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Onto-semiotic analysis of Colombian engineering students' mathematical connections to problems-solving on vectors: A contribution to the natural and exact sciences

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Abstract

The mathematical connections Colombian engineering students activated when they solved vector problems were explored. The study was based on the extended theory of connections and the onto-semiotic approach. We followed a qualitative methodology that consisted of three stages: (1) selection of engineering students as participants; (2) application of a questionnaire with 15 tasks on vectors to the participating students; and (3) analysis of these data based on the theoretical articulation. The results show that students perform arithmetic operations with vectors, find the scalar and vector product, the norm of a vector, the angle between vectors, and unit vector based on mathematical connections (*procedural, meaning, different representations,* and *implication*), detail from an onto-semiotic point of view. However, some students have difficulty finding the angle between vectors because they misuse the norm. Furthermore, the new metaphorical connection based on mnemonics activated by the "*law of the ear*" is reported. The connections activated by engineering students to solve problems about vectors may have been influenced by the explanations provided by their calculus teacher, who promotes connections for the teaching and learning mathematical concepts.

Keywords: mathematical connections, onto-semiotic approach, vectors, engineering students, higher education

INITIAL CONSIDERATIONS

Mathematical connections play an essential role in different sociocultural environments and educational levels because they contribute to the mathematical understanding of students and teachers. It means that when a person uses mathematics in their daily life or solves mathematical problems (intra or extra-mathematical), in most cases, they establish connections by following a step-by-step procedure involving meanings, properties, propositions, representations, arguments, etc. (Berry & Nyman, 2003; NCTM, 2000; Rodríguez-Nieto, 2021; Rodríguez-Nieto et al., 2022a; Rodríguez-Nieto et al., 2023).

Likewise, mathematical connections have been integrated with STEAM approach to recognize and appreciate that mathematics is closely related to science, technology, engineering, and art. However, all these areas of knowledge start or emerge from a sociocultural context with an ethnomathematical essence (Rodríguez-Nieto & Alsina, 2022; Rodríguez-Nieto & Escobar-Ramírez, 2022). In this line, it makes sense to investigate the mathematical connections in engineering students, who require mathematics as a central tool for the design

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Contribution to the literature

- This article analyzes the mathematical connections of engineering students based on a combined use of extended theory of connections (ETC) and onto-semiotic approach (OSA), delving into mathematical practices, objects, processes, and semiotic functions (SFs).
- The potential of analyzing the connections in the written productions is shown that reflects on the one hand, the detailed responses of the students and, on the other, the difficulties of the students in finding the unit vector and the angle between two vectors due to their disconnection when finding the norm and do arithmetic operations.
- A new category of metaphorical (MT) connection based on mnemonics is proposed.

of structures, parts, software management, situation modeling, etc. (Mendoza-Higuera et al., 2018; Rodríguez-Gallegos, 2017).

An exhaustive exploration of mathematical connections has been conducted at various educational levels. For example, connections were studied in basic operations and verbal arithmetic problems among fifth and sixth-grade students (Frías & Castro, 2007). Additionally, Businskas (2008) investigated mathematical connections among secondary school teachers, focusing mainly on quadratic equations, and detailed a model that categorizes the connections into different representations: instruction-oriented, part-whole, procedure, and implication. In the field of pre-university education in Mexico, the research of García-García and Dolores-Flores (2018, 2021a, 2021b) stands out. His research revolved around the impact of mathematical connections on students' understanding of the derivative and integral, delving into the connections of reversibility.

At the university level, Dolores-Flores and García-García (2017) highlighted the importance of intra-mathematical and extra-mathematical connections in problem-solving by Mexican university students. In this context, procedural connections, modeling, and different representations are crucial in applying strategies to solve problems. Additionally, aiming to enhance the analysis of mathematical activities of university students and professors, Rodríguez-Nieto et al. (2022a) articulated the theory of connections with the conceptual metaphor theory, giving rise to a new category of connection called "MT mathematical connection", where mathematical concepts are connected with bodily experiences or everyday situations. For example, a function is continuous if its graph can be drawn without lifting the pencil from the paper. Rodríguez-Nieto et al. (2021a) investigated university students' understanding of the graph of the derivative. They confirmed that students achieve more accurate graphical representations of f and f' when provided with the algebraic expression of f. Conversely, difficulties arise in graphing f or f' without this algebraic information. Additionally, Rodríguez-Nieto et al. (2021b) assessed the quality of mathematical connections made by university students when solving problems related to the derivative, and the connections were classified into three levels: level 2 or advanced, indicating that the subject establishes the connection and argues why consistently; level 1 is a connection without justification; and level 0, incorrect or inconsistent mathematical procedures.

In other research, Campo-Meneses and García-García (2020), as well as Campo-Meneses et al. (2021), explored the mathematical connections made by high school and university students on exponential and logarithmic functions, highlighting that the reversibility connection is the most important, as it addresses the bidirectional processes attributed to these inverse functions. Regarding the concept of vectors, considered by teachers and researchers as one of the most abstract topics in teaching linear algebra and calculus, students manifest difficulties during the learning process. The teaching is limited to memorized algorithms that only allow them to solve specific problems (Possani et al., 2010; Salgado & Trigueros, 2014). For this reason, Flores-García et al. (2007) point out that students have difficulties finding conceptual connections in the content and visualize it as a collection of equations to be memorized. On the other hand, Gutiérrez and Martín (2015) mention that university students struggle to understand the fundamental properties of vectors because, during their school education, their learning was limited to solving non-contextualized problems without connections to applications in other areas, such as physics.

Barniol and Zavala (2016) mention that most university students need help understanding the concept of vectors, particularly when they face challenges in interpreting the dot product as a projection. In this sense, the need to continue exploring the topic and contribute to strengthening the understanding of this concept is confirmed. Flores et al. (2017) identify that one of the difficulties students have when performing operations between vectors relates to deficiencies in some basic arithmetic processes. For example, errors are observed when adding or subtracting integers, solving a first-degree equation, and even implementing the Pythagorean theorem. The above is due to the traditional concept teaching, which does not provide any didactics to help the student develop a functional understanding of operations between vectors.

On their part, Tairab et al. (2020) categorized the difficulties students face during the vector concept learning process into two main categories. The first refers to conceptual understanding because students need help distinguishing basic concepts, explicitly differentiating between a scalar and vector quantity. The second category relates to operationalizing arithmetic processes unrelated to the concept. Most students perform calculations without understanding the relationships between scalar and vector quantities or the vector's direction. Students need more connection between conceptual and procedural aspects to understand this concept. Additionally, Cárcamo et al. (2023) determined that university students exhibit difficulty studying the vector concept due to their failure to establish connections between the topic's conceptual, representational, and procedural aspects. Consequently, they cannot make correct abstractions to comprehend and apply the concept in problem-solving.

From a theoretical point of view that is essential in this research, various works have been recognized in the literature, where theories have been articulated, based on the proposals of Prediger et al. (2008)**,** who state that the pairing of strategies and to connect theories ranges from completely ignoring other theoretical frameworks to unifying different approaches locally or globally. In this context, Drijvers et al. (2013) analyzed the use of algebra for learning the concept of parameter with the theory of instrumental genesis and OSA. Artigue and Bosch (2014) delved into the notion of praxeology in research practices to model mathematical and didactic activities. Font et al. (2016) articulated APOS theory with OSA to contrast and compare the conceptualization of the notion of object, proposing a genetic decomposition of the concept of derivative for a subsequent onto-semiotic analysis.

For their part, Thanheiser et al. (2021) articulated five theoretical approaches (cognitive task demand, lesson cohesion, student contribution types, collective argumentation, and student cognitive engagement activity), to analyze a holistic perspective of classroom culture involving the teacher, students**,** and mathematical content. Ledezma et al. (2022) integrated OSA and the modeling cycle from a cognitive perspective to improve the analysis of mathematical activity in real contexts and assess mathematical practices, processes**,** and objects. However, ETC and OSA had previously been articulated to address a problem associated with understanding the derivative because students stop making connections between meanings and multiple representations. Likewise, Campo-Meneses and García-García (2023) with this same articulation analyzed the connections of a teacher on exponential and logarithmic functions. However, the phenomenon has not been analyzed on the problems associated with the understanding of vectors from the articulated ETC-OSA framework, but rather from individually specific theories.

Once the literature has been reviewed, the significance of mathematical connections has been acknowledged from various perspectives, including research about the networking of theories and the relevance of the vector concept in mathematics and other disciplines. However, the challenge of establishing connections between multiple representations and concepts persists (Cárcamo et al., 2023; Tairab et al., 2020) as students continue to encounter difficulties in working with vectors, their diverse representations, operations, and real-world applications (Barniol & Zavala, 2014; Gutiérrez & Martín, 2015). Consequently, exploring further and understanding the connections students make when characterizing the vector concept remains essential. Therefore, this research aims to delve deeper into the mathematical connections of engineering students when addressing problems involving vectors.

In addition to the difficulties that students have in solving problems about vectors (caused by stopping making mathematical connections), it is also important to study this concept because it is fundamental for mathematics subjects such as vector calculus, geometry, linear algebra, among other and diverse applications that students and teachers work on at the higher or university level in the development of physics curriculum in all universities, especially for mathematics, physics, and engineering majors, where intra-mathematical and extra-mathematical connections based on representations are explicitly evident, procedures, operations, theorems, etc.

TWO THEORETICAL LENS

This research uses the networking between ETC and OSA, a framework for analyzing mathematical connections and other aspects involved in developing mathematical activity.

Extended Theory of Connections

A mathematical connection is understood from an integrative view of ETC and OSA as the tip of an iceberg made up of a conglomerate of practices, processes/objects (problem situations, languages, procedures, propositions, definitions, and arguments), and SFs that relate them (Rodríguez-Nieto et al., 2022b). Mathematical connections can be intra-mathematical "are established between concepts, procedures, theorems, arguments and mathematical representations of each other" (Dolores-Flores & García-García, 2017, p. 160), and extra-mathematical connections, which "establishes a relationship of a mathematical concept or model with a problem in context (not mathematical) or vice versa" (Dolores-Flores & García-García, 2017, p. 161). Next, the categories of mathematical connections used operationally to analyze the results are presented.

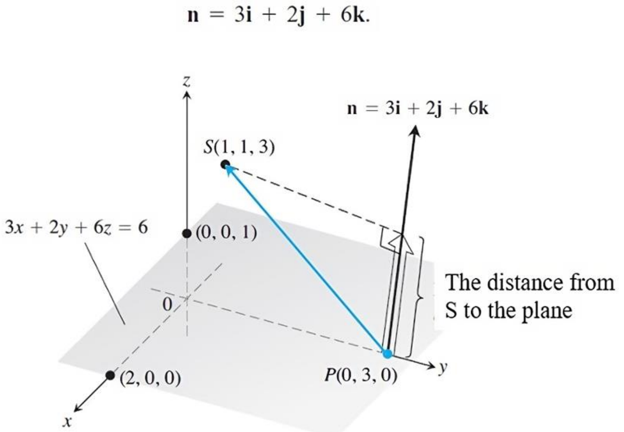

Figure 1. Connection between different alternate representations (Thomas, 2010, p. 693)

De Gamboa et al. (2020) state that extra-mathematical connections are based on intra-mathematical connections and are important for students and in-service teachers in problems-solving in the classroom. Furthermore, mathematical connections are one of the mathematical processes that foster mathematical creativity (Sánchez et al., 2022; Seckel et al., 2019). Each of the mathematical connections of ETC is described below:

1. **Modeling:** It is understood as the relationship that a subject establishes between the world of mathematics and the real world (or the daily life of students) and between mathematics and other sciences. Specifically, it can be seen as the relationship established between the mathematical concept and a task in a real context (that occurs or may occur in real life) or an application task in some discipline other than mathematics in which the subject, starting from the task build a mathematical model to solve it. When the subject builds the mathematical model, he uses various knowledge (mathematical or not) by executing multiple actions (algebraic, symbolic, graphic, etc.) to reach an answer consistent with the requirement posed (Campo-Meneses & García-García, 2023; Dolores-Flores & García-García, 2017; Evitts, 2004). For example, translating the information expressed in the statement of a problem (which requires finding the measure of the side of a box so that it has the greatest volume) to an initial mathematical expression that models the volume $v = length * width * height$.
2. **Instruction-oriented:** It refers to the understanding and use of a mathematical concept D from two (or more) previous concepts B and C (which are related), required to be understood by a person. These connection types can be recognized in two forms:

   (1) the relationship of a new topic with previous knowledge, and

   (2) the mathematical concepts, representations, and procedures connected are considered fundamental prerequisites that people must have to develop new content (Businskas, 2008).

