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Opportunity to learn about optimization problems provided by undergraduate calculus textbooks: A case study

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Abstract

This study investigated opportunities to learn about optimization problems provided by undergraduate calculus textbooks. To accomplish this, we analyzed examples and practice problems from two calculus textbooks widely used in the teaching of calculus in the United States. Findings of this study indicate that a majority of the problems in both textbooks lack realistic and essential contexts, have matching or missing information, and require a low cognitive demand to solve. Setting up objective functions is either not necessary or it is straightforward for most problems in the two textbooks. In addition, both textbooks provide ample opportunities to interpret critical numbers and extrema in context. Overall, findings of this study suggest the existence of a relationship between known students' difficulties with solving optimization problems and the opportunities to learn about optimization problems provided by calculus textbooks. Implications for several stakeholders, including calculus textbook authors and calculus instructors are discussed.

Keywords: opportunity to learn, textbook research, textbook analysis, calculus, optimization problems

INTRODUCTION

The concept of opportunity to learn originated in the early 1960s. Carroll (1963) defined opportunity to learn as the time allowed for learning a particular topic. This study uses Husen's (1967) definition of opportunity to learn which, according to Floden (2002), is the most common definition of opportunity to learn used in the mathematics education literature. According to Husen (1967), opportunity to learn refers to "whether or not … students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (p. 162-163). Mathematics textbooks are one such opportunity from which students can learn how to solve certain types of problems. The role of textbooks as an opportunity to learn has received considerable attention in the research literature on the teaching and learning of K-12 mathematics. Among other things, this research has investigated students' opportunity to learn about mathematical topics such as linear functions and trigonometry (Wijaya et al., 2015), addition and subtraction of fractions (Alajmi, 2012; Charalambous et al., 2010), calculus (Haghjoo et al., 2023), probability (Jones & Tarr, 2007), statistics (Pickle, 2012), reasoning and proof (Stylianides, 2009; Thompson et al., 2012), proportional reasoning (Dole & Shield, 2008), and deductive reasoning (Stacey & Vincent, 2009).

There is, however, a paucity of research that has examined the opportunity to learn about various mathematics topics provided by undergraduate mathematics textbooks, which is the motivation for this this study. Of the few available studies on the opportunity to learn provided by undergraduate mathematics textbooks, Mesa et al. (2012) examined the opportunity to learn about exponential functions, logarithmic functions, and transformations of functions provided by college algebra textbooks used at community colleges and four-year institutions in the United States. Mesa et al. (2012) found that "textbooks, independent of the type of institution in which they are used, present examples that have low cognitive demands, expect single numeric answers, emphasize symbolic and numerical representations, and give very few strategies for verifying correctness of the solutions"

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Contribution to the literature

- While much research has reported on the opportunity to learn about various mathematical topics provided by K-12 mathematics textbooks, there is a paucity of similar research at the undergraduate level. A growing body of research has reported on students' difficulties with solving optimization problems in introductory university calculus.
- Among other things, the current study reports on the analysis of two commonly used undergraduate calculus textbooks in the United States with a focus on opportunities to learn about optimization problems.
- Additionally, the current study compares the opportunities to learn about optimization problems provided by these textbooks with students' difficulties with solving optimization problems previously reported in the research literature.

(p. 76). Findings by Lithner (2004) indicate that most of the exercises found in an undergraduate calculus textbook commonly used in a Swedish university "… may be solved by mathematically superficial strategies, often without actually considering the core mathematics of the book section in question" (p. 405).

Findings of a recent study (Mkhatshwa, 2022b) on the opportunity to learn about ordinary and partial derivatives provided by commonly used undergraduate calculus textbooks in the United States revealed that a majority of the exercises given these textbooks do not provide opportunities to engage in higher levels of covariational reasoning. Proponents of covariational reasoning (Carlson et. al., 2002) have hypothesized that students who are able to engage in higher levels of covariational reasoning tend to develop robust understandings of rate quantities such as derivatives in calculus (cf. Nagle et al., 2013; Tyne, 2016). In another study, Chang et al. (2016) investigated the nature and prevalence of coordination of multiple representation tasks (i.e., tasks that require students to translate functions from one representation to another) in a reformed calculus textbook. Chang et al. (2016) reported that the prevalence of these tasks differs by chapter and by topic in the textbook. Specifically, Chang et al. (2016) found that tasks involving translating algebraic functions to verbal functions are prevalent in the early chapters of the textbook while tasks involving translating algebraic functions to graphical functions frequently occur in later chapters of the same textbook.

Thus, in order to build on the few studies that have investigated the opportunity to learn provided by mathematics textbooks at the undergraduate level, the current study examined the opportunity to learn about univariate optimization problems (UOPs) provided by two textbooks commonly used in the teaching of calculus in the United States. According to Mkhatshwa (2022a), a UOP is an optimization problem where the objective function (i.e., the function whose maximum or minimum value(s) is to be found) is a real-valued function of a single variable. Our motivation to examine the opportunity to learn about optimization problems provided by calculus textbooks emanates from the fact that optimization problems form an integral part of the study of first-semester differential calculus at the undergraduate level in the United States.

