MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2023, 19(9), em2325
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/13508

Optimizing the surface of orthohedra with virtual reality in primary school

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Received 15 April 2023 ▪ Accepted 08 July 2023

Abstract

Despite its importance for mathematics, science and technology, the conceptualization and calculation of volumes and surfaces of geometric solids is a source of difficulties, both in primary and secondary school. Immersive virtual reality (IVR) is a powerful resource to overcome these difficulties and promote learning with understanding that enables students to go beyond current curricular contents. This paper presents a design-research study in 6th grade of primary school, comprising three cycles, that allowed schoolchildren aged 11-12 to tackle a final challenge: the optimization of the surface area of orthohedra of a given volume. The design of the cycles, their implementation and the results obtained are described. Reflections are made on the benefits and drawbacks involved in using IVR in the classroom, and on the methodological strategies that enabled the students to successfully overcome the challenge posed.

Keywords: geometry, area, volume, optimization, virtual reality

INTRODUCTION

Measurement plays a fundamental role in mathematics, science, and technology, as well as in our daily lives. It is therefore important to overcome teaching and learning processes that focus on performing arithmetic calculations and using formulas, without students having a sense of what they are doing. This approach, still common in many primary and secondary classrooms, is a source of many difficulties and errors (Ryan & Williams, 2007; Smith et al., 2011).

In particular, the measurement of surfaces and volumes constitutes a relevant topic in the mathematics curriculum from primary to high school, since, in addition to its applicability, it provides a rich context for the extension of students' knowledge of arithmetic, geometric reasoning and spatial structuring (Battista, 2003). However, research shows a widespread difficulty of students with the measurement of these magnitudes, which is evidenced by

(a) not knowing the meaning and structure of the units of measurement,

(b) incorrectly enumerating the cubic units that constitute a solid form,

(c) ignoring hidden cubes, and double counting corner and edge cubes in prisms,

(d) focusing more on cube faces, ignoring the 3D nature of solids,

(e) confusing the magnitudes surface area and volume in geometric solids and confusing their formulae,

(f) applying incorrect formulae when calculating the surface area of a rectangular prism, and

(g) considering that figures with the same volume have the same surface area, and vice versa, etc. (Battista, 2003; Battista & Clements, 1996; Ben-Haim et al., 1985; Rupnow et al., 2022; Tan Sisman & Aksu, 2016; Voulgaris & Evangelidou, 2004).

We agree with Novak (2009) that the teaching of measurement, rather than being fragmented, opaque and rule-dominated, should be "conceptually transparent" to students. Battista (2003) states that students need to develop two skills to conceptually calculate the volume and surface area of a geometric solid: understanding and visualizing the structure of the solid, as well as linking the formulae to the structure of the solid, conceptualizing the numerical operations. Rupnow et al. (2022) found four threads of understanding that students had to coordinate to

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Contribution to the literature

- This article attempts to address the reported gap to understand the potential of VR technologies in enhancing mathematics education. It does so by providing empirical evidence of the effects of introducing IVR in a 6th grade class to address a problematic, but fundamental topic, with implications for science and technology; namely, the measurement of volume and surface area of solids.
- Students could link different representations to undenstand the structure of an orthohedron and come up with formulae of their own devise to calculate the surface and volumen of orthohedra. Moreover, students could see that solids with the same volume do not necessarily have the same surface area, and could minimise the latter for a given volume in the case of orthohedra.
- For this achievement to take place, the teacher had to cope with the overexcitement that IVR produced in young students and the markedly individual use of this technology. The problem was solved by working in small groups and designing a narrative that included students not wearing the googles and controllers to command the actions of the person inside the virtual scenario, with the help of manipulatives and other representations.

develop an effective algorithm to calculate prism volume. In the interpretation thread, students had to be able to interpret a volume measurement question as one of counting cubes or volume units. In the structuring thread, students needed to coordinate the use of volume units through a three-dimensional spatial structuring scheme using composite units. In the representation thread, students needed to understand the representation in which the problem is presented, including the three-dimensional intention of two-dimensional representations and the utility of length measurements for counting volume units. In the numeration thread, students needed to count efficiently and recognize the utility of repeated addition, skip-counting, or multiplication. The results of this study show that it was essential for students to develop understandings in the interpretation, representation, and structuring threads, but ultimately the coordination of understandings in these three threads and the numeration thread led to the most complete understanding of volume calculation. At this level, students might be observed to construct a volume calculation algorithm that matches the standard volume formula ($V=l\times w\times h$). The authors found the coordination among threads a complex interaction. Along the same lines, Tan Sisman and Aksu (2016) highlight that the relationships between the magnitudes of length, surface area and volume must be understood.

To promote meaningful teaching that takes all the above into account, the coordinated use of different representations is a fundamental basis (Lowrie & Logan, 2018; Tumová & Vondrová, 2017). These representations include numerical, graphical, manipulative materials and those provided by currently available technologies. One of the technologies that arouses most interest nowadays is virtual reality. According to Martín-Gutiérrez et al. (2017), studies in the scientific literature link virtual technologies with improvements in students' academic performance and motivation, students' social and collaborative skills, and students' psychomotor and cognitive skills. These authors distinguish between non-immersive, semi-immersive and immersive virtual reality (IVR), the latter being "a scientific and technical domain that uses computer science and behavioral interfaces to simulate in a virtual world the behavior of 3D entities, which interact in real time with each other and with one or more users in pseudo-natural immersion via sensorimotor channels" (Fuchs et al., 2011, p. 8). IVR software generates new methods of learning in which, through dynamic interaction, the touch (motor), visual and auditory senses intervene. It is possible to think that we are at the start of a new revolution in the educational field, the so-called "kinesthetic revolution". At present, there is a demand for studies on the integration of IVR in the classroom and its impact on teaching and learning processes (Demitriadou et al., 2020; Oguz, 2022; Radianti et al. 2020).