   For example, when the in-service teacher tells the students that, to work on the derivative of a function, they must first remember the concepts of function and slope of a line.
3. **Procedural:** This mathematical connection is evident when rules, algorithms, or formulas are used to arrive at a result (García-García, 2019; García-García & Dolores-Flores, 2021b). For example, if a line is not vertical and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points other than the line, then the slope of the line can be found using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, with $x_2 - x_1 \neq 0$.
4. **Part-whole:** This connection type occurs when someone identifies that A is a generalization of B, where B is a particular case of A. For example, the function $P(x) = x^3 - x^2 - 9x - 9$ is a particular case of the general expression $f(x) = ax^3 + bx^2 + cx + d$ (Businskas, 2008). These relationships can be of inclusion when a mathematical concept is contained in another (García-García, 2019).
5. **Implication:** This type of connection is based on a logical relationship if-then (A→B) (Businskas, 2008; Mhlolo, 2012). If a function f is increasing on an open interval (a, b), then f' is positive on that same interval.
6. **Different representations:** This can be alternate or equivalent (Businskas, 2008). It is alternate if a student represents a mathematical concept in two or more different ways in different registers of representation: graph-algebraic, verbal-graph, etc. For example, an alternate representation is shown in **Figure 1**, where the vector $n = 3i + 2j + 6k$ graphed. While an equivalent representation is a transformation within the same register (algebraic-algebraic, graph-graph, symbolic-symbolic, etc.). For example, $n = 3i + 2j + 6k$ is equivalent to $n = \langle 3, 2, 6 \rangle$ in the algebraic or symbolic semiotic register.
7. **Feature:** It is identified when the subject manifests some characteristics of the concepts or describes its properties in terms of other concepts that make them different or similar to others (Eli et al., 2011; García-García & Dolores-Flores, 2021a). For example, García-García and Dolores-Flores (2021a) affirm that when the person mentions some elements of a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0$ (derivative function or antiderivative function) are coefficients (all, a_i, with $i = 0, 1, 2, 3, \ldots, n$), literal or variables (in this case, the "x") and exponents of the variables $(n, n-1, n-2, \ldots, 1)$.

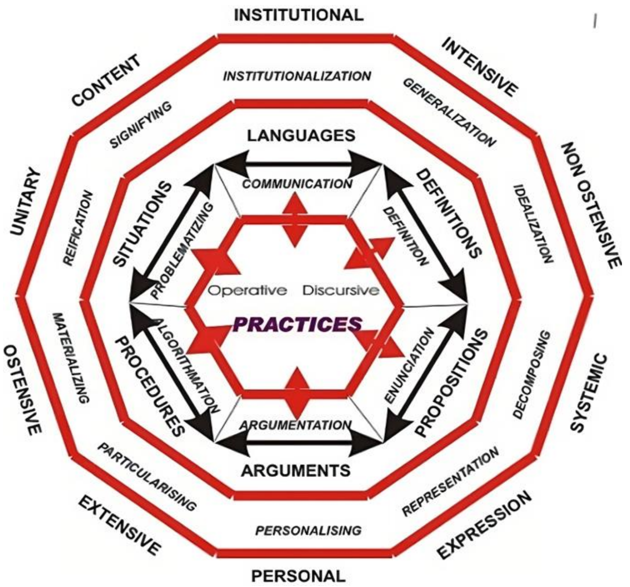

Figure 2. Schematization of mathematical knowledge from an OSA view (adapted from Font & Contreras, 2008)

8. **Meaning:** This mathematical connection is presented "when students attribute a meaning to a mathematical concept as long as what it is for them (which makes it different from another) and what it represents; it can include the definition that they have built for these concepts" (García-García, 2019, p. 131). In this sense, students express what the mathematical concept means to them, including their context of use or their definitions (García-García, 2019). In this research, we assume that this type can be more general; that is, we accept the existence of a mathematical connection between meanings. This type manifests when the students relate different meanings attributed to a concept to solve a specific problem. For example, Stewart (1999) "the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$" (p. 153).
9. **Reversibility:** It is present when a subject starts from a concept A to get to a concept B and invert the process starting from B to return to A (García-García & Dolores-Flores, 2021a). For example, this connection is established when the bidirectional relationship between derivative and integral, as operators, is recognized and when fundamental theorem of calculus is used to link both concepts (García-García & Dolores-Flores, 2018).
10. **MT:** It is understood as the projection of the properties, characteristics, etc., of a known domain to structure another less known domain. For example, when the teacher or the student uses verbal expressions such as "*travel through the graph without lifting the pencil from the paper,*" that implicitly suggests the conceptual metaphor "the graph is a path" (Rodríguez-Nieto et al., 2022a).

Onto-Semiotic Approach

It is an inclusive theoretical approach that emphasizes a person's mathematical knowledge and how mathematical instruction is carried out, which arose from the need to clarify, connect, and improve theoretical and methodological notions of several theories used in the Mathematics Education field. From this approach, it is essential to describe mathematical activity from an institutional or personal perspective, modeled in terms of practices and configuration of primary objects (PO) and processes activated in said practices (Drijvers et al., 2013). For Godino and Batanero (1994), mathematical practice is understood as "any situation or expression (…) carried out by someone to solve mathematical problems, communicate the solution obtained to others, validate it or generalize it to other contexts and problems" (p. 334). They include objects used in a broad sense to refer to any entity that, in some way, is involved in mathematical practice and can be identified as a unit (Font et al., 2013). Six PO are considered:

(1) problem situations,

(2) representations,

(3) definitions,

(4) propositions,

(5) procedures, and

(6) arguments.

These interconnected objects form the configuration of PO (Godino et al., 2019).

PO that emerge in mathematical practice can do so in different ways, which are the result of the different ways of seeing, speaking, operating, etc., on PO, which allows us to speak of primary personal or institutional objects, ostensive or non-ostensive, unitary, or systemic, intensive, or extensive, and of content or expression (Godino et al., 2007). Now, a configuration is a heterogeneous set or system of interrelated objects, which can be institutional (*epistemic*) or personal (*cognitive*) (Godino et al., 2019).

According to Godino et al. (2007), the set of PO emerges in mathematical activity through the activation of primary mathematical processes (communication, problem posing, definition, enunciation, procedures (algorithms), and argumentation) derived from the application of the process-product perspective to said PO, these precise ones occur together with those derived from applying the process-product duality to the five dualities mentioned above (institutional/personal, ostensive/non-ostensive, expression/content, extensive/intensive and unitary/systemic): personalization-institutionalization; materialization-idealization; representation-meaning; synthesis-analysis; generalization-particularization (Font et al., 2013; Godino et al., 2007) (**Figure 2**).

Another critical component in OSA is the notion of SF that allows practices to be associated with the processes and objects activated and the construction of an operational notion of knowledge, mathematical understanding, meaning, and competence (Godino et al., 2007). An SF is a triadic relationship between an antecedent (expression/object) and a consequent (content/object) made by a person (person or institution) according to a specific criterion or correspondence code (Font, 2007). SFs are inferred when mathematical activity is viewed from the expression/content duality.

In Rodríguez-Nieto et al. (2022b), it is stated that the notion of SF (OSA) is more general than the notion of mathematical connection (ETC) since the connections are considered particular cases of SFs of a personal or institutional nature. In ETC, the mathematical connection may or may not be accurate, revealing from the perspective of OSA that when a subject makes a correct connection, it coincides with the institutional one. When it is incorrect, it is personal.

Particularly in Godino (2022, p. 8), the importance of addressing the meanings of mathematical objects by subjects in given circumstances is expressed, referring to the semiotic-cognitive problem, which is related to the knowledge understood in OSA as the "set of" relations that the subject (person or institution) establishes between objects and practices, relationships that are modeled through the notion of SF.

These two theoretical frameworks are used because ETC allows the use of typology of mathematical connections and with OSA it is possible to detail the connections in terms of practices, processes, objects, and SFs that relate them and to deepen a type of understanding of the subject with respect to a mathematical concept.

Networking Between Extended Theory of Connections & Onto-Semiotic Approach

The networking of theories allows us to explore and understand how their contributions can be linked successfully (or not), respecting their underlying conceptual and methodological principles. This helps to understand and detail the complexity of the phenomena involved in the teaching and learning processes of mathematics (Kidron & Bikner-Ahsbahs, 2015; Prediger et al., 2008).

In the context of this research, the work of Rodríguez-Nieto et al. (2022b) was considered, who present a networking between ETC and OSA, which investigates three aspects: the nature of mathematical connections from the points of view of ETC and OSA; They explore how the connections of the subjects' productions (written and verbal) are inferred from both theoretical frameworks. To this end, they carried out a content analysis of the central publications of both theories, identifying principles, methods and paradigmatic research questions and asked if there are concordances and complementarities between ETC and OSA of mathematical connections that allow a more detailed analysis of mathematical connections. Furthermore, for the analysis they followed the typical steps to develop networking of theories (Drijvers et al., 2013; Kidron & Bikner-Ahsbahs, 2015; Radford, 2008), which include the selection and description of episodes, the identification of mathematical connections using ETC and OSA.

In this way, Rodríguez-Nieto et al. (2022b) identified that there is agreement between both theories, given that they use content analysis as a method. However, the thematic analysis of ETC uses a typology of mathematical connections established a priori, while the analysis carried out with OSA uses various tools. In this networking, the data were first analyzed in terms of practices, configurations of PO and SFs that relate them to as proposed by OSA. Finally, parts of mathematical activity (e.g., practices, PO, and SFs) were encapsulated as a type of connection proposed in ETC.

Although the analysis methods are different, the main conclusion is that both theories complement each other to carry out a more detailed analysis of the mathematical connections. In particular, the detailed analysis carried out with OSA tools visualizes a mathematical connection, metaphorically speaking, as the tip of an iceberg of a conglomerate of practices, processes, PO activated in these practices, and SFs that relate them, which allows for a comprehensive analysis detailing the structure and function of the connection. For this reason, in this research a more detailed analysis of mathematical connections will be used to analyze the productions of engineering students.

Vector Concept

Mathematical applications often concern magnitudes with both quantity (or intensity) and direction. An example of a magnitude is velocity. Thus, the velocity of an airplane has a quantity (how fast it flies) and direction, which determines its course. Other examples of such quantities are force, displacement, and acceleration. Physicists and engineers understand a vector as a directed straight-line segment, and these magnitudes that have quantity and direction are known as vector magnitudes.

On the other hand, a quantity that has quantity, but no direction is called a scalar quantity. Examples of scalar quantities are length, area, volume, cost, utility, and speed. The study of vectors is called vector analysis (Leithold, 1998).

A vector is represented by a directed line segment (or briefly directed segment), as a line segment starting from a point P and reaches a point Q, denoted by $\overrightarrow{PQ}$. is called the initial point, and the point Q is called the end point (Leithold, 1998) (**Figure 3**).

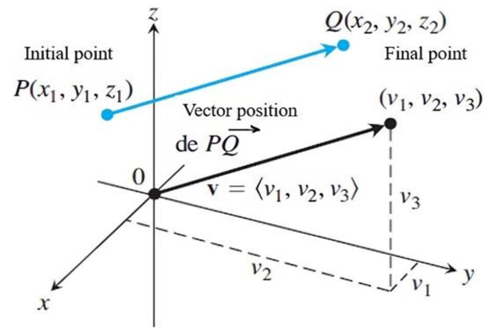

Figure 3. Vector as a directed line segment (Thomas, 2010)

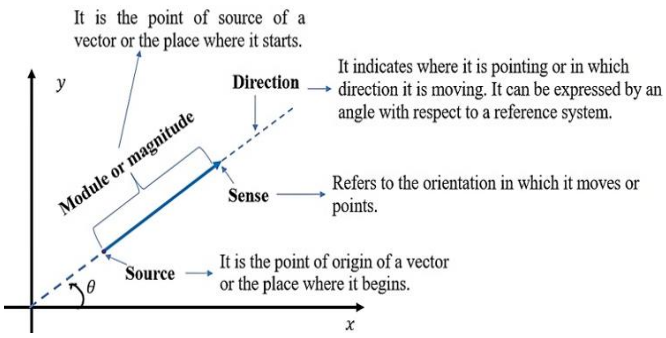

Figure 4. Elements of a vector (Thomas, 2010)

Definition of vector in plane & space

A vector in a two-dimensional plane, denoted by $\vec{A}$, is an ordered pair of real numbers and can be represented as $\vec{A} = (a_1, a_2) \in \mathbb{R}^2$. Similarly, a vector in three-dimensional space consists of an ordered triple of real numbers, which we can symbolize as $\vec{A} = (a_1, a_2, a_3) \in \mathbb{R}^3$. To correctly define a vector in the plane and space, it is essential to establish the structure of these vectors in a Cartesian coordinate system. The graphic representation of vectors is done using arrows or line segments that indicate their direction and magnitude. In a Cartesian coordinate system, vectors are drawn from a source point to an end point, and their length represents the magnitude of the vector. Furthermore, the elements of a vector are the characteristics that define its magnitude and direction (**Figure 4**).