RELATED LITERATURE AND RESEARCH QUESTIONS

The Role of Textbooks in Students' Learning of Mathematics

Without a doubt, textbooks play a significant role in students' learning of mathematics. According to Reys et al. (2004), "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Similar arguments were made by other researchers (cf. Alajmi, 2012; Kolovou et al., 2009). Findings by Begle (1973) indicate that "most student learning is directed by the text rather than the teacher" (p. 209). Begle's (1973) findings are consistent with the findings of other researchers such as Törnroos (2005) and Wijaya et al. (2015). Remarking on the role of textbooks in the teaching and learning of mathematics, Charalambous et al. (2010) asserted that "textbooks afford probabilistic rather than deterministic opportunities to learn mathematics" (p. 118). That is, students' opportunities to learn mathematics are not only limited to course textbooks as there may be other opportunities (e.g., course lectures) through which students could learn mathematics besides mathematics textbooks.

Types of Context

The term, context, has been defined in several ways by different researchers in mathematics education. According to White and Mitchelmore (1996), "in calculus, the context of an application problem may be a realistic or artificial "real-world" situation, or it may be an abstract, mathematical context at a lower level of abstraction than the calculus concept that is to be applied" (p. 81). White and Mitchelmore's (1996) understanding of context is consistent with that of other researchers (cf. Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2005). According to Wijaya et al. (2015), a mathematical problem (hereafter, problem) can have one of three types of context, namely relevant and essential, camouflage, or bare (only mathematical symbols). Alajmi (2012) refers to bare problems as problems that are situated in a “pure mathematics context” (p. 243). Problems with a camouflage context “are merely dressed up bare problems, which do not require modeling because the mathematical operations needed to solve the problem are obvious” (Wijaya et al., 2015, p. 45). A relevant and essential context is also referred to as a realistic context in the research literature (e.g., de Lange, 1995; van den Heuvel-Panhuizen, 2005). Part of our investigation in this study consisted of examining different types of contexts in examples and practice problems on optimization problems provided in the two textbooks we analyzed in the current study.

Types of Information

Several researchers have identified three types of information a problem could have, namely matching, missing, or superfluous (cf. Maass, 2007, 2010; Wijaya et al., 2015). A problem is said to have matching information if all the information required to solve it is included in the problem statement. A problem has missing information if some of the information needed to solve the problem is not immediately available to the solver i.e., the solver has to deduce this information from the problem statement. A problem with superfluous information is one in which the problem statement contains the necessary information needed to solve it, in addition to extraneous or irrelevant information that is not helpful in solving the given problem. Wijaya et al. (2015) argued that:

> Providing more or less information than needed for solving a context-based task [problem] is a way to encourage students to consider the context used in the task [problem] and not just take numbers out of the context and process them mathematically in an automatic way (p. 45).

Maass (2010) recommends that students should be given problems that would enable them to deal with these three different types of information. Part of our investigation in this study consisted of examining different types of information in examples and practice problems on UOPs provided in the two textbooks we analyzed in the current study.

Types of Cognitive Demand

Much research has reported on three types of cognitive demands commonly found in tasks provided in mathematics textbooks, namely reproduction, connection, and reflection (cf. Charalambous et al., 2010; Kolovou et al., 2009; Mesa et al., 2012; Wijaya et al., 2015). Reproduction tasks are routine problems that require the lowest level of cognitive demand to solve. These problems can often be solved by recalling memorized mathematical algorithms. On the other hand, connection problems are non-routine in nature and may require flexibility with using multiple representations (i.e., algebraically, numerically, graphically, or verbally) to solve. In their investigation of the opportunity to learn provided by 10 college algebra textbooks in the United States, Mesa et al. (2012) reported that “textbooks, independent of the type of institution in which they are used, present examples that have low cognitive demands …” (p. 76). Reflection tasks require the highest level of cognitive demand to solve. According to Wijaya et al. (2015), reflection tasks “include complex problem situations in which it is not obvious in advance what mathematical procedures have to be carried out” (p. 46). Among other things, the current study examined different types of cognitive demands in examples and practice problems on UOPs provided by the two textbooks we analyzed in the study.

Research on Students’ Understanding of UOPs

Although the focus of the current study is on the opportunity to learn about UOPs provided by calculus textbooks, it is important to discuss the existing research literature on students’ understanding of UOPs for comparison. Specifically, we want to be able to make a comparison (if and when possible) between what students are able (or not able) to do when working with UOPs as documented in the research literature and the opportunities they have to learn about UOPs provided by calculus textbooks.

A number of studies have found that setting up the objective function is particularly challenging for students when tasked with solving UOPs (cf. LaRue & Infante, 2015; Swanagan, 2012). Swanagan (2012) reported on a student who incorrectly used the equation of the parabola $y = x^2$ as the objective function in a UOP that was about finding a point on the parabola closest to the point (1, 0). In the same study, another student confused the objective function with a perimeter function in a UOP that was about finding the minimum cost to fence a rectangular plot of land.

A related line of research has reported that setting up the objective function when tasked with solving UOPs that have real-world contexts tends to be difficult for students even if the objective function is simple and students are familiar with the context of the UOP (cf. LaRue & Infante, 2015).