Focusing on the use of this technology in mathematics learning, there are works from the first decade of the 21st century (Kaufmann et al., 2000; Kaufmann & Schmalstieg, 2006; Song & Lee, 2002). Several of them point to successful advances in the integration of IVR in classrooms (Allcoat & von Mühlenen, 2018; Radianti et al., 2020). In this regard, Tang et al. (2020) claim that IVR can improve students' geometric analysis skills and creativity compared to traditional approaches. Demitriadou et al. (2020) proved that students were able to understand the geometric solids fully, as well as to perceive the difference between the objects of the three-dimensional space and those of the two-dimensional area. Rodríguez et al. (2021) and Moral-Sánchez et al. (2023) show some successful examples at different educational levels using the three-dimensional dynamic geometry software NeoTrie VR (or NeoTrie). Employing the same software, Morales and Codina (2020) show that elementary school students are able to employ flexible approaches that foster the development of structural visual reasoning and improve their argumentative ability. On the other hand, works such as those of Elkjaer and Thomsen (2022) and Hwang and Hu (2013) with IVR in the learning of mathematics in general, and geometry in particular, show that its use improves both affectivity and motivation towards mathematics in students. However, these results are not conclusive and there are studies that do not obtain significant differences using the IVR resource in this subject (Carbonell-Carrera & Saorin, 2017; Kang et al., 2020; Silva-Díaz et al., 2021). Hence, the need to continue providing empirical evidence that sheds light on the implications of including this resource in the classroom (Dilling & Sommer, 2021).

Moreover, in their review of recent developments of augmented reality (AR) and virtual reality (VR) technologies and their impact on mathematical learning, Cevikbas et al. (2023) note that geometry emerged as the most popular subject domain, given that this technology allows for the visualization of geometric objects in the real sense. Nonetheless, since visualization plays a significant role in the learning of different mathematics subjects, the authors claim the effectiveness of VR technologies for learning subjects in various foundational areas of mathematics. Measurement is one of these areas, which constitutes a critical component of elementary school mathematics, linking number with space (Rupnow et al., 2022). Further research is needed in this domain, as well evidence concerning the effects of VR on the problem-solving skills of learners.

In their study, Cevikbas et al. (2023) also point out the need to explore the potential benefits and drawbacks of VR technologies in mathematics learning based on empirical studies. This would help ensure that the recent developments and research trends, as well as the successful implementation of these technologies, are guided by empirical evidence rather than hype and speculation. In order to do so, together with large-scale studies that can help visualize the big picture in terms of the impact of VR technologies on mathematics learning, the authors demand in-depth qualitative research studies that facilitate comprehensive examination of this impact. Indeed, we must consider that the way virtual technologies are used will influence the learning outcomes. It is possible to use this resource to access knowledge as being a passive viewer, or just as following a list of instructions as in a traditional lab practice. However, the cornerstone of virtual technologies is immersion and interactivity, which require the design of Virtual Learning Environments with pedagogical scaffolding in order to design virtual learning scenarios to obtain maximum learning benefits (Martín-Gutiérrez et al., 2017). Design-based research studies are particularly well suited for that purpose, since they allow us to take an in-depth view and to address the "how" question, focusing on decisive factors that determine the eventual benefits in specific cases (Armstrong et al., 2022; Collins et al., 2004; Swan, 2020).

Figure 1. Tools table in NeoTrie (Source: Authors' own elaboration)

In this article, under the design-research paradigm, we present a teaching experiment (Prediger et al., 2015) in which IVR geometry software NeoTrie VR is introduced in a 6th grade class in Spain (11-12 years of age). Assuming that young children have the potential to learn mathematics that is complex and sophisticated, and that there is much to gain by engaging young children in mathematical experiences (Clements & Sarama, 2011), we posed the class the following challenge: to optimize the surface area of rectangular prisms (orthohedra) of a given volume. To this end, a series of activities were designed in which a scaffolding based on the use of IVR, supported by manipulative materials, allowed the challenge to be successfully met.

Next, we present the virtual reality software NeoTrie VR and the physical materials used in the experience; we describe the methodological principles and the scaffolding underlying the activities; we analyze the results of the implementation; and, finally, we reflect on the benefits and drawbacks of the resources employed, as well as on the strategies that can be used to take advantage of its potential in the classroom.

MATERIALS AND METHODS

Materials

The main resource used in this innovative proposal is IVR software NeoTrie VR for the teaching-learning of geometry, which is already being used in primary, secondary and university classes (Cangas et al., 2019; Rodríguez et al., 2021; Rodríguez, 2022). At the time of the teaching experiment, it used hardware consisting of a virtual reality headset and their corresponding controllers (oculus rift) to recreate a three-dimensional scenario within which different tools and geometric figures can be interacted with (**Figure 1** and **Figure 2**).

It allows students to have an immersive experience, where they can visualize, create, and manipulate objects over which to explore, reason and make decisions based on results to reach their learning goals, increasing their

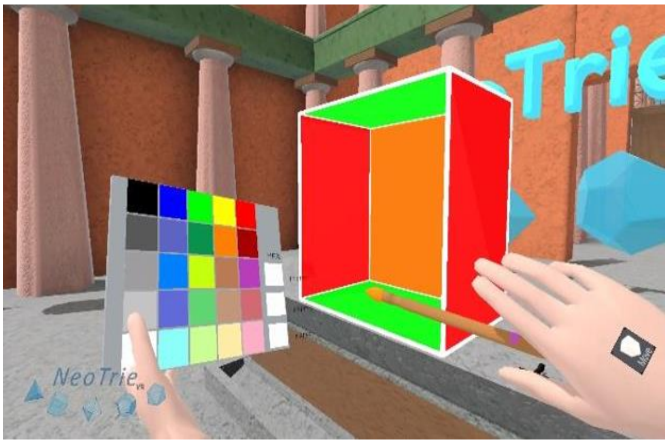

Figure 2. Interaction with geometric figures (Source: Authors' own elaboration)

Figure 3. Students using IVR & manipulative materials (Source: Authors' own elaboration)

learning performance and cognitive skills (Kotranza et al., 2009).

Among the numerous functions of NeoTrie, in this experiment we mainly made use of the creation of flat and spatial figures; the invocation of these figures by voice; editing of figures to modify their characteristics; the painting of edges and faces of polyhedra; movement by teleportation or flight; the measurement of edges, surfaces and volumes; and the gluing of figures through their faces. Another fundamental aspect of the software is the enormous motivational power it exerts on pupils of this age group, which allows the creation of an intense link between affect and cognitive development.