It is important to note that a vector in an n-dimensional space is represented as $\vec{A} = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n$. In what follows, let us consider vectors in n-dimensional space. Now, we define the length or norm of a vector $\vec{A}$, denoted by $\|\vec{A}\|$ as $\|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + \cdots + a_n^2}$. The norm of a vector is always a non-negative value. On the other hand, the fundamental operations that can be performed with vectors are shown: the addition of vectors and multiplication by a scalar. A scalar is simply a real number (Thomas, 2010).

METHODOLOGY

This research is qualitative (Cohen et al., 2018). It seeks to identify the mathematical connections in solving problems about vectors by engineering students. Three stages were followed: the first refers to the selection of the participants (engineering students); The second deals with the collection of data through a questionnaire and participant observation; and, in the third stage, the data were analyzed to recognize the mathematical connections of the students.

Participants & Context

The participants consisted of 58 students (aged 18-20 years) from different engineering fields from a university on the Colombian Caribbean Coast who were studying vector calculus and participated in this research. These students have previous mathematics training in calculus I and calculus II courses related to teaching and learning functions, limits, derivatives, integrals, and their real-world applications.

In addition, they are aware of this research for educational purposes and have chosen to participate voluntarily.

Data Collection

The data were collected by creating a questionnaire with 15 tasks involving operations with vectors and their meaning (**Table 1**), which was structured based on the literature review and the contents of calculus books (Thomas, 2010).

In addition, the performance indicators and contents reflected in the syllabus of the first semester of vector calculus were considered. Then, the questionnaire was implemented in three classes in a sectioned manner. For example, the professor in charge of this activity (the first author of this article) explained the contents, and then, for thirty minutes, he applied the tasks corresponding to said theme (five tasks per class). After the students solved the tasks, some voluntarily participated in class, exposing their way of solving them on the blackboard.

This questionnaire was reviewed and validated by expert teachers (one in charge of linear algebra and another who teaches physics) including the authors of the article, who analyzed the writing, relevance of the tasks, consistency with the level of complexity of the students and that it really, the tasks were in line with the indicators suggested in vector calculus syllabus.

At a theoretical level, this questionnaire was discussed with the experts to make known that for its construction the cognitive semiotic problem of OSA was considered, where it began by asking about the meaning of the concept of vector and simultaneously sought to investigate the connection of meaning (ETC).

Generally, these tasks aim for the student to have knowledge of the meaning of the vector, as well as the competence to operate with vectors and strengthen their bases to solve situations of application problems developed in vector calculus.

Table 1. Questionnaire with tasks

Items
1. For you, what does a vector mean?
2. How is a vector represented?
3. Given the vectors $\vec{A}=(3,-1)\in\mathbb{R}^2$ and $\vec{B}=(-4,5)\in\mathbb{R}^2$, calculate $\vec{A}+\vec{B}$.
4. Given the vectors $\vec{A}=(4,-2)\in\mathbb{R}^2$ and $\vec{B}=(6,-3)\in\mathbb{R}^2$, calculate: $\vec{A}-\vec{B}$.
5. Given a vector $\vec{A}=(4,-5)$. Calculate: $3\vec{A}$ and $5\vec{A}$.
6. Given vectors $\vec{A}=(4,3)\in\mathbb{R}^2$; $\vec{B}=(-2,1)\in\mathbb{R}^2$; $\vec{C}=(8,2)\in\mathbb{R}^2$. Determine scalars h & k such that $\vec{C}=h\vec{A}+k\vec{B}$.
7. Find a unit vector that has the same direction and sense of $\vec{A}=(2,-1,2)\in\mathbb{R}^3$.
8. Given the vectors $\vec{A}=(3i+j)$ & $\vec{B}=(-2i+4j)$, obtain a unit vector that has the same direction and sense of $\vec{A}+\vec{B}$.
9. Given the vectors $\vec{A}=(4,2,-6)\in\mathbb{R}^3$ and $\vec{B}=(-5,3,-2)\in\mathbb{R}^3$. Calculate $\vec{A}\cdot\vec{B}$.
10. Determine the angle between the vectors $\vec{A}=(2,2,-1)\in\mathbb{R}^3$ and $\vec{B}=(5,-3,2)\in\mathbb{R}^3$.
11. Given the vectors $\vec{A}=(2,1,-1)\in\mathbb{R}^3$ and $\vec{B}=(5,-4,2)\in\mathbb{R}^3$. Demonstrate that $\vec{A}$ and $\vec{B}$ are orthogonal.
12. Let $\vec{A}=(4,-2,1)\in\mathbb{R}^3$ and $\vec{B}=(3,5,-2)\in\mathbb{R}^3$. Calculate: $\vec{A}\times\vec{B}$.
13. Let $\vec{A}=(1,5,2)\in\mathbb{R}^3$ and $\vec{B}=(4,7,-1)\in\mathbb{R}^3$. Calculate: $\vec{A}\times\vec{B}$.
14. Find the angle θ between the vectors $\vec{A}=(1,5,-2)$ and $\vec{B}=(3,-1,4)$.
15. Given vectors $A(1,0,1), B(2,0,-1)$ & $C(0,1,4)$, area of triangle they determine, ABC, is given by $\left(\frac{1}{2}\right)\lvert\overrightarrow{AB}\times\overrightarrow{AC}\rvert$.

Table 2. Phases of ETC-OSA data analysis (Rodríguez-Nieto et al., 2022c)

Phases		Description
1	Transcription of interviews or organization of written productions	The students' written productions were organized so that the researchers became familiar with the students' responses. Additionally, this process is critical to ensure that the researcher further analyzes, reads, and interprets the information collected.
2	Temporal narrative	The student's resolution of the task is mathematically explained. It contains the practices carried out by the student and some important primary objects identified in the narrative.
3	Mathematical practice	The mathematical practices or sequence of actions are described, regulated by institutionally established rules useful for solving a problem. In these mathematical practices the foundation of each mathematical connection of ETC is evident.
4	Cognitive configuration	It is the system of primary mathematical objects that a subject mobilizes as part of the mathematical practices developed to solve a specific problem (Godino et al., 2019). Actually, these primary objects are a fundamental part of the connection because they are generally the beginning (antecedent) and end (consequent) of its structure.
5	Semiotic functions	Semiotic functions are established between primary objects of cognitive configuration. In this way, the mathematical connections proposed by ETC are formed and visualized.

Data Analysis

In this research, the data were analyzed considering the integrative method resulting from the networking between ETC and OSA as evidenced in Rodríguez-Nieto et al. (2022b), using elements of thematic analysis (Braun & Clarke, 2006) and the onto-semiotic analysis, where initially these two methods are similar, that is, the data transcriptions (ETC) and OSA narrative are the same because they allow the interviews to be seen in text format or, if the data are written productions, they are organized to make them more accessible to the researcher.

Then, the rest of the data analysis remains regarding practices, processes, configurations of PO, and SFs (**Table 2**). It should be noted that the analysis followed in this work is more linked to the use of OSA tools and ETC connection categories.

Based on the analysis method described in **Table 2**, the results or connections established by the engineering students are reported.

FINDINGS

This section details some students' mathematical connections used when solving problems involving vectors. Narrative, mathematical practices, processes, PO, and configuration for some tasks (e.g., 1, 2, and 7) will be reported jointly, considering the mathematical activity of several students who proceeded similarly.

Temporal Narrative

Student P1 was assigned task of meaning of a vector to which he responded that it is a line segment that consists of magnitude, direction, and sense. Later, in the second task, he was asked how to represent a vector, and P1 responded with a graph and an example with different representations. In the third task referring to addition of vectors, P1 followed rule of adding component by component of vectors $\vec{A}$ and $\vec{B}$, so result was $\vec{A}+\vec{B}=(3+(-4),(-1)+5)=(-1,4)$. Other tasks are narrated in mathematical terms, highlighting procedures, definitions, representations, etc.

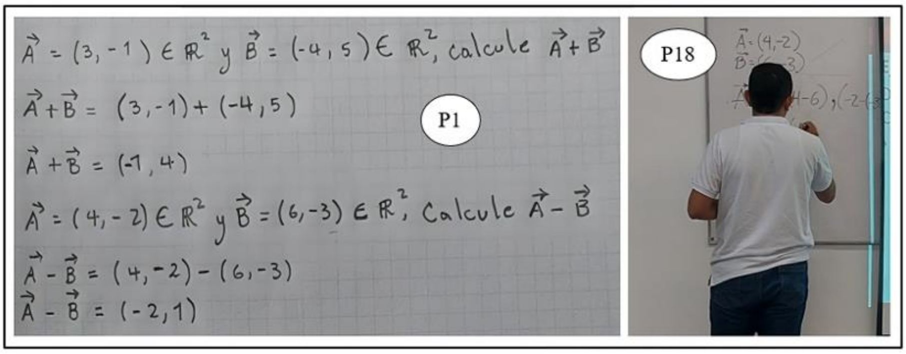

Figure 5. Addition & difference of vectors (Source: Authors' own elaboration)

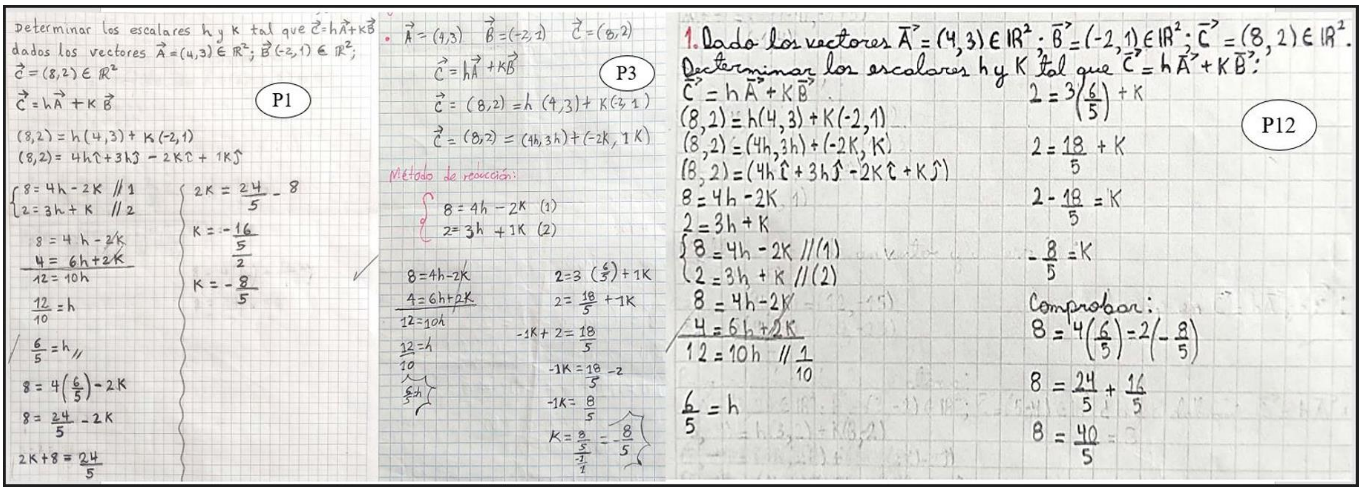

Figure 6. Written evidence in resolution of task 6 (Source: Authors' own elaboration)

Mathematical Practice System

In this section, we present the students' sequenced actions to carry out the proposed tasks.

Mp1: Student P1 read and understood the tasks of the questionnaire, in this case, task 1, answering that a vector is a line segment with magnitude, sense, and direction. In the same way, the other students gave their meaning about the vector.

Mp2: P1 responded to task 2 by representing the vector graphically as a ray and assigned it a symbolic representation by $\vec{\boldsymbol{v}}$.