Evidence from research suggests that calculating/interpreting critical numbers or calculating/interpreting extrema when solving UOPs that have real-world contexts is problematic for students (cf. Brijlall & Ndlovu, 2013; Dominguez, 2010; Mkhatshwa, 2019; Swanagan, 2012). Two students in Swanagan’s (2012) study relied on guesswork to determine extrema in a cost minimization UOP. Most of the 94 students in Dominguez’s (2010) study struggled with interpreting extrema in a UOP situated in a profit maximization context. In another UOP situated in a profit maximization context, Mkhatshwa (2019) reported on three students who confused extrema (i.e., maximum profit) with a critical number (i.e., the number of units that must be produced and sold to maximize profit). Brijlall and Ndlovu (2013) reported similar results in another UOP situated in a volume minimization context.

Findings by Mkhatshwa (2019) and Borgen and Manu (2002) indicate that justifying/verifying extrema is challenging for students. Sixteen of the 24 students in Mkhatshwa's (2019) had difficulty verifying extrema while reasoning about a UOP situated in a profit maximization context. Borgen and Manu (2002) reported on a student who correctly used calculus to calculate the minimum value of the quadratic function $y = 2x^2 - x + 1$. However, when the student was asked to explain how she knew the function has a minimum value and not a maximum value, she incorrectly stated that this was because the coefficient of the linear term of the function is negative.

Research Aim

Several studies have reported on difficulties typically exhibited by students when working with optimization problems. These difficulties include setting up the objective function, calculating and interpreting the objective function's critical and extreme value(s), and verifying whether the objective function attains a maximum/minimum value at a particular critical value (cf. Borgen & Manu, 2002; LaRue & Infante, 2015; Mkhatshwa, 2019; Swanagan, 2012). It has long been established that the content students learn in mathematics classrooms and how they learn it is often directed by course textbooks (cf. Begle, 1973; Reys et al., 2004; Wijaya et al., 2015).

We are not aware of any study that has investigated opportunities to learn about optimization problems provided by first-semester calculus textbooks (where optimization problems are ordinarily covered at the undergraduate level), let alone how these opportunities may be related to students' difficulties with solving optimization problems. The aim of the current study, a case study that examined opportunities to learn about optimization problems provided by two widely used calculus textbooks in the United States, is to address this knowledge deficiency. The current study investigated the following research questions:

1. What types of context (relevant and essential, camouflage, no context), information (matching, missing, superfluous), and cognitive demands (reproduction, connection, reflection) are found in undergraduate calculus textbooks?
2. How do reported students' difficulties with solving optimization problems compare to the opportunities to learn about optimization problems provided by calculus textbooks?

METHODOLOGY

Calculus Textbooks Analyzed

This qualitative study reports on opportunities to learn about optimization problems provided by two textbooks that are commonly used in the teaching of regular calculus (commonly known as engineering calculus) and applied calculus in the United States. Details about the textbooks are provided in **Table 1**.

There are generally three flavors of calculus offered at the undergraduate level in the United States, namely regular calculus, business calculus, and life sciences calculus. In some institutions, a business calculus or life sciences calculus course may be referred to as an applied calculus course. After consulting with major calculus textbook publishers in the United States, we determined that the textbooks analyzed in the current study are among the most commonly used textbooks in the teaching of regular or applied calculus, hence their selection. Additionally, we selected these textbooks because their latest editions are recent, and they have many previous editions, suggesting that they have undergone several revisions over the years and may thus present better learning opportunities for students compared to similar textbooks with fewer editions. We remark that in the United States, regular calculus is typically taken by students pursuing STEM (science, technology, engineering, and mathematics) degrees, especially engineering, while applied calculus is generally taken by students pursuing non-STEM degrees such as economics.

Data Sources and Method of Data Analysis

To answer our research questions, we analyzed two sources of data. These sources of data are examples and practice problems on optimization problems given in each of the sections identified in **Table 1**. **Table 2** provides counts of the number of examples and practice problems we analyzed in each section identified in **Table 1**.

Table 1. Analyzed textbooks

Textbook name	Author(s)	Textbook abbreviation	Section(s) analyzed	Textbook publisher
Calculus: Early transcendentals (9th ed)	Stewart, J., Clegg, D., & Watson, S. (2021)	RC	4.7: Optimization problems	Cengage Learning
Applied calculus (6th ed)	Hughes-Hallett et al. (2018)	AC	4.3: Global maxima and minima	Wiley

Table 2. Counts of expository sections, examples, and practice examples analyzed

Textbook abbreviation	Section	Examples	Practice problems
RC	4.7	6	88
AC	4.3	3	60

Table 3. Analytical framework reproduced from Wijaya et al. (2015)

Task characteristic	Sub-category	Explanation
Type of context	No context	-Refers to only mathematical objects, symbols, or structures.
	Camouflage context	-Experiences from everyday life or common sense reasoning are not needed. -The mathematical operations needed to solve the problems are already obvious. -The solution can be found by combining all numbers given in the text.
	Relevant & essential context	-Common sense reasoning within context is needed to understand & solve problem. -The mathematical operation is not explicitly given. -Mathematical modeling is needed.
Type of information	Matching	-The task contains the exact amount of information needed to find the solution.
	Missing	-Task contains less information than needed so students need to find missing information.
	Superfluous	-Task contains more information than needed so students need to select information.
Type of cognitive demand	Reproduction	-Reproducing representations, definitions, or facts. -Interpreting simple and familiar representations. -Memorization or performing explicit routine computations/procedures.
	Connection	-Integrating and connecting across content, situations, or representations. -Non-routine problem solving. -Interpretation of problem situations and mathematical statements. -Engaging in simple mathematical reasoning.
	Reflection	-Reflecting on and gaining insight into mathematics. -Constructing original mathematical approaches. -Communicating complex arguments and complex reasoning.