The experience with IVR was supplemented with manipulative materials, such as cubic units, paper boxes and magnetic polygons to construct prisms and their plane developments (**Figure 3**). A blackboard was also available to make graphic representations and calculations.

Methods

Under the paradigm of design research (Armstrong et al., 2022; Prediger et al., 2015), we depart from the conjecture that pupils in the upper levels of primary school are capable of reasoning and solving metric

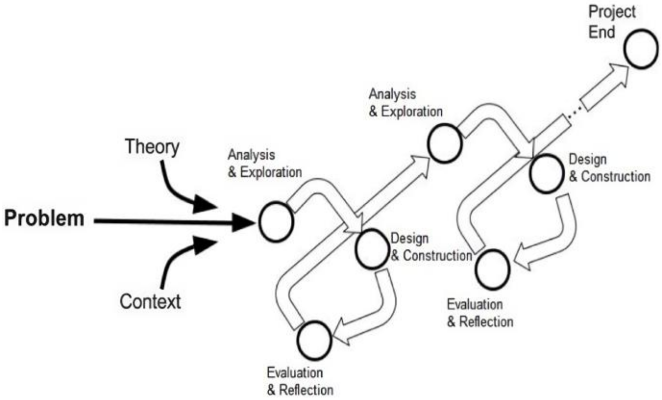

Figure 4. Iterative process of design-based research (Armstrong et al, 2022)

problems at a higher level than usual. To help them develop this capability, we adopt a socio-constructivist, inductive approach, based on coordinating interpretation, representation, and spatial structuring of orthohedra, and linking it with arithmetic operations and formulas. Currently available technologies, such as virtual reality, together with manipulative materials are key resources for this approach.

In order to test this conjecture, a series of activities were designed to provide students in a Spanish 6th grade class with the necessary knowledge and tools to tackle the following challenge: "Given a fixed volume, determine what will be the dimensions of the orthohedron that requires a minimum surface area for its manufacture". For the pedagogical scaffolding, fixed volumes were established to find out different orthohedra with that volume and to calculate their surface area. But beyond looking for correct results, the goal was for students to infer that the optimization of a minimum surface area for the volume of any orthohedron is determined by the one with equal faces, i.e., the hexahedron or cube.

In what follows, we present the context in which the experience was carried out and the activities designed to scaffold the final challenge, following the design methodology cyclic scheme (**Figure 4**).

Contextualization

The teaching experiment was carried out at a public school in the south of Spain that is committed to innovation. University of Almería had been collaborating with the school and, on this occasion, the authors of the article: a primary teacher, a researcher in mathematics education and one of the developers of the software NeoTrie VR worked collaboratively to carry out the experiment reported in this article with a 6th grade class.

In this class, there were 22 students, nine girls and 13 boys, with diversity in their learning pace. They were used to working in pairs and small groups, in a participative and collaborative way. Thanks to their teacher's approach in mathematics, they were used to posing and solving problems from a critical perspective, willingly facing challenges. Only a couple of students had had previous experience with virtual reality, but all of them were familiar with new technologies and their applications. On this occasion, we counted on the added value of the motivation and positive emotions that IVR generates among young people (Martín-Gutiérrez et al., 2017; Olmos-Raya et al., 2018). The group was enthusiastic and eager to know what challenge they would have to face through the use of this technology.

Virtual reality was introduced in the classroom, together with manipulative materials, through a teaching methodology consisting of students working in small groups, each group on a different activity from a different subject, during the same class. The teaching experiment consisted of three sets of mathematics activities lasting approximately one hour, each one corresponding to a research cycle. The groups rotate through all the activities in three weeks intervals. This allows time between sessions for analyzing and adjusting the design of the activities in micro-cycles within the cycles, which is a feature of the research methodology adopted (Collins et al., 2004; Swan, 2020).

In addition, in order to assess the students' prior level of knowledge, an individual pre-test was carried out. This made it possible to diagnose the main difficulties and to adjust the design. The items, taken from Pittalis and Christou (2010) and Pitta-Pantazi and Christou (2010), refer to the visualization of orthohedra and calculating their volumes and surfaces. The same test was repeated at the end of the experiment, as a post-test.

The teaching sequence and the cycles of the experiment are described below.

Teaching Sequence

The results of the pre-test made it possible to identify a series of generalized difficulties in the students, which coincide with those already recorded in the literature among which we highlight the following:

1. Confusion between the concepts of surface area and volume.
2. Difficulty in abstracting parts that are not visually accessible in a two-dimensional representation of three-dimensional objects.
3. Misconception of direct formulas to solve area and volume problems.
4. Lack of knowledge/forgetting area and volume units of measurement.

Based on this diagnosis, we established the following objectives for the teaching sequence:

1. Perceiving the difference between the volume and the surface of an orthohedron.
2. Distinguishing between units of area and volume and using them appropriately.
3. Understanding and applying the multiplicative principle to find the volume of an orthohedron.
4. Relating the formula "length x width x height" to the counting of cubic units.
5. Understanding and applying the multiplicative principle to find the surface of an orthohedron.
6. Devising appropriate strategies for calculating the surface of an orthohedron in the most efficient way.
7. Noting that orthohedra with the same volume can have different surface areas.
8. Constructing orthohedra of different dimensions for a given volume.
9. Indicating the dimensions of all possible orthohedra of a given volume.
10. Finding out what is the minimum surface area that can enclose a given volume for an orthohedron.

These objectives serve as a reference for the design, observation, and assessment of the activities outlined in **Table 1**. The results of experience are described below.

RESULTS

Based on the data obtained through direct observation, the diary of the teacher, the video recordings of the sessions and part of the tests carried out, it was possible to give an account of the degree to which the objectives of each cycle had been met, as well as the advantages and disadvantages of the methodology used. This had implications for the next cycle. The following is a summary of what happened in the cycles.

First Cycle

The first cycle is focused on the introduction and use of NeoTrie and the review of previous knowledge. Given the time available and considering that only one person in the group can wear the virtual reality goggles at any given time, it was decided to set up two rotational shifts. Thus, each member starts with approximately 10/12 minutes with the VR goggles and the controllers, while the rest follow their steps through the laptop monitor. Once all the members of the group have experimented with virtual reality once, a second, shorter shift begins (around five minutes) so that they can enjoy the activity again.