Mp3: P1 mentioned that a vector has components $\vec{\boldsymbol{v}} = (\hat{\imath}, \hat{\jmath}, \hat{k})$ a symbolic representation.

Mp4: P1 provided an example of a vector $\vec{\boldsymbol{v}} = (\mathbf{3}, \mathbf{4})$ P1, furthermore, illustrated it graphically on a Cartesian coordinate plane.

Mp5: P1 solves task 3 by adding the components of vectors $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = (\mathbf{3}, -\mathbf{1},) + (-\mathbf{4}, \mathbf{5}) = (-\mathbf{1}, \mathbf{4})$. To accomplish this, P1 considered rules of signs (**Figure 5**).

Mp6: P1 solve task 4 by subtracting the components of vectors $\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} = (\mathbf{4}, -\mathbf{2}) - (\mathbf{6}, -\mathbf{3}) = (-\mathbf{2}, \mathbf{1})$. Similarly, P1 applied the rules of signs (**Figure 5**).

Mp7: To solve task 5, P1 used the distributive property to multiply the scalar by each of the components of the vectors; on the one hand, $3\vec{\boldsymbol{A}} = \mathbf{3}(\mathbf{4}, -\mathbf{5}) = (\mathbf{12}, -\mathbf{15})$ and on the other, $5\vec{\boldsymbol{A}} = \mathbf{5}(\mathbf{4}, -\mathbf{5}) = (\mathbf{20}, -\mathbf{25})$.

Mp8: Solved task 6 since he found the values h y k. To do this, he first considered the structure of the vector $\vec{\boldsymbol{C}} = \boldsymbol{h}\vec{\boldsymbol{A}} + \boldsymbol{k}\vec{\boldsymbol{B}}$, and then substituted the vectors, as follows: $(8,2) = h(4,3) + k(-2,1)$.

Mp9: Subsequently, he multiplied the vector $\vec{A}$ by the scalar h and the vector $\vec{\boldsymbol{B}}$ by the scalar k, highlighting the components of the vectors: $(8,2) = 4h\hat{\imath} + 3h\hat{\jmath} - 2k\hat{\imath} + 1k\hat{\jmath}$.

Mp10: P1 structured two equations with two unknowns, considering the vectors' components (**Figure 6**).

Mp11: To find the values of h and k, P1 implemented the reduction method to solve 2×2 linear equations by multiplying equation 1 by one (yielding: $8 = 4h - 2k$) and multiplying equation 2 by 2 (yielding: $4 = 6h - 2k$).

Mp12: P1 added equations 1 and 2 to eliminate the k terms and obtained the equation $12 = 10h$.

Mp13: P1 used the multiplicative inverse or MT clearance to find the value of $h = \frac{12}{10}$, which is simplified as equal to $h = \frac{6}{5}$.

Mp14: P1 replaced the value of $h = \frac{6}{5}$ in equation 1, yielding the expression $8 = 4\left(\frac{6}{5}\right) - 2k$.

Mp15: P1 performed arithmetic operations, obtaining different equivalent representations $8 = 4\left(\frac{6}{5}\right) - 2k = 8 = \frac{24}{5} - 2k$.

Mp16: P1 applied the properties of additive inverses (for example, the additive inverse of -2k is 2k) to obtain $2k + 8 = \frac{24}{5}$.

Mp17: P1 applied the properties of additive inverses (for example, the additive inverse of 8 is -8) to deduce $2k = \frac{24}{5} - 8$, and then performed operations with fractions to obtain $2k = \frac{-16}{5}$.

Mp18: To find the value of k, P1, and P3 among other students, apply the law of extremes or what is metaphorically called the *"law of the ear,"* obtaining $k = \frac{\frac{-16}{5}}{2} = -\frac{8}{5}$. In this way, P1 found the values of *h* and *k*.

> P1: The first thing I did was set up the equation. I equaled vector C to the multiplication of *h* by vector A plus the multiplication of *k* by vector B. I performed the operation and then separated the terms *h* and *k* into two different equations. After that, I multiplied the equations by a number that favored me to eliminate one of the unknowns later. I multiplied equation 1 by one and equation 2 by 2. In this case, the k was canceled, and I was left with the equation 12=10*h*. I replaced that 12 over 10 equal to *h* in equation 1 to find the second unknown *k* equal to -8/5 and *h* equal to 6/5; those are the scalars.
>
> Interviewer (I): Could you explain how you got the value of *k*?
>
> P1: I subtracted the eight that I was adding from the other side to leave the k alone. I still had the number 2. I divided it, and with the law of the ear, I multiplied the ends by the ends and the middles by the media. In this case, it would be -16 times 1 and 5 times 2. I would stay with -8/5 because half of 16 is 8, and half of 10 would be 5.

However, P12 did not use the "*law of the ear*" to find the result of *k*.

Mp19: To solve task 7, for example, students P1, P2, P3, P4, P5, P6, P7, P9, P10, P11, P12, P13, P14, P15, P17, P18, P19, P20, P22, P23, and P28 proceed similarly, initially stating if the vector $\vec{A} = (a_1, a_2, a_3) \in \mathbb{R}^3$ o ($\vec{A} = a_1 i + a_2 j + a_3 k$), the unit vector can be determined using the following formula: $\vec{U} = \frac{a_1}{\|\vec{A}\|} i + \frac{a_2}{\|\vec{A}\|} j + \frac{a_3}{\|\vec{A}\|} k = \frac{\vec{A}}{\|\vec{A}\|}$.

Mp20: Then, students found the norm of the vector $\vec{\boldsymbol{A}} = (\mathbf{2}, -\mathbf{1}, \mathbf{2})$, as follows: $\|\vec{A}\| = \sqrt{(2)^2 + (-1)^2 + (2)^2}$.

Mp21: Students developed arithmetic operations of potentiation and radiation to find the result of the norm: $\sqrt{4+1+4} = \sqrt{9} = 3$.

Mp22: Next, they substituted the components of $\vec{A}$ and the norm $\sqrt{9} = 3$ in the formula: $\vec{U} = \frac{2}{\sqrt{9}} i - \frac{1}{\sqrt{9}} j + \frac{2}{\sqrt{9}} k$.

Mp23: The students then expressed the unit vector in Cartesian coordinates: $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$, an equivalent representation of $\vec{U} = \frac{2}{\sqrt{9}} i - \frac{1}{\sqrt{9}} j + \frac{2}{\sqrt{9}} k$.

Mp24: The participants verified the unit vector by calculating its norm, which equaled 1 (**Figure 7**).

Mp25: In solving task 8, students initially considered the sum of vectors $\vec{\boldsymbol{A}} = (\mathbf{3}\boldsymbol{i} + \boldsymbol{j})$ y $\vec{\boldsymbol{B}} = (-\mathbf{2}\boldsymbol{i} + \mathbf{4}\boldsymbol{j})$ component by component, resulting $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = (1, 5)$. Then, to find the unit vector, they first found the norm using arithmetic operations, potentiation, radiation, and equivalent representations: $\|\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}\| = \sqrt{1^2 + 5^2} = \sqrt{1 + 25} = \sqrt{26}$.

Mp26: The students applied the formula to construct the unit vector $\vec{U} = \frac{1}{\sqrt{26}} i + \frac{5}{\sqrt{26}} j$.

Mp27: The students verified their procedures by calculating the norm of the vector $\vec{U} = \frac{1}{\sqrt{26}} i + \frac{5}{\sqrt{26}} j$, as follows: $\|\vec{U}\| = \sqrt{\left(\frac{1}{\sqrt{26}}\right)^2 + \left(\frac{5}{\sqrt{26}}\right)^2} = \sqrt{\frac{1}{26} + \frac{5}{26}} = \sqrt{\frac{26}{26}} = 1$. This result confirmed that they determined the unit vector with the same direction and magnitude as $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$ (**Figure 8**).

Mp28: In task 9, all students found the scalar product between the vectors ($\vec{\boldsymbol{A}} = (\mathbf{4}, \mathbf{2}, -\mathbf{6})$ and $\vec{\boldsymbol{B}} = (-\mathbf{5}, \mathbf{3}, -\mathbf{2})$) by multiplying component by component $(4)(-5) + (2)(3) + (-6)(-2)$.

Mp29: The students performed arithmetic operations to obtain the scalar: $-20 + 6 + 12 = -2$ (**Figure 9**).

Mp30: To solve task 10, students considered the formula. Therefore, they first found the scalar product between the vectors $\vec{\boldsymbol{A}} = (\mathbf{2}, \mathbf{2}, -\mathbf{1})$ and $\vec{\boldsymbol{B}} = (\mathbf{5}, -\mathbf{3}, \mathbf{2})$, obtaining $\left((\mathbf{2} * \mathbf{5}) + \left(\mathbf{2} * (-\mathbf{3})\right) + (-\mathbf{1} * \mathbf{2})\right)$.

Mp31: Then, students applied arithmetic operations and the law of signs to obtain the scalar: $10 + (-6) + (-2) = 2$.

Mp32: The students found the norm of the vector $\vec{\boldsymbol{A}} = (\mathbf{2}, \mathbf{2}, -\mathbf{1})$, which resulted: $\|\vec{A}\| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$.

Mp33: the students determined the norm of the vector $\vec{\boldsymbol{B}} = (\mathbf{5}, -\mathbf{3}, \mathbf{2})$, which was: $\|\vec{B}\| = \sqrt{(5)^2 + (-3)^2 + (2)^2} = \sqrt{25+9+4} = \sqrt{38}$.

Mp34: Once the students found the scalar product and the norms of the vectors $\vec{\boldsymbol{A}}$ and $\vec{B}$, they substituted the values into the formula $\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\| \cos(\theta)$ working it in two equivalent ways: 1) $2 = (\sqrt{9})(\sqrt{38}) \cos(\theta)$ for the case of P1, P2, P13 y P18 or, 2) $\cos(\theta) = \frac{2}{(\sqrt{9})(\sqrt{38})}$ for P23.

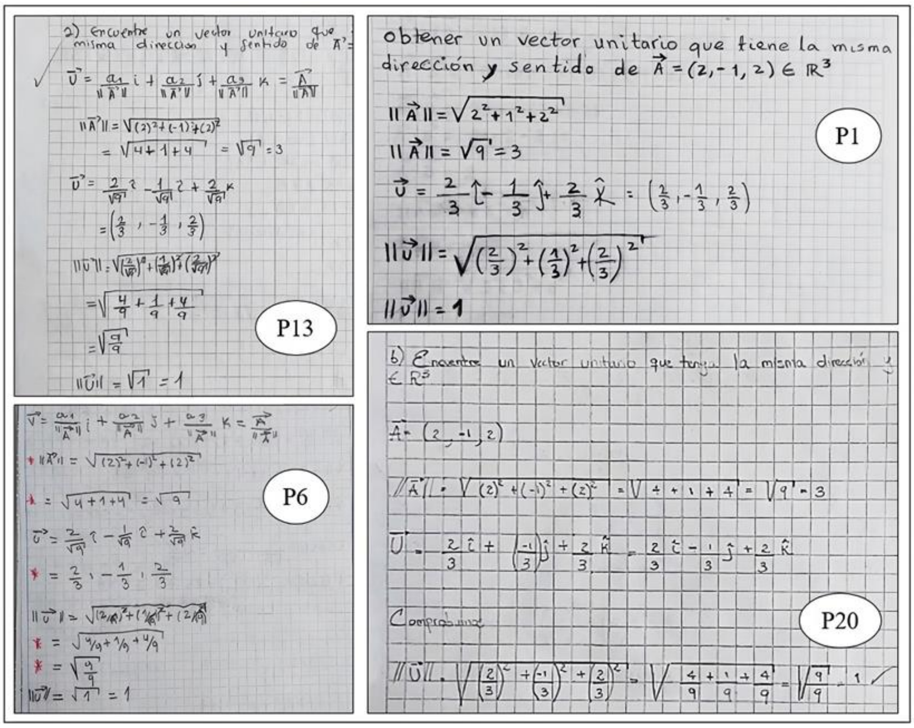

Figure 7. Procedures to find unit vector (Source: Authors' own elaboration)

