,000

Table 4. Structural Understanding of Literal Expressions – Differences in Achievement on Intial and Final Tests (Ranks)

	i5 - z1	i6 - z1	z2 - i4	z4vg - i9
Negative Ranks	37	41	0	3
Positive Ranks	2	1	48	28
Ties	18	15	9	26

This is corroborated by the fact that we noticed a drastic change in nature of students' iconic representations after the application of the Model. Namely, students schematically represented the structure of literal expressions and those schemes were algebraic in nature. From all perceived and stated problems during iconic representation of the structure of expressions on initial test, we identified only *misunderstanding of the structure of the expression* on the final test.

After the application of the Model, we compared students' success rates on tasks which pertain to literal expressions and in which students were asked to write down literal expressions and accept them as a solution. The results indicated (tables 3 & 4) that differences in students' achievement on initial and final tests (parallel version of the tasks) were statistically significant at level 0.01.

We shall take a closer look at the obtained results:

Students showed statistically significantly better results on the final test compared to the initial one, when they wrote down literal expressions based on a word problem. The task on the initial test was:

Sanja has 36 stickers more than Emma. Emma has x stickers. Write down the expression which shows the number of Sanja's stickers.

Only 29.3% of students solved it correctly. There was also a parallel version of the task on the final test:

The jacket cost 125 dinars more than the trousers. The trousers cost x dinars. Write down the expression which shows the price of the jacket.

91.2% of students solved it correctly. Comparing these tasks on initial and final tests, where in the first step students should write down the expression and then represent the structure of the expression with a picture (scheme), thus showing the meaning of the expression, we concluded that students were statistically significantly more successful on the final test (at significance level 0.001).

Students showed statistically significantly better results on the final test compared to the initial one and when they manipulated literal expressions as objects without connecting them to meaning. While on the initial test the task: *Write down the expression which is 2 times bigger than the expression $x + 5$,* was solved correctly only by 17.2% of students, parallel version of the task on the final test (*Write down the expression which is 2 times bigger than the expression $x + 30$)* was solved correctly by 59.6% of students. Students were significantly more successful in the manipulation of literal expressions as independent objects after the application of the Model. For instance, when students respond to the instruction *Write down the expression which is 2 times bigger than the expression $x + 30$*, by writing down the expression $2 \cdot (x + 30)$ and accept it as the solution to the task, it means that they accepted the literal expression as a mathematical object and not solely as a process.

After the analysis of the solutions to the tasks from the initial test which pertain to exploring the ways in which students accepted literal expressions, we can conclude the following:

1) Solutions are directed towards the answer, i.e. towards finding the value of the expression: variable is given arithmetic value or the expression is equaled with the unknown ($x + 36 = x$).

2) Variable is ignored, i.e., letter is ignored or no significance is given to its existence.

3) Letter is treated as a concrete object.

4) Misunderstanding of mathematical structure of the problem and the structure of the expression.

The first three categories correspond to Kuchemann's (1981) initial categories. We shall analyze the solution of the aforementioned example.

The solution (Fig.6.) illustrates the misunderstanding of mathematical structure of the problem and the structure of the expression as well as procedural understanding of the expression (Kieran 1992; Kieran 2004; Sfard 1991; Sfard and Linchevski 1994). The solution is directed towards the answer, i.e. towards finding the value of the expression. Students could not accept the expression as the solution to the task, and so they equaled the expression with the unknown ($36 – x =$

x). In these examples, predominant mistake was the fact that the letter was given numerical value.

The following type of response (Fig.7), which illustrated the mistake of ignoring algebraic letters, showed that students did not accept literal expression (neither as an object, nor as a process). Students could not accept the expression as the solution to the task and therefore they equaled the expression with the unknown (36 – x = x). In these examples predominant mistake was that the letter was given numerical value. Only 29.3% of students solved the aforementioned task correctly. Students did not recognize the relation between two quantities and could not state the number of stickers with the expression x + 36. We believe that students could understand the relation „36 more“, but that the acceptance of literal expression as a solution is the cause of the problem (Kieran 1992; Crowley et al. 1994, etc.).

After the analysis of incorrect solutions to the tasks in which students were asked to write down the expression based on schematic representation of the structure of the expression, we noticed the same, aforementioned mistakes. Still, "the predominant mistake in those examples was that the letter was treated as a concrete object" (Fig.8). This type of mistake is stressed in Kuchemann's (1981) research. That students cannot accept literal expression as an object is indicated by the solutions to the same task in which students wrote down the literal expression correctly, but when devising word problems which correspond to that expression, they regarded the letter as a concrete object. A student, who wrote down the expression 3·m +3·n, devised the following word problem which corresponded to the picture: *Each of the 3 boxes contains 1 small ball, while other 3 boxes contain 1 big ball. How many balls are there altogether?* This type of mistake corresponds to the notion of algebraic syntax which Crowley et al. (1994) denote as "fruit salad".