The data (optimization examples and practice problems) were analyzed in two stages. In the first stage of the analysis, we coded examples and practice problems on optimization problems given in each textbook using a textbook analytical framework proposed by Wijaya et al. (2015).

The framework, reproduced in **Table 3**, has three dimensions of analysis, namely type of context, type of information, and type of cognitive demand. We coded a total of 157 tasks, consisting of all the examples and practices problems noted in **Table 2**. In particular, we coded 94 tasks from the RC textbook, consisting of six examples and 88 practice problems noted in **Table 2**. A total of 63 tasks from the AC textbook, consisting of the three examples and 60 practice problems noted in **Table 2** were coded. In analyzing the tasks, two independent researchers coded the tasks and compared their findings to ensure the reliability of the results. They discussed differences in coding until they reached agreement. Following is an illustration of our coding using three tasks that were selected from at least one of the textbooks analyzed in the current study.

> **Example 1 (Hughes-Hallett et al., 2018, p. 188):** Find the global maximum and minimum of $f(x) = x^3 - 9x^2 - 48x + 52$ on the interval $-5 \leq x \leq 14$.
>
> **Solution:** We have calculated the points of this function previously using
>
> $$f'(x) = 3x^2 - 18x - 48 = 3(x+2)(x-8),$$
>
> so $x = -2$ and $x = 8$ are critical points. Since the global maxima and minima may occur at critical points or at endpoints of the interval, we evaluate f at these four points:
>
> $$f(-5) = -58$$
>
> $$f(-2) = 104$$
>
> $$f(8) = -396$$
>
> $$f(14) = 360$$
>
> Comparing these four values, we see that the global maximum is 360 and occurs at $x = 14$, and that the global minimum is -396 and occurs at $x = 8$.

We coded example 1

(1) as having no context because the problem in the example is bare i.e., it neither has a camouflage context nor a realistic and essential context,

(2) as having matching information because it contains the exact amount of information needed to solve it, and

(3) as a reproduction task because the strategy required to solve it requires performing an explicit routine procedure.

Practice problem 34 (Hughes-Hallett et al., 2018, p. 192): A grapefruit is tossed straight up with an initial velocity $50\ ft/sec$. The grapefruit is five feet above the ground when it is released. It's height, in feet, at time t seconds is given by $y = -16t^2 + 50t + 5$. How high does it go before returning to the ground?

We coded practice problem 34

(1) as having a camouflage context because the operations needed to solve the problem are obvious and the context can be ignored when solving the problem,

(2) as having matching information because it contains the exact amount of information needed to solve it, and

(3) as a reproduction task because the strategies required to solve it requires performing explicit routine procedures.

Practice problem 70 (Stewart et al., 2021, p. 346): A company operates 16 oil wells in a designated area. Each pump, on average, extracts 240 barrels of oil daily. The company can add more wells but every added well reduces the average daily output of each of the wells by eight barrels. How many wells should the company add in order to maximize daily production?

We coded practice problem 70

(1) as having a relevant and essential context because reasoning with the context of the task is needed to understand and solve the problem,

(2) as having missing information because students will have to deduce some missing information such as an algebraic expression for the number of barrels that each well will produce before they can solve the problem posed in the question, and

(3) as a reflection task because solving it requires engaging in complex reasoning. Specifically, finding the production function in terms of the number of wells is not straightforward i.e., requires students to engage in complex reasoning.

In the second stage of the analysis, we used the themes from the literature on students' understanding of optimization problems reviewed earlier. To reiterate, common themes from the literature are that setting up the objective function, finding and interpreting critical numbers/extrema, and verifying extrema is problematic for students when working with optimization problems (cf. Borgen & Manu, 2002; LaRue & Infante, 2015; Swanagan, 2012). We examined the nature of the opportunities provided in the two textbooks with regard to setting up the objective function. Specifically, we examined the nature (routine or non-routine) of the objective function in each example or practice problem, as well as whether or not the objective function has multiple critical numbers. In addition, we examined whether or not the textbooks place an emphasis on the need to interpret critical numbers/extrema, especially in examples or practice problems that have a camouflage or realistic and essential context. Finally, we examined whether or not the textbooks encourage verifying extrema such as using the second derivative test to verify the location of a relative maximum/minimum for a given objective function.

Transparency and replicability of the current study was considered through the lens of Aguinis and Solarino (2019) who proposed 12 criteria for ensuring transparency and thus the replicability of qualitative research studies. The criteria we consider to be applicable and thus used in the current study include specifying the qualitative methodology (case study) used in the study, specifying sampling procedures (i.e., how the textbooks were selected) and the unit of analysis (examples and practice problems), and explaining how the data were summarized, including a summary of how inter-rater reliability was established.

RESULTS

The results of coding optimization tasks using Wijaya et al.'s (2015) textbook analytical framework (**Table 3**) are summarized in **Table 4**. To reiterate, a total of 157 tasks were coded. Of these tasks, 94 tasks are from the RC textbook, and 63 tasks are from the AC textbook.