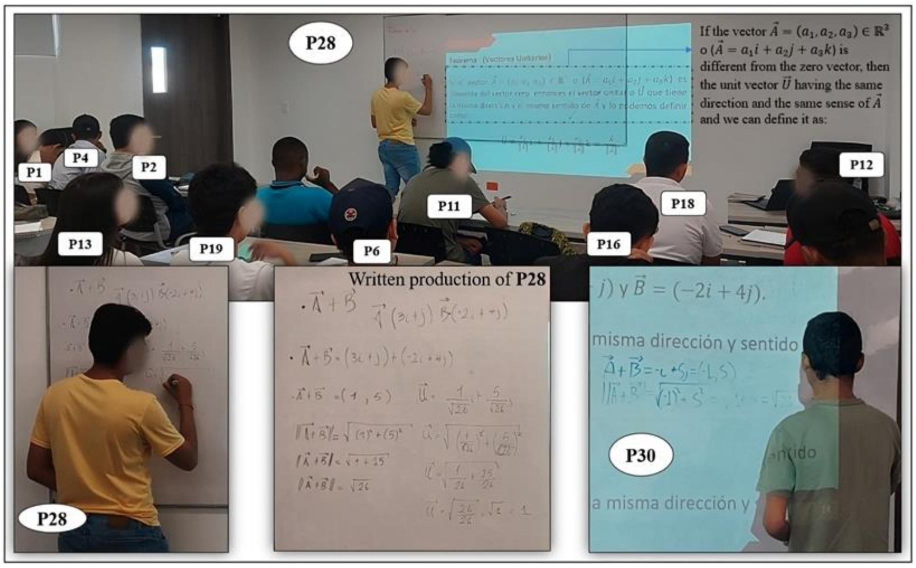

Figure 8. Participation in P28 & P30 (Source: Authors' own elaboration)

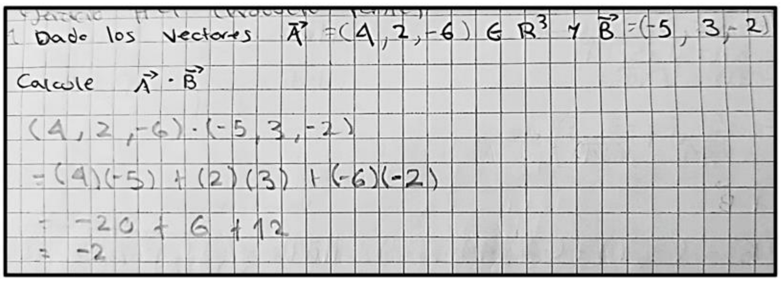

Figure 9. Written production of P26 to obtain scalar (Source: Authors' own elaboration)

Mp35: The students applied the inverse cosine of the argument to find the value of the angle: $\theta = cos^{-1}\left(\frac{2}{(\sqrt{9})(\sqrt{38})}\right)$.

Mp36: With the calculator, they found the measure of the angle $\theta = 83.79°$ (**Figure 10**).

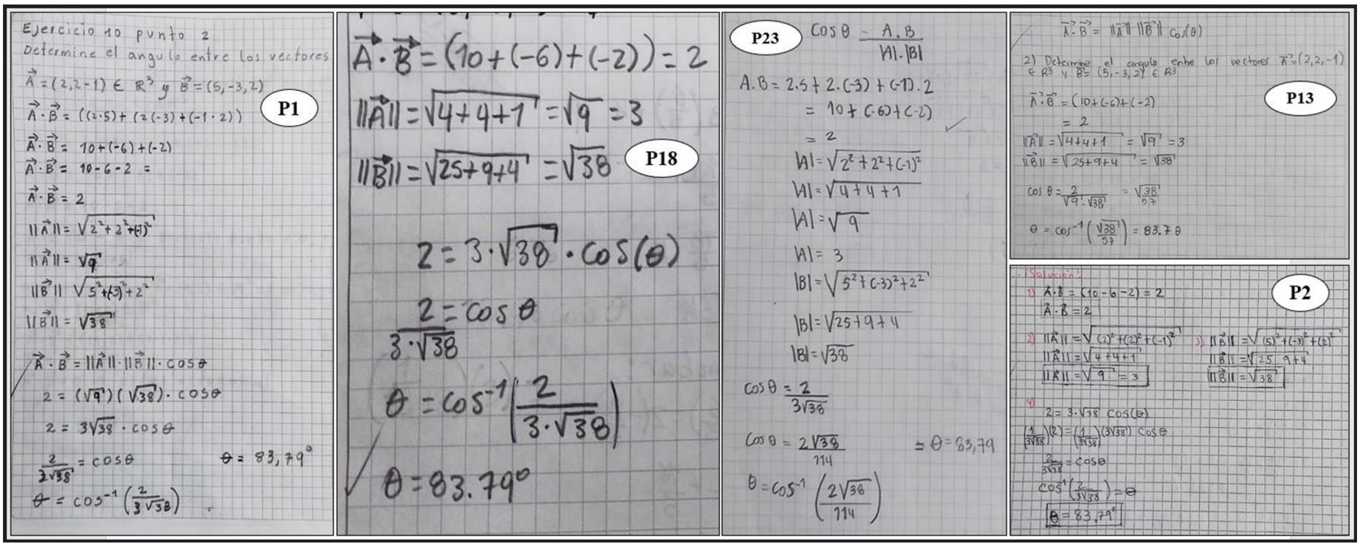

Figure 10. Procedure to find angle between vectors (Source: Authors' own elaboration)

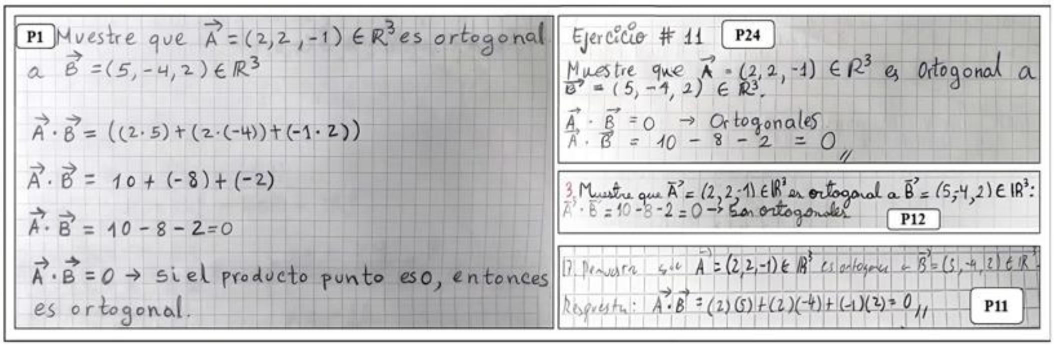

Figure 11. Evidence of how to find orthogonal vector (Source: Authors' own elaboration)

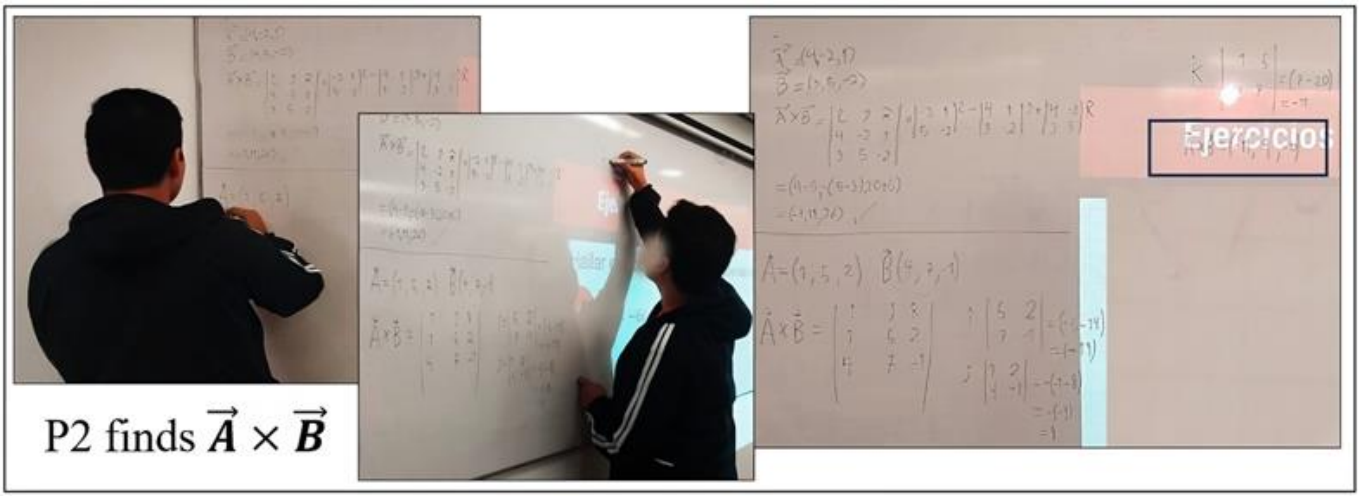

Figure 12. P2 finding vector product (Source: Authors' own elaboration)

Mp36: In solving task 11, the students applied the scalar product between the vectors $\vec{A} = (2, 2, -1)$ and $\vec{B} = (5, -4, 2)$, as follows: $\vec{A}\cdot\vec{B} = ((2 * 5) + (2 * (-4)) + (-1 * (2))$ (**Figure 11**).

Mp37: **Figure 11** shows how the students did arithmetic operations and obtained $\vec{A}\cdot\vec{B} = 10 - 8 - 2 = 0$. P1 stated that if the scalar product is 0, it is orthogonal (an expression that activates the implication connection).

Mp38: The students solved task 12 and task 13 using the same method (**Figure 12**). For example, P2, to solve task 13, proceeded with the method of determinants, where he initially applies the definition of the vector product and constructs the matrix $\begin{vmatrix} i & j & k \\ 1 & 5 & 2 \\ 4 & 7 & -1 \end{vmatrix}$.

Mp39: Then, the student multiplied each element of the first row by the determinant of the submatrix that

Figure 13. Evidence of resolution of task 14 (Source: Authors' own elaboration)

results from eliminating the first row for i and the first column for i. That is, he calculates the cofactor C_{11} and multiplies it by the first element of the first row $i\begin{vmatrix}5 & 2\\ 7 & -1\end{vmatrix} = (-5-14) = -19$.

Mp40: The student calculates the cofactor C_{12} and multiplies it by the second element of the first row, which is $-j\begin{vmatrix}1 & 2\\ 4 & -1\end{vmatrix} = -(-1-8) = 9$. It is important to note that the signs for the cofactors are alternate; for example, for $C_{12} = (-1)^{1+2} = (-1)^3 = -1$ indicates that the second cofactor's sign is negative.

Mp41: The student finds the cofactor C_{13} and multiplies it by the third element of the first row, referring to $k\begin{vmatrix}1 & 5\\ 4 & 7\end{vmatrix} = (7-20) = -13$.

Mp42: The student P2 states that the vector product is: $\vec{A} \times \vec{B} = (-11, 9, -13)$.

Mp43: To solve task 14 on the angle between the vectors $\vec{\boldsymbol{A}} = (\mathbf{1}, \mathbf{5}, -\mathbf{2})$ and $\vec{\boldsymbol{B}} = (\mathbf{3}, -\mathbf{1}, \mathbf{4})$, students initially applied the formula: $\|\vec{A} \times \vec{B}\| = \|\vec{A}\|\|\vec{B}\|\operatorname{sen}(\theta)$.

Mp44: The students used the method of determinants, applied the definition of the vector product, and constructed the matrix $\begin{vmatrix}i & j & k\\ 1 & 5 & -2\\ 3 & -1 & 4\end{vmatrix}$.

Mp45: Then, the students took each element of the first row and multiplied it by the determinant of the submatrix; that is, they calculated the cofactor C_{11} and multiplied it by the first element of the first row $i\begin{vmatrix}5 & -2\\ -1 & 4\end{vmatrix} = (20-2) = 18$.

Mp46: P25 and P28 calculated the cofactor C_{12} and multiplied it by the second element of the first row, i.e., $-j\begin{vmatrix}1 & -2\\ 3 & 4\end{vmatrix} = -(4+6) = -10$.

Mp47: The students found the cofactor C_{13} and multiplied it by the third element of the first row, referring to $k\begin{vmatrix}1 & 5\\ 3 & -1\end{vmatrix} = (-1-15) = -16$. It is concluded that $\vec{A} \times \vec{B} = (18, -10, -16)$.

Mp48: P25 and P28 applied the norm of $\vec{A} \times \vec{B}$ equal to $\sqrt{680} = 2\sqrt{170}$.