From all perceived and stated problems, upon writing down literal expressions and accepting the expressions as the solutions to the task, we identified on the final test only misunderstanding of mathematical structure of the problem and the structure of the expression.

Many studies (Kuchemann 1981; Kieran 1996; Stacey and Macgregor 1997; Radford et al. 2005, etc.) highlighted the problem of understanding the literal expression among the students in the final grades of primary school, which was confirmed by the results of our initial test. However, the results of our study show that it is possible to develop procedural as well as structural understanding of literal expressions in younger elementary school students. The results of the initial testing indicate that the independent students' creation of representations to express meanings of literal expressions result with models that are not suitable for generalization (which corresponds to the results of Van Oers 2001; Terwel et al. 2009). By connecting the verbal, visual and symbolic representation, which we see as the scaling up the level of abstraction in the process of co-construction of representations (Ainsworth 2006), it is possible to develop the ability for modelling. As a result of this process, students independently designed the models that were adequate ground for generalizations and understanding of the fundamental notion of algebraic expression. It is worth noting that

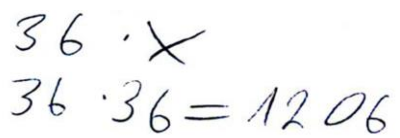

Figure 6. Misunderstanding of the structure of the problem; variable was given numerical value

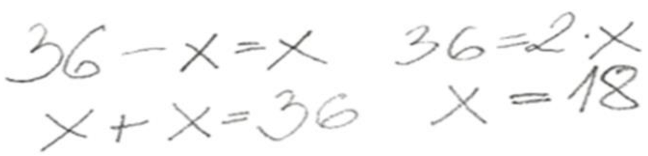

Figure 7. The letter was given numerical value. The expression was equaled with the unknown

Figure 8. The letter is treated as a concrete object

the study used visual representations of a high level of abstraction, devoid of details of real situations (Cai 2004; Koedinger et al. 2008; Terwel et al. 2009). The research results show that students, who on the initial test designed the models with many perceptive details from the real environment, during the final test created abstract visual models, which together with verbal representations have contributed to the structural understanding of the concept of literal expression.

CONCLUSION

The results of research questions that we raised in our paper indicate that lower grade students (aged 9-10) are able to develop procedural/structural comprehension of literal expressions and the concept of a variable. Iconic models that the students devised to express the meaning of literal expressions on the initial test are non-algebraic in nature. Predominant tendency noted in their results is the iconic representation of each unit of counting with images corresponding to real objects. Such tendencies lead to understanding the variable as a concrete object. We suppress excessive concreteness in our model by using schemes which were devised in such a manner as to represent a high level of generalization and abstraction and enable visual interpretation of abstract situations. We believe that the schemes instigate the creation of adequate mental images, which lead to symbolic representation that is procedural in nature. The return from the field of automatic memory to the field of images, realised as iconic symbols, would be feasible if the learning process went in the same direction, i.e. from images to symbolic codes. Thus, symbol operations are meaningful when they are followed by evoking inner representations (mental images) or when drawings of iconic representations express the full meaning. Excluding iconic representation of symbolic algebraic claims leads to misunderstanding and difficulties in acquiring such content (Radford 2011; Mason et al. 2007, etc.).

The results of the initial test revealed that students did not have a developed meaning of literal expressions as objects and characteristic mistakes that occurred in our students' answers correspond to those already stated in other research (Freudenthal 1962; Kieran 1996; Kieran 2004; Sfard 1991; Sfard and Linchevski 1994; Crowley et al. 1994, etc.). We found the cause of such mistakes in teaching methods which are focused on procedural comprehension of this content and in formal approach which excludes the development of meaning of algebraic symbols. After the application of the Model, students showed that they accepted literal expression as a mathematical object, which has its own meaning, and accepted the letter in the role of a variable. In order to accept literal expression as an independent object, which has its own meaning, numerical, arithmetic expressions have to be accepted as objects, and not only as a stimulus for counting (Freudenthal 1962; Kieran 1996; Kieran 2004; Sfard, 1991). The expression 8 + 4 has to be accepted as an object which relies on the meaning in the form of spatial additive schemes, and not solely as an invitation to calculate the sum, i.e. write „= 12“. Only on such grounds will the students be able to comprehend the structure and meaning of literal expressions.

Literal expressions and literal relations should be represented to students as sets of tasks to be understood in both ways: as generalizations of individual cases and as specifications of general rules. Crowley et al. see the problem in the fact that students comprehend the expression $x + 3$ as the process and not as an object. In order to deal with such an expression, it is necessary not only to attach meaning to it, but also that the meaning could correspond to it as a process (evaluation when x is known) and as an object which can be manipulated. Flexibility is required which would regard symbol as a „procept“(Crowley et al. 1994).

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