The first-round deals with the basic contents of polygons and polyhedra, previously worked on autonomously by the students. It is dedicated to learning the different tools and commands of the software, as well as the movement within the virtual space of NeoTrie. The second shift is established in a more flexible way, leaving the students free to explore, to observe their own

Table 1. Planning teaching sequence

Activities	Objectives
First cycle: Familiarization with the software & previous concepts	
-Introduction to virtual reality. -Practicing with the controls and the main menu. -Movement within NeoTrie. -Recognition, invocation, & creation of basic, flat, & spatial figures. -Using tools to point out parts & elements of the figures. -Personalizing, painting, & free drawing in virtual spaces.	-Gaining experience and fluency with NeoTrie VR software and its main tools. -Recognizing, creating, and classifying polygons. -Recognizing, invoking, and classifying basic polyhedral. -Recognizing & naming different parts of a geometric figure. -Exploring students' interests in relation to use of VR.
Second cycle: Deepening of the contents needed for the completion of the challenge	
-Filling orthohedra with cubic units. -Calculating orthohedron volumes. -Calculating orthohedron surfaces. -Free design of geometrical shapes & free use of tools in NeoTrie.	-Inferring multiplicative principle for measurement of volumes. -Learning to calculate volumes & surfaces of orthohedra. -Visualizing and understanding plane developments and projections of simple figures in space, through virtual reality. -Proving that orthohedra with the same volume can have different linear dimensions and different surfaces, through virtual reality and manipulative material. -Practicing 3D drawing with NeoTrie.
Third cycle: Optimization	
-Final challenge proposal based on the optimization of the surface for a given volume. Two levels of challenge: -First level: Finding the different orthohedra for a given volume (8 & 27 cubic units). Finding the one with the smallest surface area. Inferring optimization. -Second level: Finding the non-integer dimensions of the minimum area orthohedron for a given volume (12, 30, & 60 cubic units).	-Relating all that has been previously learned. -Learning to calculate dimensions of a minimum area for a given volume. -Inferring the "efficiency" of square figures and cubic bodies, for minimum perimeters and areas, respectively (whether the edge is an integer or not). -Improving spatial representation skills. -Drawing in 3D and representing orthohedra of given dimensions with NeoTrie.

interests and to stimulate the autonomous and voluntary use of such an attractive tool.

The results about content learning are positive. The software allows a practical and playful learning of different polygons and polyhedra, as well as their elements and properties. Concepts such as "vertex", "edge" or "face" are easily identifiable. The agility with which the students create, invoke, paint, and modify the figures and their parts, enables the consolidation of basic concepts and elements.

Despite this relative success, issues arise that demand further reflection. On the one hand, the teacher's ability to improvise in order to meet the different levels of attention, motivation and mastery required to use the program becomes apparent. On the other hand, the question of time management comes up. All the groups feel that the time they have available for the experience is scarce. This issue, rather than being an obstacle, speaks clearly of the strong motivational conditioning factor generated by such an activity There is only one case of a student who, due to shyness and fear of not doing well, does not seem interested in trying the experience; but even this student, once motivated, asks for as much time as possible to practice the activity.

In relation to the previous point, the viewing of the videos reveals a problem: the over-excitement of the pupils. It is important for the teacher to identify and control situations in which the students, conditioned by the fun and attractive part of the activity, relax their concentration due to an excess of interest in the tool, giving priority to leisure over the didactic basis that the teacher is trying to prioritize. In short, it is necessary to distinguish and manage the difference between motivation and over-excitement. This situation is not negative as such, nor does it require turning the activity into something more tedious, as if fun were a negative characteristic of the experience. But it is important to note that pre-planning, even for spare time within NeoTrie, is essential in order not to lose control of the activity and, at the same time, to take advantage of the motivational and didactic potential of the software.

On the other hand, it was found that the groups that have a good theoretical basis make better use of their time within the activity, both the student who interacts with NeoTrie and the peers who observe. This is particularly relevant because it points to the solution of another of the difficulties encountered: managing the interest and attention of group members who are not using the software at any given time.

The first small groups activity is concluded with the following ideas for the design of future ones:

1. Before each activity, a short, but very clearly designed time should be devoted to a theoretical explanation of the concepts to be worked on in that activity.
2. It is necessary to include a narrative or gamified design that involves the members of each group while they are not working in the immersive

Table 2. First cycle

Familiarization	
Activity development	Advantages
-Guided demonstration of tools and resources. -Practice with autonomous previously generated content. -Time planning, two shifts with virtual reality: 1st. Theoretical, introductory (10-12 minutes). 2nd. Practical, free, and flexible (5 minutes).	-High intrinsic motivation in students. -Easy consolidation of content. -Identification of spatial situations that are impossible or difficult to generate outside a virtual space. -Playfulness and attraction component.
Disadvantages	Conclusions
-Overexcitement. -Difficulties with time control. -Loss of focus on those who are not using NeoTrie. -Requires improvisation and adjustment skills on the part of the teacher. -Levels of prior knowledge differentiate the efficiency of practice time and group work.	-Reviewing shift and time management. -Searching for a narrative thread, not necessarily playful, involving the whole group. -Preliminary theoretical explanations to support the work in virtual reality. -Use of manipulative material and other representation systems to complement virtual reality.

environment of NeoTrie. To develop it, we refer to Pedraz (2017), who provides guidelines to achieve an experience that integrates the playful and the cognitive, based on a narrative or a design of collaborative challenges.

3. It is advisable to complement the work in virtual reality with manipulative material and other representations systems, which can be used by the members of the group who are not wearing the VR goggles and controllers at a given moment, in order to support the wearer.

Table 2 summarizes the results of the first cycle.