Mp49: The students found the norm of the vector $\vec{\boldsymbol{A}} = (\mathbf{1}, \mathbf{5}, -\mathbf{2})$, which resulted: $\|\vec{A}\| = \sqrt{(1)^2 + (5)^2 + (-2)^2} = \sqrt{1+25+4} = \sqrt{30} = 5.48$

Mp50: the students also found the norm of a vector, which is $\vec{\boldsymbol{B}} = (\mathbf{3}, -\mathbf{1}, \mathbf{4})$ and obtained: $\|\vec{B}\| = \sqrt{(3)^2 + (-1)^2 + (4)^2} = \sqrt{9+1+16} = \sqrt{26} = 5.1$

Mp51: The students replaced the values obtained in the vector product and its norm in the formula $2\sqrt{170} = \sqrt{30}\sqrt{26}\operatorname{sen}(\theta)$.

Mp52: P25 and P28 performed operations with radicals and obtained $2\sqrt{170} = 2\sqrt{195}\operatorname{sen}(\theta)$.

Mp53: The students applied arithmetic operations to find the value of $\operatorname{sen}(\theta) = \frac{2\sqrt{170}}{2\sqrt{195}}$.

Mp54: The students applied the inverse sine and found the value of the angle $\theta = \operatorname{sen}^{-1}\left(\frac{2\sqrt{170}}{2\sqrt{195}}\right) = 69{,}01°$ (**Figure 13**).

Mp55: To solve task 15 on the area of the triangle, students found distances between the given points or the vectors $\overrightarrow{AB}$ y $\overrightarrow{AC}$. For example, students ensure that the coordinates of the vector $\overrightarrow{AB}$ are the coordinates of the end point (point B) minus the coordinates of the origin point (point A): if $A(1,0,1)$ and $B(2,0,-1)$, then $\overrightarrow{AB} = (2-1, 0-0, -1-1) = (1,0,-2)$.

Mp56: To find the vector $\overrightarrow{AC} = (0-1, 1-0, 4-1) = (-1,1,3)$.

Mp57: Then, the students applied the method of determinants, in turn applying the definition of the vector product, and constructed the matrix $\begin{vmatrix}i & j & k\\ 1 & 0 & -2\\ -1 & 1 & 3\end{vmatrix}$.

Mp58: The students considered each element in the first row and multiplied it by the determinant of the submatrix: $0 + 2 = 2$.

Mp59: The students calculated $-(3-2) = -1$.

Mp60: The students performed operations on the submatrix, resulting in $(1+0) = 1$. Consequently, they obtained that $\overrightarrow{AB} \times \overrightarrow{AC} = (2, -1, 1)$.

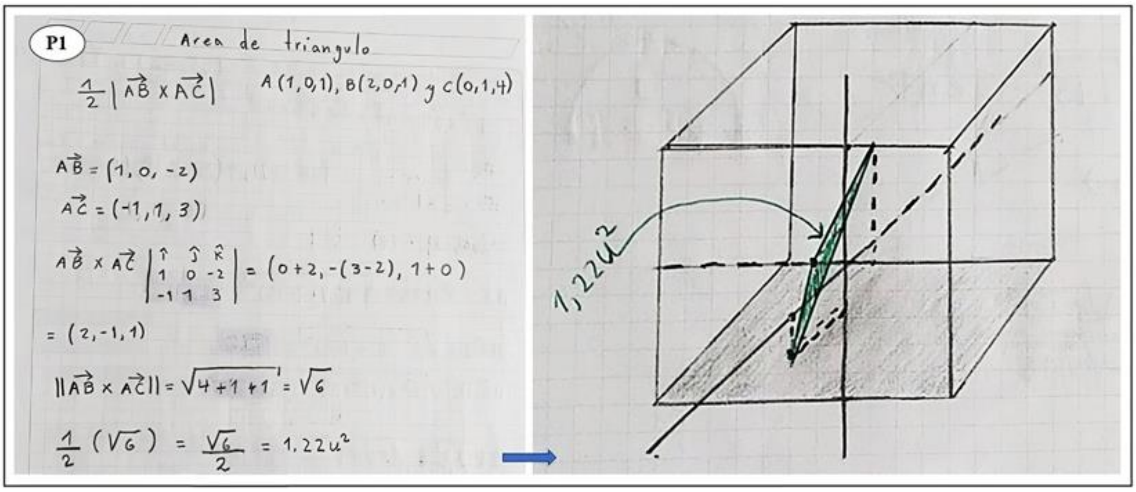

Figure 14. Area & representation of triangle (Source: Authors' own elaboration)

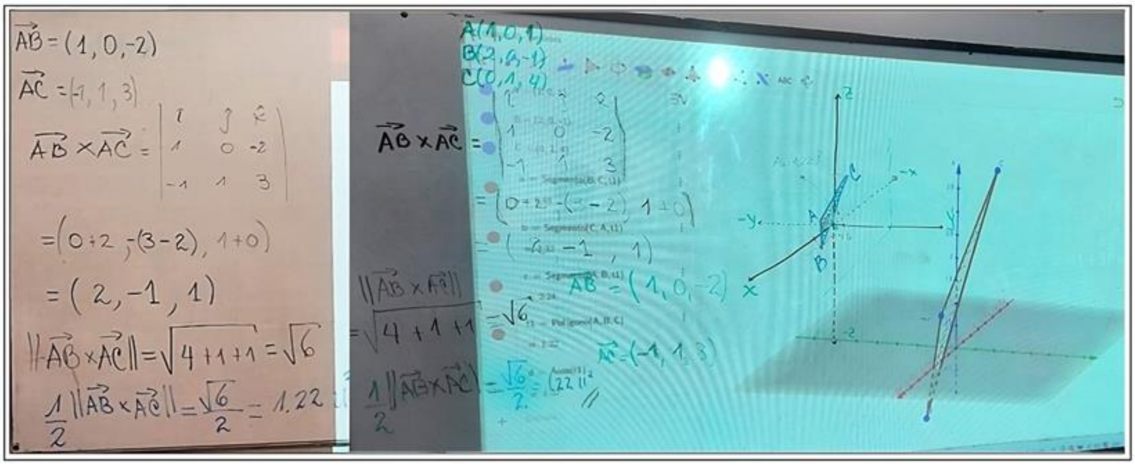

Figure 15. Procedure to find & represent area of triangle (Source: Authors' own elaboration)

Mp61: The students found the norm of $\overrightarrow{AB} \times \overrightarrow{AC}$ equal to $\sqrt{4+1+1} = \sqrt{6}$.

Mp62: The students substituted the values obtained into the formula $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$, obtaining $\frac{\sqrt{6}}{2} = 1.22\ u^2$.

Mp63: The students manually graphed the triangle in their response to the questionnaire and on the board (**Figure 14**).

Mp64: The students graphed the triangle using GeoGebra, thanks to the discussions generated after completing the written questionnaire (**Figure 15**). Students and teachers need to work with GeoGebra (**Figure 15**) because it allows "visualizing mathematical objects, and permanently evidence the connections established by a person in the graphical and algebraic-symbolic views" (Rodríguez-Nieto, 2021, p. 273). Furthermore, to address the topic of vectors, it is essential to resort to different representations due to their magnitude, direction, and the sense that people must understand.

Cognitive Configuration

Since several students participated in this research, the cognitive configuration of PO presented in **Table 3** will only be exemplified with the mathematical activity of some students who proceeded similarly when solving some tasks.

Table 3. Cognitive configuration of primary objects

Ops' description
Task 1
There are 15 proposed tasks, but tasks 1, 2, & 7 will be considered in this configuration. T1: For you, what does a vector mean? T2: How is a vector represented? T7: Find a unit vector that has the same direction and sense of $\vec{\mathbf{A}} = (\mathbf{2}, -\mathbf{1}, \mathbf{2}) \in \mathbb{R}^3$.
Linguistic elements (LE)
Verbal: Vector, point, line segment, unit vector, norm, magnitude, direction, sense, angle, scalar, reduction method, linear equations, multiplicative inverse, additive inverse, potentiation, square root, inverse cosine, etc.

Table 3 (Continued). Cognitive configuration of primary objects

Ops' description

Symbolic: $\vec{v}$; $5\vec{v}$; $\vec{v} = (\hat{\imath}, \hat{\jmath}, \hat{k})$; $\vec{v} = (3,4)$; $\vec{A} + \vec{B} = (3,-1,) + (-4,5) = (-1,4)$; $\vec{A} - \vec{B} = (4,-2) - (6,-3) = (-2,1)$; $\vec{C} = h\vec{A} + k\vec{B}$; $(8,2) = h(4,3) + k(-2,1)$; $(8,2) = 4h\hat{\imath} + 3h\hat{\jmath} - 2k\hat{\imath} + 1k\hat{\jmath}$; $8 = 4h - 2k$; $4 = 6h - 2k$; $12 = 10h$; $h = \frac{12}{10}$; $h = \frac{6}{5}$; $8 = 4\left(\frac{6}{5}\right) - 2k$; $8 = 4\left(\frac{6}{5}\right) - 2k = 8 = \frac{24}{5} - 2k$; $2k + 8 = \frac{24}{5}$; $2k = \frac{24}{5} - 8$; $2k = \frac{-16}{5}$; $k = \frac{-\frac{16}{5}}{2} = -\frac{8}{5}$; $\vec{U} = \frac{a_1}{\|\vec{A}\|}i + \frac{a_2}{\|\vec{A}\|}j + \frac{a_3}{\|\vec{A}\|}k = \frac{\vec{A}}{\|\vec{A}\|}$; $\vec{A} = (2,-1,2)$; $\|\vec{A}\| = \sqrt{(2)^2 + (-1)^2 + (2)^2}$; $\sqrt{4+1+4} = \sqrt{9} = 3$; $\vec{U} = \frac{2}{\sqrt{9}}i - \frac{1}{\sqrt{9}}j + \frac{2}{\sqrt{9}}k$; $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$; $\vec{A} = (2,2,-1)$ y $\vec{B} = (5,-3,2)$; $\left((2*5) + \left(2*(-3)\right) + (-1*2)\right) = 10 + (-6) + (-2) = 2$; $\vec{A} = (2,2,-1)$; $\|\vec{A}\| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$; $\vec{A} = (5,-3,2)$; $\|\vec{A}\| = \sqrt{(5)^2 + (-3)^2 + (2)^2} = \sqrt{25+9+4} = \sqrt{38}$; $\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\|\cos(\theta)$; $2 = (\sqrt{9})(\sqrt{38})\cos(\theta)$; $\cos(\theta) = \frac{2}{(\sqrt{9})(\sqrt{38})}$; $\theta = cos^{-1}\left(\frac{2}{(\sqrt{9})(\sqrt{38})}\right)$; $\theta = 83.79°$ …

Graphic: see **Figure 14**, **Figure 15**, & **Figure 16**.

Concepts/definitions (CD): Previous concepts: Vector, point, line segment, norm, magnitude, direction, sense, angle, scalar, reduction method, linear equations, multiplicative inverse, additive inverse, potentiation, square root, unit vector, inverse cosine ...

Definition 1 (D1): A vector is a line segment with magnitude, sense, and direction.
D2: Unit vector is a vector with a magnitude equal to 1.
D3: Orthogonal vector: Two vectors are orthogonal if the dot product between them equals zero.

Propositions/properties: Previous propositions: Arithmetic operations, determinants, systems of linear equations, etc.

Proposition 1 (Pr1): A vector is represented by an arrow (task 2).
Pr2: Unit vector is $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ equivalent to $\vec{U} = \frac{2}{\sqrt{9}}i - \frac{1}{\sqrt{9}}j + \frac{2}{\sqrt{9}}k$. (Task 7).
Pr3: Measure of the angle θ is $83.79°$ (task 10).

Procedures: Main procedure 1 (Mpc1): Represent the vector (task 2).