Types of Context Provided in Tasks

As can be seen in **Table 4**, it is rather disappointing that a majority of the optimization tasks (38% in the RC textbook and 75% in the AC textbook) in both textbooks have no context. One would expect the AC textbook to have more tasks with either a camouflage context or

Table 4. A summary of textbook analysis results

Textbook abbreviation	Type of context	Type of information	Type of cognitive demand
RC	No context: 36 (38%)	Matching: 53 (56%)	Reproduction: 51 (54%)
	Camouflage content: 26 (28%)	Missing: 41 (44%)	Connection: 34 (36%)
	Relevant & essential context: 32 (34%)	Superfluous: 0 (0%)	Reflection: 9 (10%)
AC	No context: 47 (75%)	Matching: 61 (97%)	Reproduction: 60 (95%)
	Camouflage content: 14 (22%)	Missing: 2 (3%)	Connection: 1 (2%)
	Relevant & essential context: 2 (3%)	Superfluous: 0 (0%)	Reflection: 2 (3%)

relevant and essential context by virtue of it being an applied calculus textbook.

On the contrary, and as can be seen in **Table 4**, tasks with a relevant and essential context are almost nonexistent in the AC textbook. The RC textbook provides a substantial number of tasks with a relevant and essential context-this is much appreciated as it provides opportunities for students to make sense of mathematical ideas in real-world contexts, thereby demonstrating the importance of mathematics in real life.

Types of Information Provided in Tasks

Both textbooks are dominated by optimization tasks that have matching information. Specifically, 56% of the tasks in the AC textbook have matching information and 97% of the task in the AC textbook have matching information. A notable number of tasks have missing information in the AC textbook. This is commendable as it provides opportunities for students to make sense of the context of the tasks in order to deduce missing information that is necessary to solve any optimization task with missing information. Similar opportunities are lacking in the AC textbook. Lastly, neither textbook has tasks with superfluous information. In other words, none of the two textbooks provide opportunities for students to make sense of the context of the tasks in order to distinguish between extraneous information and important information that is necessary to solve any optimization task with superfluous information.

Types of Cognitive Demands Provided in Tasks

The number of reflection tasks i.e., tasks with a high cognitive demand is extraordinarily low in both textbooks. Precisely, only 10% of the tasks in the RC textbook are reflection tasks and only 3% of the tasks in the AC textbook are reflection tasks. According Wijaya et al.'s (2015) framework, the shortage of reflection tasks indicates that the two textbooks do not provide opportunities to "… communicate complex arguments and complex reasoning" (p. 52). Sadly, a majority of the tasks in both textbooks (54% in the RC textbook and 95% in the AC textbook) are reproduction tasks i.e., they are tasks that only require "…memorization or performing explicit routine computations/procedures" (p. 52) to solve.

Opportunities to Set Up Objective Functions in Tasks

Previous research on students' thinking about optimization problems indicates that setting up the objective function is often one of the stumbling blocks for students when tasked with solving optimization problems (cf. LaRue & Infante, 2015; Swanagan, 2012). Our analysis of the optimization problems in both textbooks revealed that a majority of the problems in both textbooks either provide the objective function (e.g., practice problem 34 reproduced before) or that setting up the objective function is straightforward such as finding the area function of a rectangular field when given its perimeter. In fact, only 5% of the 63 optimization tasks provided in the AC textbook require setting up an objective function to solve. The 5% of tasks that require setting up an objective function are practice problem 50, practice problem 51, and practice problem 52 on pages 192 and 193 in Hughes-Hallett et al. (2018). On the contrary, nearly 71% of the 94 optimization tasks in the RC textbook require setting up an objective function to solve. The objective functions are however routine and setting them up (or recalling them) is straightforward. Following is a reproduction of two tasks from the RC textbook that require setting up an objective function to solve:

Practice problem 3 (Stewart et al., 2021, p. 342): Find two positive numbers whose product is 100 and whose sum is a minimum.

Practice problem 18 (Stewart et al., 2021, p. 343): A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

Finding the objective function in practice problem 3 is straightforward, while the objective function in practice problem 18 can simply be found by recalling the formula for calculating the volume of a rectangular prism. Taken together, opportunities for students to make sense of realistic problem situations and to set up complex objective functions are minimal in both textbooks.

Opportunities to Interpret Critical Numbers and Extrema

As previously noted, a substantial number of the optimization tasks in both textbooks have no context, and hence do not provide opportunities to interpret critical numbers or extrema in either a camouflage or relevant and essential context. We note, however, that all the other tasks that either have a camouflage or relevant and essential context provide ample opportunities for students to interpret critical numbers or extrema. Following is a reproduction of one such tasks:

Practice problem 57 (Hughes-Hallett et al., 2018, p. 193): The quantity of a drug in bloodstream t hours after a tablet is swallowed is given, in mg, by

$$q(t) = 20(e^{-t} - e^{-2t}).$$

(a) How much of the drug is in the bloodstream at time $t = 0$?

(b) When is the maximum quantity of the drug in the bloodstream? What is that maximum?

(c) In the long run, what happens to the quantity?

The prompt "when is the maximum quantity of the drug in the blood stream" in part (b) is a direct instruction that emphasizes the need to interpret a critical number i.e., the amount of time it would take for the quantity of the drug to be maximized in the bloodstream. Similarly, the prompt "What is that maximum" in part (b) is another direct instruction that promotes the need to interpret extrema i.e., the maximum quantity of the drug in the bloodstream.