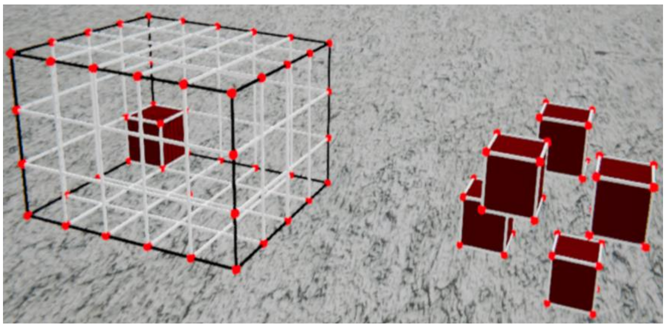

Figure 5. Orthohedron & cubic units to calculate its volume in NeoTrie (Source: Authors' own elaboration)

Second Cycle

The second cycle activity has three aims. First, to get students to infer the multiplicative principle for calculating volumes and areas; second, to establish their own strategies for calculating the surface area of prisms; and third, to realize that orthohedra with the same volume can have different surface areas.

In order to achieve these objectives, first of all, three orthohedra of different dimensions, but of equal volume (60 cubic units), are presented in NeoTrie. Each group has to calculate the volume of three orthohedra using the software virtual tools (**Figure 5**). One more rule is included: the student interacting with the software cannot give the answer to the question. The other members of the group must try to carry out these calculations and guide the partner who has the controls; for this purpose, they have at their disposal cent cubes, magnetic pieces and a blackboard to make drawings and calculations.

All the groups manage to solve the challenge posed, successfully overcoming some initial difficulties (for example, once the first floor of cubes has been counted, in order to calculate the number of cubes that fill the volume, some are not clear whether the result should be multiplied by all the cubes iterated on the edge corresponding to the height, or whether the cube they have already considered should be removed). The different groups arrive in different ways at devising a strategy with which to calculate the volume of an orthohedron, either through the formula "height x length x width" or by realizing that, once the area of a face has been established, if it is multiplied by the axis not used in the calculation of that surface, the volume is also obtained. We assume that this achievement is conditioned by the use of physical manipulatives and, above all, by working in virtual space, where they can "fly" around the figure, rotate it and position themselves as they wish in relation to it. These actions mean that, for most of the participants in the activity, the orthohedra do not have a given base, which leads to a flexible understanding and acceptance of the multiplicative principle. They first calculate the surface of one face and then multiply it by the remaining dimension.

Another aspect that emerges during this cycle is that the difficulty students usually have in understanding the concept of surface when referring to three-dimensional figures is more lexical than conceptual. Thus, when setting the problem in context ("How much paper do we need to wrap this shoebox?"), they have no difficulty in understanding what they are being asked. This understanding is probably influenced by the ease of painting the faces of the prisms, on the one hand, and being able to fill them in, on the other, using NeoTrie. It is also detected that the groups devise different strategies to find the solution. Some of them infer that to

Figure 6. Collaborative resolution of the activities in the second activity (Source: Authors' own elaboration)

Table 3. Second cycle

Deepening	
Activity development	Advantages
-Guided discovery of different procedures for finding the volume and surface area of orthohedra. -Calculation of volumes and surfaces. Realization that orthohedra with the same volume can have different dimensions and surface areas. -Elements of prisms, plane developments and projections. -Free time, spatial drawing.	-Collaborative work, involvement of the whole group. -Consolidation of motivation and curiosity. -Inferences obtained by different strategies. -Easy visualisation of problems. -Clarification of one's own thinking, by having to guide the partner using Neotrie. -Development of didactic skills to support the learning of teammates.
Disadvantages	Conclusions
-Need for attention to the response to the narrative by certain groups and particular students. -The playful part can still prevail over the didactic part. -Design and improvisation effort to respond to individual needs and doubts. -Difficulty in sharing responsibilities. -Hardware and software troubleshooting. -Lack of control over individual acquisition of knowledge.	-Narrative line is fundamental; you have to set a goal & a path. -Traditionally established theoretical concepts are confronted with knowledge acquired through intuition and one's own inference. -Inference is quickly learned and consolidated. -Complementary work with manipulative materials, well-articulated, is useful and helps to manage many of the difficulties that arise.

find this total surface area they do not need to find the area of the six faces, but with three of them (if they are different), or even two of them (when it is a prism with a square base), and a series of additions or multiplications, they can find the surface area they are looking for.

The results of this activity, once again, are quite positive. This time, the performance of collaborative work in the proposed tasks is striking. The didactic strategy of establishing that the student wearing VR goggles must be guided by the rest allows the whole group to be involved at the same time, carrying out their learning collectively and tutoring among peers (**Figure 6**).

However, for the teacher, it is a major effort to manage the narrative of the challenge, according to the characteristics of the different groups and students. It is always necessary to make sure that the playful aspect of the physical and virtual materials does not distract attention from the didactic objective. Sometimes, technical problems arise with software and hardware that must be solved on the fly. Furthermore, attention needs to be paid to the difficulties that appear in some groups regarding the distribution of responsibilities and, although all the groups manage to successfully complete the proposed tasks, it is difficult to control in situ the degree of assimilation by the individual students.

Among the implications of this cycle for the next and final cycle, we highlight the need to stick to a narrative in pursuit of the final objective, paying particular attention to articulating the collaborative dynamics with materials in a way that contributes to the achievement of the objective. **Table 3** summarizes the results of this second research cycle.

Third Cycle

The third cycle corresponds to the activity in which students must tackle the challenge of optimizing the surface of any orthohedron, using the necessary tools, both theoretical and practical, which have been worked on in the previous cycles. To help them in this task, an information layer is enabled in NeoTrie, which, when active, dynamically reflects the volume and surface area of any geometric figure, among other data (**Figure 7**).