Auxiliary procedure 1.1 (Apc 1.1): Draw the Cartesian plane.
Apc1.2: Locate the end point of the vector (3, 4).
Apc1.3: Draw the arrow (directed segment) from (0, 0) to (3, 4) and symbolically called P1 $\vec{V} = (3,4)$.
Mpc2: Find the unit vector (task 7).
Apc2.1: Use the formula: $\vec{U} = \frac{a_1}{\|\vec{A}\|}i + \frac{a_2}{\|\vec{A}\|}j + \frac{a_3}{\|\vec{A}\|}k = \frac{\vec{A}}{\|\vec{A}\|}$.
Apc2.2: Substitute the components of the vector to find its norm $\vec{A} = (2,-1,2)$, $\|\vec{A}\| = \sqrt{(2)^2 + (-1)^2 + (2)^2}$.
Apc2.3: Do arithmetic operations of potentiation and establishment to find the norm: $\sqrt{4+1+4} = \sqrt{9} = 3$.
Apc2.4: Replace the components of $\vec{A}$ and the norm $\sqrt{9} = 3$ in the formula: $\vec{U} = \frac{2}{\sqrt{9}}i - \frac{1}{\sqrt{9}}j + \frac{2}{\sqrt{9}}k = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.
Mpc3: Find the angle between the vectors $\vec{A} = (2,2,-1)$ y $\vec{B} = (5,-3,2)$.
Apc3.1: Use the formula $\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\|\cos(\theta)$.
Apc3.2: Find the scalar product between the vectors $\vec{A}$ & $\vec{B}$, obtaining $\left((2*5) + \left(2*(-3)\right) + (-1*2)\right)$.
Apc3.3: Apply the arithmetic operations and law of signs to obtain the scalar: $10 + (-6) + (-2) = 2$.
Apc4: Find the norm of the vector $\vec{A} = (2,2,-1) = \|\vec{A}\| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$.
Apc5: Find the norm of the vector $\vec{B} = (5,-3,2) = \|\vec{B}\| = \sqrt{(5)^2 + (-3)^2 + (2)^2} = \sqrt{38}$.
Apc6: Substitute the values obtained in the formula $\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\|\cos(\theta)$ working it in two equivalent ways: 1) $2 = (\sqrt{9})(\sqrt{38})\cos(\theta)$ y 2) $\cos(\theta) = \frac{2}{(\sqrt{9})(\sqrt{38})}$.
Apc7: Apply the inverse cosine of the argument to find the value of the angle: $\theta = cos^{-1}\left(\frac{2}{(\sqrt{9})(\sqrt{38})}\right) = 83.79°$ (**Figure 10**).

Arguments: **Argument 1 (A1) for task 2:** Thesis: Vector is represented with an arrow.
Reason 1 (R1): P1 constructs a Cartesian coordinate plane and locates the point (3, 4).
R2: P1 drew an arrow from the point (0, 0) to (3, 4).
R3: In response to task 1 P1, other participants understand the vector as a line segment.
Conclusion: The vector is represented by an arrow or line segment.
Argument (A2) for task 3: Thesis: The unit vector is: $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.
R1: Initially, P1 and other participants used the formula: $\vec{U} = \frac{a_1}{\|\vec{A}\|}i + \frac{a_2}{\|\vec{A}\|}j + \frac{a_3}{\|\vec{A}\|}k = \frac{\vec{A}}{\|\vec{A}\|}$.
R2: Then, they found the norm $\|\vec{A}\| = 3$.
R3: Finally, through arithmetic operations, P1 and other participants found the unit vector following the formula.
Conclusion: based on the set of ratios, the unit vector is $\vec{U} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.

Based on the information in **Table 3**, SFs related to PO will be constructed. The links between the objects that support the mathematical connection are visualized (**Figure 17**).

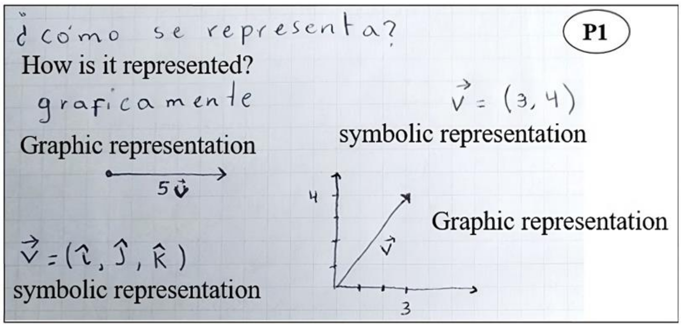

Figure 16. Graphic & symbolic representation of vector (Source: Authors' own elaboration)

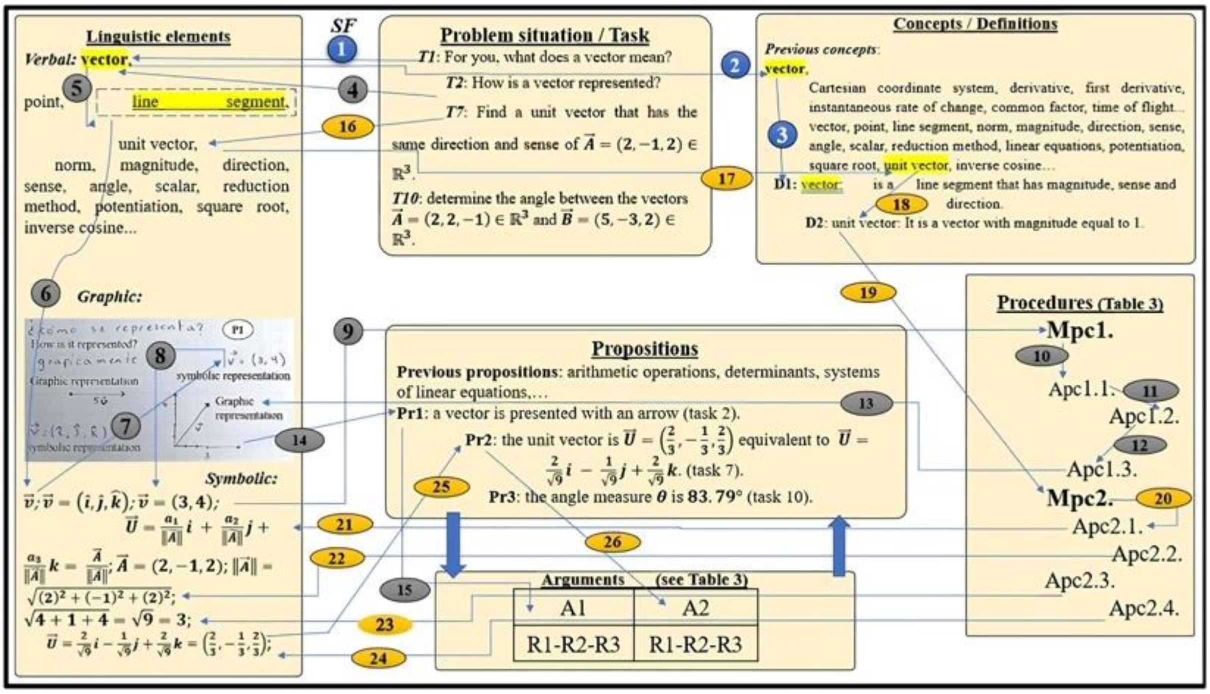

Figure 17. Semiotic functions made by P1 (Source: Authors' own elaboration)

Next, **Table 4** shows the mathematical connections established by P1 when solving tasks 1, 2, and 7, considering OSA tools, where the mathematical connection is made up of a conglomerate of practices, processes, objects (**Figure 17**) and SFs that relate them (in **Figure 17** in blue for task 1; gray for task 2 and in orange color for task 7).

Table 4 shows the mathematical connections established by P1 to solve tasks 1, 2, and 7, mainly based on mathematical practices.

Mathematical connections are essential to achieve an adequate and consistent result, but when a mathematical practice fails, it is directly reflected in the processes, objects, and SF. **Figure 18** presents an example of a mathematical connection of meaning.

Table 4. Detailed analysis of connections established by P1 when solving task

Mp	Processes	Objects	SF	Mathematical connections (ETC)
Mp1	-Signification/understanding -Problematization -Enunciation	Explain that a vector is a line segment with magnitude, sense, and direction.	SF1 SF2 SF3	Meaning Feature
Mp2	-Problem-solving -Enunciation -Representation	P1 stated that vector is represented graphically using a straight or ray segment & associated with symbolic expression $\vec{v}$.	SF4 SF5 SF6	Feature Different representations
…	…	…	…	…
Mp4	Problem-solving -Particularization -Representation -Enunciation	P1 proposed a particular case of a vector $\vec{v} =$ (3,4) and represented it graphically in a Cartesian coordinate plane. Pr1 and A1.	SF7 SF8 SF9 SF10	Part-whole Different representations Feature

Table 4 (Continued). Detailed analysis of connections established by P1 when solving task

Mp	Processes	Objects	SF	Mathematical connections (ETC)
Mp4	-Argumentation		SF11 SF12 SF13 SF14 SF15	
…	…	…	…	…
Mp19	-Problem-solving -Algorithmizing -Enunciation	P1 and other participants selected the formula $\vec{U}=\frac{a_1}{\lVert\vec{A}\rVert}i+\frac{a_2}{\lVert\vec{A}\rVert}j+\frac{a_3}{\lVert\vec{A}\rVert}k=\frac{\vec{A}}{\lVert\vec{A}\rVert}$ to find the unit vector.	SF16 SF17 SF18 SF19 SF20 SF21	Procedural Meaning
Mp5	-Problem-solving -Algorithmizing	Find the norm of the vector $\vec{A}=(2,-1,2)$ substituting the components into the formula: $\lVert\vec{A}\rVert=\sqrt{(2)^2+(-1)^2+(2)^2}$	SF22	Procedural
Mp6	-Problem-solving -Algorithmizing	Using arithmetic operations to find the norm: $\sqrt{4+1+4}=\sqrt{9}=3$.	SF23 SF13 SF14	Meanings
Mp7	-Problem-solving -Algorithmizing -Representation -Argumentation	Substitute the values obtained into the formula: $\vec{U}=\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$, which is an equivalent representation of $\vec{U}=\frac{2}{\sqrt{9}}i-\frac{1}{\sqrt{9}}j+\frac{2}{\sqrt{9}}k$. Pr2 and A2.	SF24 SF25 SF26	Procedural Different representations (equivalents) Implication
…	…	…	…	…

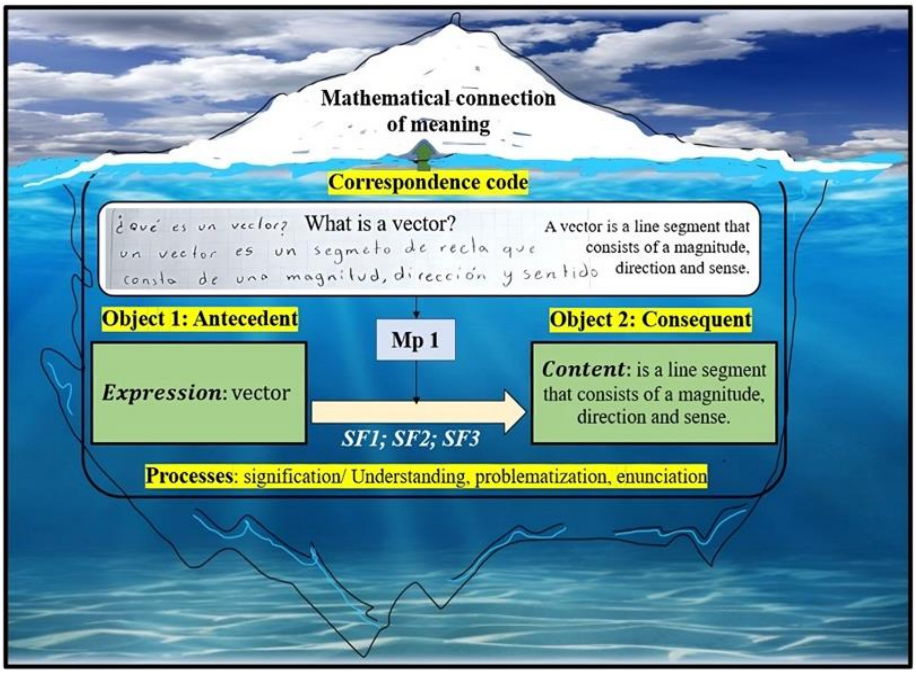

Figure 18. Mathematical connection of meaning (Source: Authors' own elaboration)

Notably, in this research, just as there are participants who successfully solved the problems, some also stopped establishing some connections, which is the leading cause of errors and difficulties in working with the vector concept. For example, to solve task 7, students P8 and P25 made personal connections (OSA view) or errors that led them to obtain an inadequate result (**Figure 19**).

On the other hand, to solve task 10, some participants, such as P3, P20, P15, P11, and P24, presented difficulties when finding the angle between vectors, which is caused by the failure to establish procedural connections such as finding the norm by the wrong use of formula, operate with radicals and extract square root (**Figure 20**).