Opportunities to Verify/Justify Extrema

Motivated by research on students' thinking about optimization problems that shows that verifying/justifying extrema (i.e., showing that a critical number(s) of the objective function gives a maximum/minimum value of the objective function) is problematic for students, we examined opportunities to verify/justify extrema provided in the two textbooks. Our examination revealed that that of the 94 optimization tasks in the RC textbook, consisting of 88 practice problems and six examples, verifying/justifying extrema using the first derivative test, second derivative test, or the closed interval method (Stewart et al., 2021) was encouraged in five of the examples, but was not encouraged in any of the practice problems. Following is a reproduction of one of the examples, where extrema was justified:

Example 6 (Stewart et al., 2021, p. 341): A store has been selling 200 TV monitors a week at $350 each. A market survey indicates that for each $10 rebate offered to buyers, the number of monitors sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize revenue?

Solution: If x is the number of monitors sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units, the price is decreased by $10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} * 10$ and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the first derivative test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \frac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of $125.

In the preceding solution to example 6, Stewart et al. (2021) specifically mentioned using the first derivative test to justify extrema through the comment: "This value of x [450]gives an absolute maximum by the first derivative test …" (p. 341). Stewart et al. (2021) further noted that an informal graphical approach i.e., observing that the graph of the objective function [revenue function] is a parabola that opens downward could be used to justify that the objective function attains its maximum value at the critical number $x = 450$. Analysis of the 63 optimization tasks in the AC textbook, consisting of 60 practice problems and three examples, revealed that verifying/justifying extrema using the second derivative test or the closed interval method was encouraged in two of the examples, but was not encouraged in any of the practice problems. The solution to example 1 presented in the Data Sources and Method of Data Analysis section shows how, for instance, Hughes-Hallettt et al. (2018) used the closed interval method to verify/justify extrema in this example.

DISCUSSION AND CONCLUSIONS

There are six findings from this study. Following is a discussion of each of these findings in light of the themes from the literature review. First, 66% of the tasks in the RC textbook either have no context or they have a camouflage context. 97% of the tasks in the AC textbook either have no context or they have a camouflage context. In other words, both textbooks provide limited opportunities for students to make sense of mathematical ideas in relevant and essential contexts. We especially note that although tasks with a camouflage context are typically preferred over tasks with no context at all, camouflage contexts are not viewed as authentic because, while they dress up the mathematical operations, they do not really require deep thought about the context and the relationship to the mathematics involved.

Second, all the tasks in both textbooks either have matching information or they have missing information. In fact, Mass (2010) argued that it is important for students to be exposed to tasks with all types of information, namely missing, matching, and superfluous. While tasks with matching information are easy to solve in that all the information needed to solve them is provided, Wijaya et al. (2015) remarked on the benefits of providing opportunities for students to work on tasks that either having missing information or superfluous information:

> Providing more or less information than needed for solving a context-based task [problem] is a way to encourage students to consider the context used in the task [problem] and not just take numbers out of the context and process them mathematically in an automatic way (p. 45).

The absence of tasks with superfluous information in both textbooks is concerning. Specifically, the absence of these type of tasks suggests that opportunities meant to encourage students to consider the context in order to determine what information is useful and what is not when solving optimization are absent in both textbooks.

Third, less than 10% of the tasks in both textbooks combined are reflection tasks, that is, higher-order, non-routine tasks that require a high cognitive demand to solve. A majority of the tasks in both textbooks are reproduction tasks i.e., tasks with a low cognitive demand. These tasks only require memorization or performing explicit routine computations/procedures to solve. Consequently, there are limited opportunities for students to engage in "communicating complex arguments and complex reasoning" (Wijaya et al., 2015), which can only be found in reflection tasks. We note that the RC textbook has a considerable amount (36% of the 94 tasks in the textbook) of connection tasks i.e., tasks that require a higher level of cognitive demand to solve compared to reproduction tasks in that they are non-routine and may require multiple representations.

Fourth, findings from research on students' understanding of optimization problems indicates that among other things, setting up the objective function is a difficult step for students when tasked with solving optimization problems (cf. LaRue & Infante, 2015; Mkhatshwa, 2019; Swanagan, 2012). This is especially true when setting up of the objective function requires complex reasoning and making sense of the problem. It would seem that some of the students' struggles with setting up non-routine objective functions observed in the literature may be related to the opportunities to set up such objective functions provided in the two textbooks. Precisely, analysis of the optimization problems in both textbooks revealed that either the objective function is provided or that setting up the is straightforward such as finding the area function of a rectangular field when given its perimeter in a majority of the problems found in both textbooks. In fact, only 5% of the 63 optimization tasks provided in the AC textbook require setting up an objective function to solve. While 71% of the 94 optimization tasks require setting up an objective function, setting up these objective functions is straightforward, and often involves recalling geometric formulas such as the formula for calculating the area of a circle.

Fifth, all the tasks that either have a camouflage or relevant and essential context provide ample opportunities for students to interpret critical numbers or extrema. To reiterate, 61% of the optimization tasks in the RC textbook either have a camouflage or relevant and essential context. Twenty-five percent of the tasks in the AC textbook have a camouflage or relevant and essential context. Given that both textbooks provide an adequate number of opportunities to interpret critical numbers and extrema in context, an argument could be made that students' reported difficulties with interpreting critical numbers or extrema likely do no stem from the lack of such opportunities in calculus textbooks (cf. Brijlall & Ndlovu, 2013; Dominguez, 2010; Mkhatshwa, 2019; Swanagan, 2012).