Thus, once the dimensions that each group wants to work on for their orthohedra have been decided, they can have these data directly. To facilitate inference,

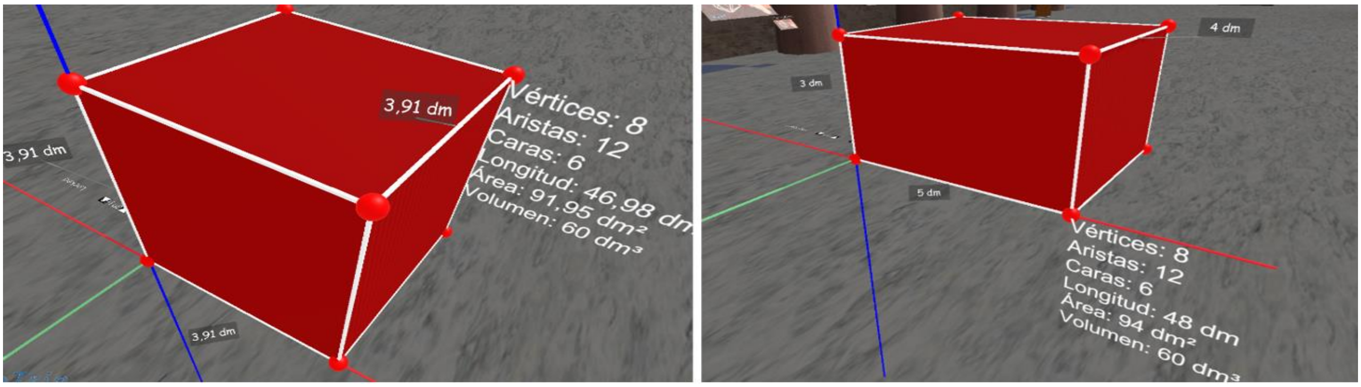

Figure 7. Information layer of two ortohedra (Source: Authors' own elaboration)

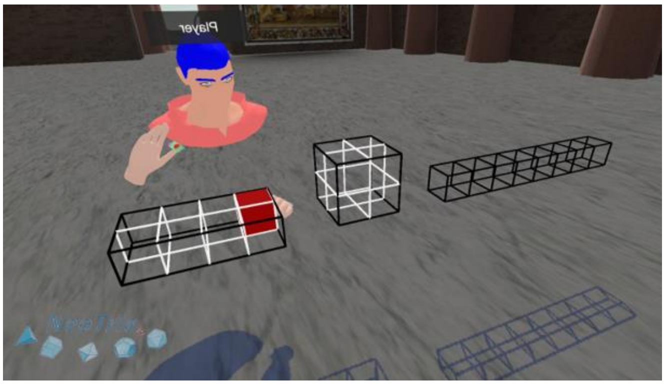

Figure 8. Orthohedra of volume eight cubic units (Source: Authors' own elaboration)

Figure 9. Group assesses possible orthohedra of volume eight cubic units (Source: Authors' own elaboration)

different volumes are considered with which to pose the challenge, establishing two levels of difficulty:

1. A first basic level, with volumes with cubic integers values, in order to have edges with integer values. A volume of 8 cubic units is chosen, as it is an affordable number and allows a simple but sufficient variety of orthohedra (2x2x2, 4x2x1, 8x1x1). A volume of 27 cubic units is also considered, in case a second cubic integer value is needed.
2. A second level, with values that do not have a hexahedron option with integer values for its edges. For this other option, volumes of 12, 30 and 60 cubic units are chosen, as they are values that allow several possible integer dimensions due to their variety of divisors, and whose edges in the case of considering a hexahedron have a relatively simple calculation by approximation.

The activity starts by asking each group to find all possible orthohedra with eight cubic units volume (**Figure 8**).

Once they have been found, the next step is to assess which of these orthohedra requires the smallest surface area. Before doing so, the pupils are asked to indicate, without actually calculating it, which of them they think will require the least surface area. The students have at their disposal all the materials that they have been working on during the proposal (**Figure 2** and **Figure 9**). There is a natural inclination to use the manipulatives to

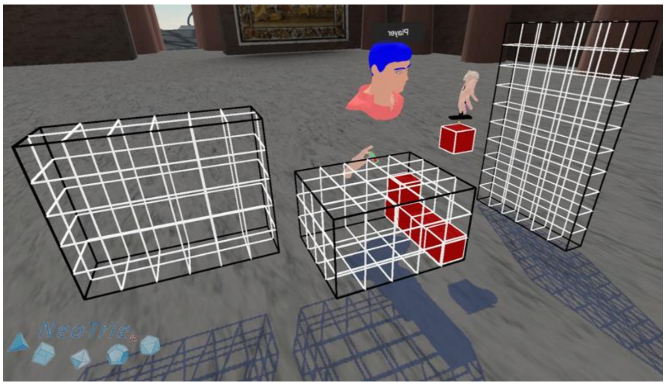

Figure 10. Orthohedra of volume 60 cubic units (Source: Authors' own elaboration)

begin with and the NeoTrie virtual tools, both graphical and analytical, to demonstrate and calculate.

This first level of challenge, again, is overcome by all groups. In case that not all members of a group manage to solve the challenge, those students who get it first are the ones who, driven by emotion, explain the strategy and inference discovered to the rest of their classmates.

For the second level of challenge, the same problem is stated as in the first one (find all possible orthohedra of the given volume), but this time with values whose cube root is not integer (**Figure 10**).

This question is solved by all the groups from the consolidation of the multiplicative principle, using the manipulative materials or the blackboard to share their ideas with the other members; although not all the groups do it with the same ease and do not use the same strategies.

Having established that none of the orthohedra corresponding to the volumes of the second level is a hexahedron (if the dimensions of their sides are considered integers), the students are able to infer that the shape of the orthohedron with the minimum surface area for each value of the volume would be a cube, but its edge would not be an integer. They are then asked to approximate the edge of the figure that most closely resembles this hypothetical cube of minimum surface area for a given volume by using decimal numbers. This question generates more obstacles. While most of the students are clear that they are looking for a numerical value that when multiplied by itself three times yields the initial volume, they find the calculations difficult. Without activating the figure information layer provided by NeoTrie, only one group give correct answers by approximation.

At this point, it can be stated that the use of virtual reality offers several advantages for solving the optimization challenge proposed to our students. On the one hand, it is familiar, intuitive and the possibility of using it stimulates their desire to face the challenge in a playful way, without fear or pressure. On the other hand, it allows them to understand the approach to a surface optimization problem, which, in principle, requires a good level of abstraction and spatial visualization at this age level. Although the use of manipulative materials could be sufficient to understand the problem, the possibility of flying over the figures and changing their position with respect to them, modifying their point of view, favors a flexible approach to understand and calculate the volume and the surface area of a given orthohedron. By realizing the link between the multiplicative principle and the graphical representation, students were able to "compose formulas" and perform the calculations in the way that suits them best. In addition, NeoTrie facilitates the visualization of such calculations as the dimensions of the edges of the orthohedra vary dynamically, and hence the inference of the dimensions of the minimum surface area orthohedron for a given volume. The advantages of the immersive virtual environment increase when dealing with more complex processes, such as volumes whose cube root is not an integer.