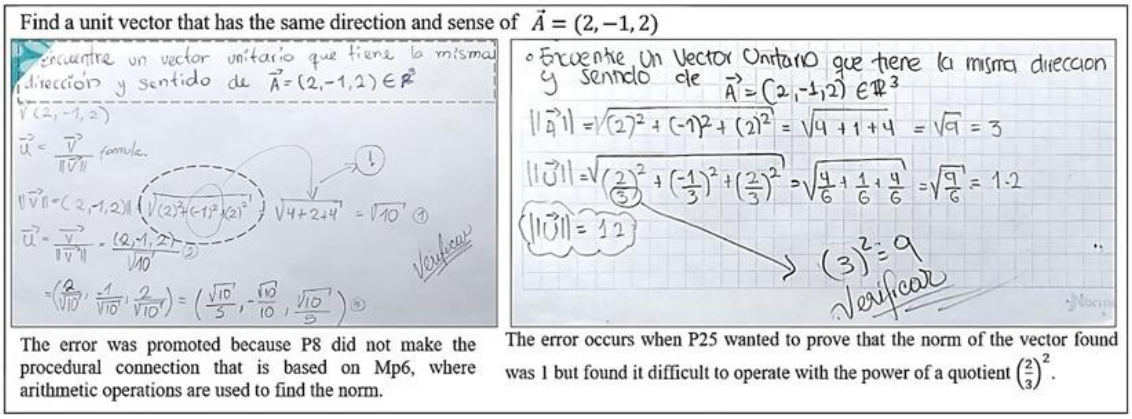

Figure 19. Errors caused by P8 & P25 in resolution of task 7 (Source: Authors' own elaboration)

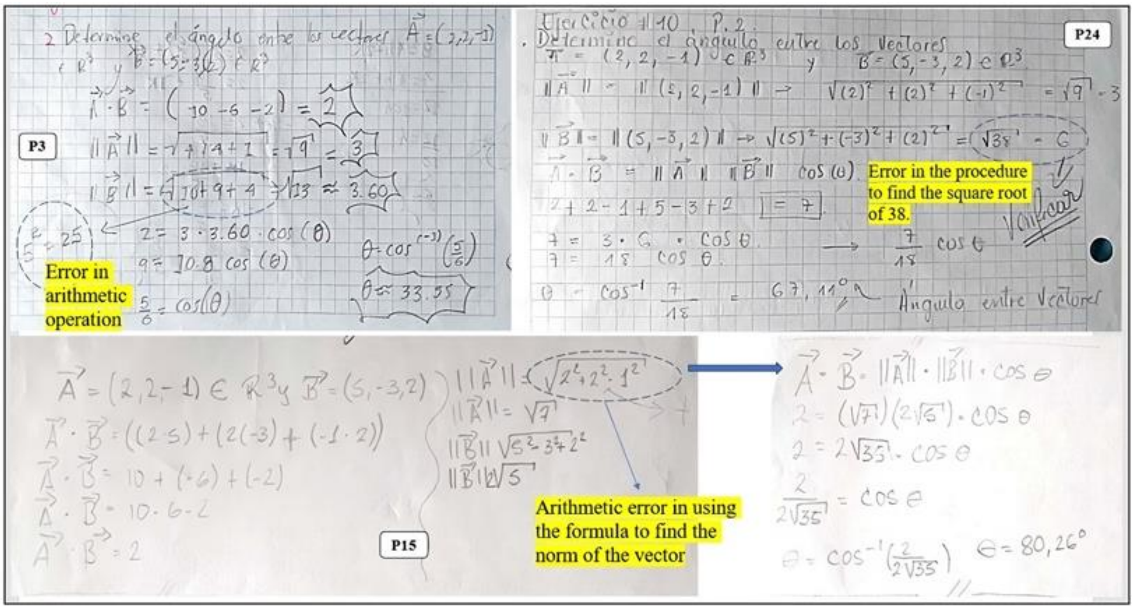

Figure 20. P3, P15, & P24 caused errors in resolution of task 10 (Source: Authors' own elaboration)

In addition to the errors mentioned above, in this research, other relevant results were found at a theoretical-practical level, given that in some episodes for P1's speech and his written production, they could not be categorized into some priori mathematical connection. Therefore, an episode of task 6 was deeply analyzed, where P1 stated the following:

> "There, the eight that I was adding I subtracted from the other side to leave the k alone, I still had the number 2, what I did was divide it and with the law of the ear I multiplied the ends by the ends and the middles by the media. In this case, it would be -16 times 1 and 5 times 2, I would stay with $-\frac{8}{5}$ because half of 16 is 8, and half of 10 would be 5".

In this extract of the transcription, there is an expression that cannot be categorized with any of the types of connections proposed a priori in the theoretical foundation, which led us to investigate this type of expressions "*the law of the ear*" and it was found that refers to mnemonics used by humans to memorize or remember procedures more quickly (Balbuena & Buayan, 2015; Bor & Owen, 2007; Díaz-Urdaneta & Prieto-González, 2015; Drushlyak et al., 2021; Fiallo, 2010; Hoffmann, 2018; Keller, 2016; Márquez-García, 2013; Nelson et al., 2013; Reyes-Gasperini et al., 2014; Vargas & Urzúa, 2018). Therefore, in this research, the new category for ETC is proposed as *MT connection based on mnemonics*, understood as the relationship established by the subject between a mnemonic rule (often a familiar resource) and a mathematical object, rule, or mathematical procedure to memorize and use strategically more easily.

Figure 21 proposes the new extension of ETC with MT connection based on mnemonics. Finally, the results of this research may have immediate applications in science and engineering education by providing conceptual and practical information on how to improve the presentation of mathematical concepts in vector problem-solving contexts and providing a solid foundation for the development of more effective and contextually relevant teaching strategies based on

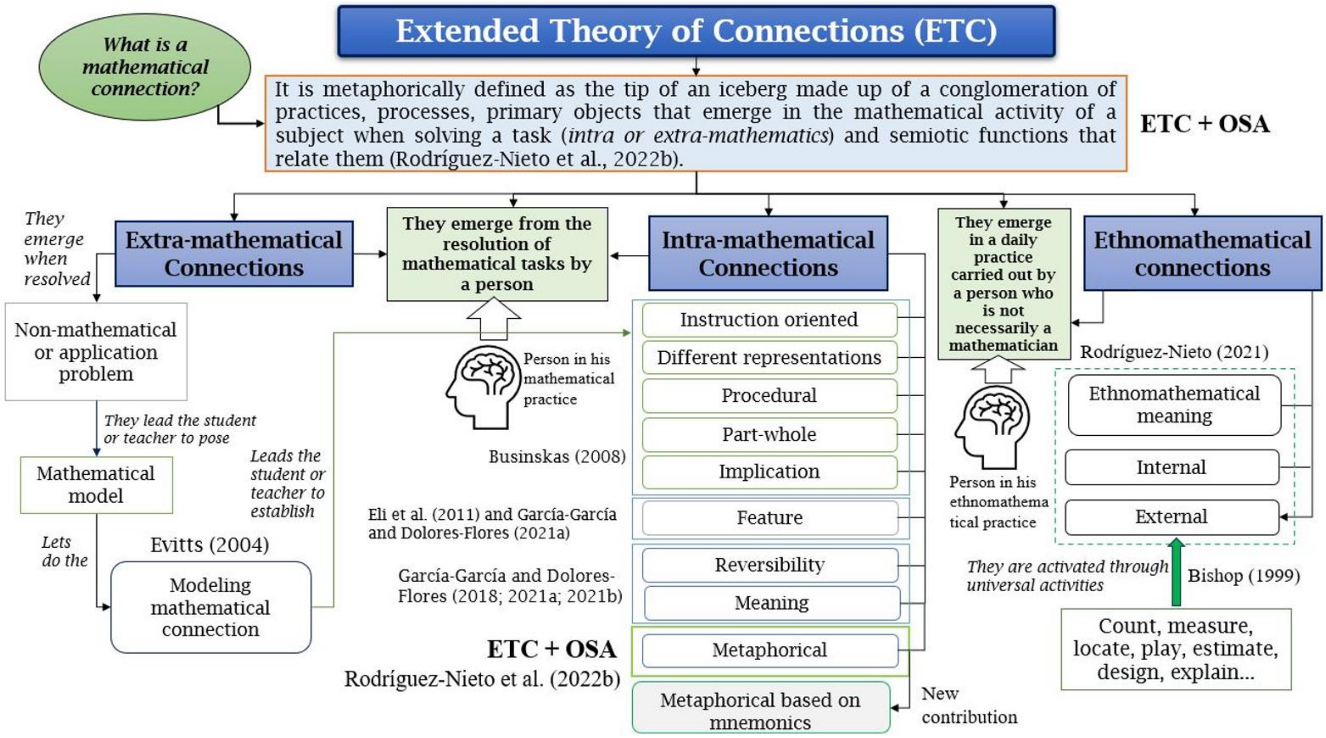

Figure 21. Synthesis of ETC (Adapted from García-García & Dolores-Flores, 2021a and Rodríguez-Nieto et al., 2022c)

mathematical connections that enrich students' procedures and their processes of meaning, problematization, communication and argumentation of their ideas. With this, the contribution to the natural and exact sciences is recognized.

DISCUSSION & CONCLUSIONS

In this research, the mathematical connections of university engineering students were analyzed when they solved problems about vectors, where the potential of their procedures and types of understanding demonstrated in the resolution of each problem was evident. Another fundamental aspect of this research is that mathematical connections are fundamental to developing any topic in mathematics and other sciences. For example, in differential and integral calculus, there are studies focused on derivatives, functions and integrals (Galindo-Illanes et al., 2022; García-García & Dolores-Flores, 2021b; Rodríguez-Nieto, 2021; Rodríguez-Vásquez & Arenas-Peñaloza, 2021; Rodríguez-Nieto et al., 2023), but they are also relevant in the study of vectors and their applications.

Particularly in this work, the primary role of students' well-organized written production is reflected. The teacher in charge of teaching the subject of vector calculus (authors of this research) always tells the participating students to consider step-by-step their reasoning, equipped with graphs, mention properties, use additive and multiplicative inverses, make graphs, use meanings, etc., in such a way that their procedures are argued and justified from mathematics and as algorithmic processes. A particular case was recognized when the students found the unit vector and proceeded to find the solution in detail. However, some students presented difficulties due to disconnections in the procedure to find the norm, as other authors have stated (e.g., Barniol & Zavala, 2014; Cárcamo et al., 2023; Gutiérrez & Martín, 2015; Tairab et al., 2020). However, for future research, a study can be promoted, where these errors are addressed and contribute to students' understanding of vectors and their use.

Another contribution of the results of this research is related to the implication of the teaching of vector calculus based on connections by the teacher of the participants in this study, which is relevant because the students have followed a path of problem solving. detailed or step by step, implicitly evidencing the influence of the connections expressed by their teacher. It is known that the emphasis of this article is not the teacher's teaching, but it could be stated that this teaching of mathematics has enhanced the ways of solving problems by students.

The importance of the results of this research is recognized, which emerged inductively and were born from the results and the emphasis placed on mathematical connections. For example, as the analysis is detailed, it was possible to observe and infer novel needs and contributions to ETC-OSA theory and practice, such as *MT connection based on mnemonics* that directly influences the teaching and learning of mathematics at any school grade, goes through the different ways of teaching mathematical concepts and is transversal to other sciences (Khatin-Zadeh et al., 2023). With this success, we notice that literature plays in favor of mnemonics and, at the same time, begs to be careful if they are used excessively, given that problems could be solved in a mechanized way. In fact, Makau et al. (2019) and Ni and Hassan (2019) used mnemonics with students because it makes it easier for them to perform mathematical procedures and they relate them to events in their daily lives to ensure understanding. This new category makes sense because the "research in mathematics education can validate these typologies, but it could also include other categories not yet identified" (García-García, 2019, p. 131).

MT connections based on mnemonics are inclusive, and to identify them we must recognize three elements:

(1) keywords that are similar to the word (or term) being referred to,

(2) in acronyms, mnemonics can also be identified when the first letter of each word is used in a list to construct another word, and

(3) acrostics are considered, which consist of constructing a sentence, where the first letter of each constitutes the term studied (Mastropieri & Scruggs, 1989).

Other authors (Maccini et al., 2007; Manalo et al., 2000; Mastropieri & Scruggs, 1989; Test & Ellis, 2005) maintain that teachers' strategies when using mnemonics are powerful because they allow the understanding of concepts and mathematical procedures and without a doubt, students are more fluent in solving problems using mnemonics, regardless of whether they have disabilities. Drushlyak et al. (2021) recommend mnemonics for the training of mathematics teachers and suggest that future studies continue working in different environments to teach and learn mathematical concepts.

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