Sixth, verifying/justifying extrema does not receive much attention in both textbooks. Specifically, of the 94 optimization tasks in the RC textbook, consisting of 88 practice problems and six examples, verifying/justifying extrema using the first derivative test, second derivative test, or the closed interval method (Stewart et al., 2021) was encouraged in five of the examples, but was not encouraged in any of the practice problems. Furthermore, of the 63 optimization tasks in the AC textbook, consisting of 60 practice problems and three examples, verifying/justifying extrema using the second derivative test, or the closed interval method was encouraged in two of the examples, but was not encouraged in any of the practice problems. Evidence from research shows that verifying/justifying extrema is problematic for students when tasked with solving optimization problems (cf. Borgen & Manu, 2002; Mkhatshwa, 2019).

The results of this study have some implications for different stakeholders, namely textbook authors, textbook selection committees, and instructors. Calculus textbook authors need to include a much broader range of optimization examples and practice problems in terms of types of context, types of information, and types of cognitive demands to maximize the learning opportunities provided by their textbooks. Calculus textbook selection committees need to select textbooks that contain a balance of optimization tasks in terms of types of context, types of information, and types of cognitive demands to avoid limiting students' opportunity to learn about optimization problems to tasks with matching information, no context, and tasks of low cognitive demand (i.e., reproduction tasks), which are dominant in the two textbooks analyzed in this study. According to Reys et al. (2004), "the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Calculus instructors would find it beneficial for students to supplement the examples and practice problems given in calculus textbooks to include tasks with superfluous information and/or tasks with higher cognitive demands in order to maximize students' opportunity to learn from such tasks, which are extra ordinarily low or absent in the textbooks we analyzed.

We caution the reader of this paper to interpret our findings in light of the limitations of the study. We only analyzed opportunities to learn about optimization problems provided by two calculus textbooks. Although these textbooks are two of the most commonly used textbooks for calculus instruction in the United States, it is possible that the opportunities to learn about optimization problems presented in these textbooks are not representative of the opportunities to learn about optimization problems provided by all regular and applied calculus textbooks. Findings from this study provide an insight on the need to consider, among other things, opportunities to work with reflection optimization tasks (i.e., tasks that require a higher cognitive demand to solve) and tasks that have superfluous information when making important decisions such as

(1) when writing a calculus textbook (in the case of textbook authors),

(2) when deciding on what calculus textbook to adopt for a particular course (in the case of textbook selection committees), and

(3) whether or not there is a need to supplement examples and practice problems provided in calculus textbooks (in the case of calculus instructors).

Given the importance of mathematics textbooks in students' learning at all levels, there is a need for more research that examines other opportunities to learn about a wide range of mathematics concepts provided by undergraduate mathematics textbooks.

In light of the findings reported in the current study, we suggest the following as potential approaches for future research on the opportunity to learn about optimization problems. First, it might be helpful for future research to consider analyzing a much larger sample of textbooks, including possibly an analysis of calculus textbooks from other countries (in addition to the United States), in order to gain a global perspective regarding the opportunities to learn about optimization problems provided by calculus textbooks. This is because textbooks from other countries may differ, compared to textbooks used in the United States, in how they present optimization problems. Second, the current study only analyzed examples and practice problems on optimization problems. It would be helpful if future research on the opportunity to learn about optimization problems provided by calculus textbooks would extend beyond analyzing examples and practices problems, and also consider examining learning opportunities provided in the narrative/expository sections of these textbooks. This might provide a perspective of how ideas related to optimization problems are developed in calculus textbooks. Third, taking into consideration Charalambos et al.'s (2010) argument that "textbooks afford probabilistic rather than deterministic opportunities to learn mathematics" (p. 118), future research might investigate learning opportunities on optimization problems provided during classroom instruction in different countries, in addition to comparing how these learning opportunities compare to learning opportunities on optimization problems provided by calculus textbooks used in these countries.

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REFERENCES

Aguinis, H., & Solarino, A. M. (2019). Transparency and replicability in qualitative research: The case of interviews with elite informants. *Strategic Management Journal, 40*(8), 1291-1315. https://doi.org/10.1002/smj.3015

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics, 79*(2), 239-261. https://doi.org/10.1007/s10649-011-9342-1

Begle, E. G. (1973). Some lessons learned by SMSG. *Mathematics Teacher, 66*(3), 207-214. https://doi.org/10.5951/MT.66.3.0207

Borgen, K. L., & Manu, S. S. (2002). What do students really understand? *The Journal of Mathematical Behavior, 21*(2), 151-165. https://doi.org/10.1016/S0732-3123(02)00115-3

Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimization problems in calculus: A South African case study. *South African Journal of Education, 33*(2), 1-18. https://doi.org/10.15700/saje.v33n2a679

Carroll, J. (1963). A model of school learning. *Teachers College Record, 64*, 723-733. https://doi.org/10.1177/016146816306400801

Chang, B. L., Cromley, J. G., & Tran, N. (2016). Coordinating multiple representations in a reform calculus textbook. *International Journal of Science and*

Mathematics Education, *14*(8), 1475-1497. https://doi.org/10.1007/s10763-015-9652-3

Charalambous, C. Y., Delaney, S., Hsu, H. Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117-151. https://doi.org/10.1080/10986060903460070

de Lange, J. (1995). Assessment: No change without problems. In T. A. Romberg (Ed.), *Reform in school mathematics* (pp. 87-172). SUNY Press.