As limitations, we can point out that IVR has a strong individual use, and it is not easy to devise and maintain dynamics in which students who do not play directly with NeoTrie keep their attention and involvement. In this respect, although the results at group level are very positive, the question remains as to the individual use and assimilation that the pupils have made of these activities. **Table 4** summarizes the third and last cycle of the research.

Individual Results

As we indicated earlier, it was not possible to collect information on individual performance with the virtual reality software. Nevertheless, we can gain some insight on the effect of the experience on individual students by means of the test they filled out at the beginning and at the end of the experience. In particular, students' responses to the following question (**Figure 11**), before and after the teaching experiment show a clear improvement on their understanding.

Table 4. Third cycle

Optimization challenge	
Activity development	Advantages
First level challenge: -Obtaining all possible orthohedra of volume eight & 27 cubic units. Calculation of surfaces and inference of the cube as the orthohedron with the smallest surface area. Second level challenge: -Obtaining orthohedra for volumes of 12, 30, & 60 cubic units. Finding non-integer dimensions of orthohedron of minimum surface area for above volumes.	-Neotrie's attractiveness influences students to want to solve challenges using it. -It facilitates trial and error, and therefore inference. -It provides speed and visualization of calculations. -It makes it possible to link multiplicative principle to graphical representation in different ways, making use of formulae for calculating surface area & volume of orthohedra more flexible. -This understanding stimulates the search for efficiency in calculation strategies.
Disadvantages	Conclusions
-Need to generate extra interest, playful or narrative, so as not to detract from learning -Despite the collaborative dynamics, the software is still largely used by individual students	-Immersive virtual reality, through NeoTrie VR, is familiar, intuitive, and highly engaging for students in learning geometry and measurement. -Its possibilities increase when dealing with more complex contents, encouraging interest of students, their cognitive flexibility, and the generation of their own strategies. -VR supports teachers, but it does not replace their work. To take advantage of its potential, knowledge, & constant adjustment is required, both at a technical level and in terms of collaborative and playful dynamics. Adaptation to the evolution of the students must be made not only at the planning level, but also at the level of real time reaction.

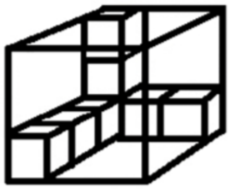

Figure 11. Pre- & post-test item (Pitta-Pantazi & Christou, 2010)

> Helena is playing with her cubes and has placed some in a box as shown in the picture. Can you figure out the number of cubes she will need to fill the box? Justify your answer.

Of the 22 students who participated in the experiment, 21 answered the pre-test and the post-test. **Table 5** shows the answers (correct or incorrect) and the justifications given to the item by the students, before and after the experience.

The results in **Table 5** show that before the teaching experiment, most students (19 out of 22) give wrong answers. Of these, most are obtained by counting (11 answers), some are not justified (four answers) and the rest (four answers) are the result of applying the multiplicative principle wrongly. Only two students answer correctly, one student who solves it by counting and one student who applies the multiplicative principle correctly and then subtracts the cubes shown in the picture.

After the teaching experiment, the situation is reversed. Only two students give wrong answers, 18 give correct answers and one student gives a partially correct answer (he correctly calculates the total number of cubes that would fit in the orthohedron but makes a mistake when counting the ones that are already drawn). Incorrect answers are obtained one by counting and one by incorrect application of the multiplicative principle. The correct answers lack justification in six cases, and the remaining 13 are obtained by application of the multiplicative principle.

DISCUSSION

The results of this experience evidence that primary grade children have the capacity to learn substantial mathematics and tackle cognitive challenges with the aid of VR technology. They are in line with those reported by Martín-Gutiérrez et al. (2017), showing how virtual technologies can contribute to improving students' cognitive skills, achievement, motivation, and collaboration. As these authors state, a careful pedagogical scaffolding is required in order to achieve a virtual learning environment that capitalizes the affordances of IVR software such as NeoTrie VR. We

Table 5. Responses to the item in the pre- & post-test for the whole class

Student	PCA	Justification	LCA	Justification
A1	0	Counting (18)	1	Does not justify (28:36 and subtracts 8)
A2	0	No answer	1	Multiplies the "squares" on the axes. (28: 3x3x4 and subtract 8 from the result)
A3	0	"I have thought about it" (4)	1	Multiply the "squares" in the middle (36)
A4	0	Counting (29)	1	28 cubes: 3x3x4 and subtract 8 from the result
A5	0	Count cubes on each side (10 on each side & 30 in total).	1	It does not justify where it gets 36 cubes, and it does say that it subtracts 8 cubes.
A6	0	Counting (67)	1	28 cubes: 12x3 and subtracts 8
A7	0	No justification. It seems from drawing that he is trying to count them (20 cubes).	0	The area of a face is 8 cubes, and there are 6 faces (48).
A8	1	Counting (28)	1	Counts "cubes in each line" (supposed to apply multiplicative principle afterwards) (28 cubes)
A9	0	Counting (20)	1	28: 3x3x4 and subtract 8
A10	0	Unwarranted (14 buckets)	1	"I visualise" (28 cubes)
A11	0	Multiplicative principle to find 2-sided cubes & then add them (42 cubes)	1	28: (4x3)x3 and subtract 8
A12	0	3+2+3 (8 cubes)	1	Does not justify (28:36 and subtract 8)
A13	1	36: 4x3x3	1	Not justified (36 and subtract 8)
A14	0	23 cubes (7+16)	0	Counting (33 cubes)
A15	0	Multiply to get "cubes of sides" (6: 3x2)	1	28: 9x4 and subtract 8
A16	0	It only considers the visible cubes to apply multiplicative principle (18: (3x 2)x3)	1	28: (3x3)x4
A17	0	No answer	0.5(*)	27: (12x3)-9 *Instead of 8, the student counts 9 cubes.
A18	0	Counting (33)	1	28: (4x3x3)-8
A19	0	Multiply "cubes on one side by cubes on other side" (21)	1	Does not justify (28: 36-8)
A20	0	Counting (16)	1	28: 4x3x3 and subtract 8
A21	0	Sum 6+6+4 (16)	1	28: (3x3x4)-8
Total	2		18.5	

Note. PCA: Prior correct answers & LCA: Latter correct answers agree with Guss et al. (2022) that opportunities for learning math are maximized when technology-based practices include core math content, focus on children's (often intuitive) thinking and learning, and design teaching practices that follow research-based knowledge, considering math as a discipline and children as individuals with unique characteristics and abilities.