Dole, S., & Shield, M. J. (2008). The capacity of two Australian eighth-grade textbooks for promoting proportional reasoning. *Research in Mathematics Education*, *10*(1), 19-35. https://doi.org/10.1080/14794800801915863

Dominguez, A. (2010). Single solution, multiple perspectives. In R. Lesh, P L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 223-233). Springer. https://doi.org/10.1007/978-1-4419-0561-1_19

Floden, R. E. (2002). The measurement of opportunity to learn. In A. C. Porter, & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational achievement* (pp. 231-266). National Academy Press.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111-129. https://doi.org/10.1023/A:1003749919816

Haghjoo, S., Radmehr, F., & Reyhani, E. (2023). Analyzing the written discourse in calculus textbooks over 42 years: The case of primary objects, concrete discursive objects, and a realization tree of the derivative at a point. *Educational Studies in Mathematics*, *112*(1), 73-102. https://doi.org/10.1007/s10649-022-10168-y

Hughes-Hallett, D., Lock, P. F., Gleason, A. M., Flath, D.E., Gleason, A. M., Luzano, G. I., Rhea, K., Connally, E., McCallum, W. G., Sahin, A., Kalaycioglu, S., Osgood, B. G., Spiegler, A. H., Lahme, B., Patterson, C. L., Tecosky-Feldman, J., Lomen, D. O., Quinney, D., Tucker, T. W., Lovelock, D., & Wooten, A. D. (2018). *Applied calculus*. Wiley.

Husen, T. (Ed.). (1967). *International study of achievement in mathematics: A comparison of twelve countries*. John Wiley & Sons.

Jones, D. L., & Tarr, J. E. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*, *6*(2), 4-27. https://doi.org/10.52041/serj.v6i2.482

Kolovou, A., van den Heuvel-Panhuizen, M., & Bakker, A. (2009). Non-routine problem solving tasks in primary school mathematics textbooks–a needle in a haystack. *Mediterranean Journal for Research in Mathematics Education*, *8*(2), 31-68.

LaRue, R., & Infante, N. E. (2015). Optimization in first semester calculus: A look at a classic problem. *International Journal of Mathematical Education in Science and Technology*, *46*(7), 1021-1031. https://doi.org/10.1080/0020739X.2015.1067844

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior*, *23*(4), 405-427. https://doi.org/10.1016/j.jmathb.2004.09.003

Maass, K. (2007). Modelling tasks for low achieving students–first results of an empirical study. In D. Pitta Pantazi, & G. Philippou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education* (pp. 2120-2129).

Maass, K. (2010). Classification scheme for modelling tasks. *Journal für Mathematik-Didaktik* [*Journal for Mathematics Didactics*], *31*(2), 285-311. https://doi.org/10.1080/10511970.2012.667515

Mesa, V., Suh, H., Blake, T., & Whittemore, T. (2012). Examples in college algebra textbooks: Opportunities for students' learning. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *23*(1), 76-105. https://doi.org/10.1080/10511970.2012.667515

Mkhatshwa, T. (2019). Students' quantitative reasoning about an absolute extrema optimization problem in a profit maximization context. *International Journal of Mathematical Education in Science and Technology*, *50*(8), 1105-1127. https://doi.org/10.1080/0020739X.2018.1562116

Mkhatshwa, T. (2022a). A study of calculus students' difficulties, approaches, and ability to solve multivariable optimization problems. *International Journal of Mathematical Education in Science and Technology*, *53*(11), 2987-3014. https://doi.org/10.1080/0020739X.2021.1927227

Mkhatshwa, T. (2022b). Quantitative and covariational reasoning opportunities provided by calculus textbooks: The case of the derivative. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2129497

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*, *6*(6), 1491-1515. https://doi.org/10.1007/s10763-013-9411-2

Pickle, M. C. C. (2012), *Statistical content in middle grades mathematics textbooks* [Unpublished doctoral dissertation]. University of South Florida.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership, 61*(5), 61-66.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics, 72*(3), 271-288. https://doi.org/10.1007/s10649-009-9193-1

Stewart, J. (2021). *Calculus: Early transcendentals*. Cengage Learning.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258-288. https://doi.org/10.1080/10986060903253954

Swanagan, B. S. (2012). *The impact of students' understanding of derivatives on their performance while solving optimization problems* [Doctoral dissertation, University of Georgia].

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253-295. https://doi.org/10.5951/jresematheduc.43.3.0253

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation, 31*(4), 315-327. https://doi.org/10.1016/j.stueduc.2005.11.005

Tyne, J. G. (2016). *Calculus students' reasoning about slope and derivative as rates of change* [Unpublished master's thesis]. The University of Maine.

van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics, 25*(2), 2-9.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27*(1), 79-95. https://doi.org/10.2307/749199

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics, 89*(1), 41-65. https://doi.org/10.1007/s10649-015-9595-1

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