In our case, the activities were designed for students to develop understanding in the interpretation, structuring, representation and numeration threads, as well as their coordination, for developing effective algorithms to calculate prism volume and surface (Rupnow et al., 2022). These activities were implemented emphasizing coherence and "conceptual transparency", as key aspects involved in measuring the surface and volume of solids (Novack, 2009). Equally important, a challenge and a narrative were introduced to provide meaning and joy to learning, as suggested by Guss et al. (2022) and Pedraz (2017).

Such a pedagogical design has allowed this group of students to properly interpreting a sophisticated and substantial mathematical problem, such as the optimization of surface in 3D objects, overcoming the difficulty in distinguishing surface and volume in solids, as well as the preconception that considering that figures with the same volume have the same surface area. They have been able to visualize and understand the structure of the orthohedron, coordinating 2D and 3D representations, both manipulative and virtual. This has been reported as a cause of struggle in literature in the teaching and learning of volume measurement (Rupnow et al., 2022). Furthermore, most students could coordinate the interpreting, structuring and representation threads with the numeration thread, identified by Rupnow et al. (2022) as the key to the most complete understanding of volume calculation. At this level, students are able to construct algorithms. In our case, the groups of students could come up with formulae of their own devise in order to calculate not only the volume of orthohedra, but also the surface. The latter is something more complex, that involves applying the multiplicative principle and combining it with other arithmetic operations, matching them with the structure of the orthohedron in a flexible way. The easy and natural way in which students could fly around the orthohedra in NeoTrie immersive environment, paint faces and sides, as well as fill them with cubes, has played a key role in distinguishing length, surface area and volume, and understanding the relationship among these magnitudes. Also in considering all possible perspectives, choosing the more convenient ones for calculations. This result is consistent with those obtained by Demitriadou et al. (2020) and Morales and Codina (2020) in elementary school, where students developed structural visual reasoning, were able to understand the geometric solids, perceived the difference between the objects of the three-dimensional space and those of the two-dimensional area, and employed flexible approaches to reason about geometrical problems.

Moreover, this approach has enabled our students to achieve a goal beyond the common reach at this age-level: to understand that solids with the same volume do not necessarily have the same surface area; and to minimize the latter for a given volume in the case of orthohedra. Some students were able to do so even when the dimensions of the corresponding cube were not integers. While it is true that this goal has been overcome collectively, individual results still show a significant improvement. The class moved from most students counting unit cubes inaccurately to the majority of them given correct answers and more than half developing an effective algorithm to calculate prism volume, by understanding and applying the multiplicative principle.

In addition to the affordances that NeoTrie provided for the development of cognitive skills, its role in students ' motivation deserves to be highlighted. The students in this experiment carried out a big cognitive effort in a climate of curiosity, interest and even excitement. But firstly, the drawback of overexcitement of young students with IVR had to be dealt with. The teacher coped with this challenge by always having in mind the objectives of the activities and how the resources could support their achievement, as well as by maintaining a delicate ongoing balance between keeping a playful sensation in students and avoiding distraction, guiding their cognitive focus and efforts. Time for concentrating and time for free use of the software was provided in each cycle. The free use of NeoTrie was not a waste of time, but helped students to explore its possibilities, to gain familiarity with it, and to maintain the interest in its use.

Another drawback was due to the markedly individual use of IVR for a classroom setting. In our case, it was solved by designing a narrative that included students not wearing the VR googles and controllers to command the actions of the person inside the virtual scenario. In addition, the teacher had to pay attention in real time to fostering the active participation of each student, considering their previous knowledge, skills and level of interest. As a result, a climate of collaboration and peer tutoring arose, where faster students were willing to share their realizations and insights with their peers, so that the whole group could achieve the common target. These results are in line with those reported by Cevikbas et al. (2023) in their review of the benefits of using AR/VR technologies in mathematics learning, where the most prevalent favorable outcomes observed were the socio-emotional ones.

CONCLUSIONS

This article attempts to address the reported gap to understand the potential of VR technologies in enhancing and transforming mathematics education, in line with the requirements of the digital age (Cevikbas et al., 2023). We agree with these authors that the relatively poor adoption of VR technology in education may be due to different factors, such as the complexity to manage it, its cost, the expertise required for its development, and, finally, a lack of awareness of its potential benefits and applications. However, as VR technologies continue to evolve and become more affordable, their use for educational purposes will likely increase. Hence the relevance of research-based interventions in mathematics classrooms that explore their possibilities and the conditions for their feasibility; in particular, those which extend the influence of visualization to mathematical areas other than geometry and employ a problem-solving approach.

In our study, we have offered evidence of the cognitive, socio-emotional and pedagogical benefits of introducing the IVR software NeoTrie VR in a primary grade class to address a problematic, but fundamental topic in mathematics education, with implications for science and technology; namely, the measurement of volume and surface area of solids. We have also reported some drawbacks of applying this type of technology in real settings with young students, and the ways we have dealt with them.

Nonetheless, we can point out different limitations in this work. On the one hand, it may be difficult to replicate, not only because of the close collaboration among the software developer, the educational researcher and the teacher to produce the design of the activities, but for the expertise of the teacher for flexibly adapting the design to the ongoing contingencies that arise in a classroom setting. We agree with Drickey (2000) that lessons that incorporate resources such as virtual reality are difficult to manage and implement efficiently. In our experience, the role of the teacher in optimizing the learning potential of this technology, adapting the pedagogical design to meet the students´ abilities, characteristics and evolving needs can hardly be overestimated. For this purpose, teachers have to develop new skills, as stated on UNESCO's (2011) framework for teacher competences: "the teaching skills of the future will include the ability to develop innovative ways of using technology to enhance the learning environment and to foster technological literacy, deepening and creating knowledge" (p. 8).

Another limitation is that we did not assess individual or small groups ways to deal with the VR software, nor described learning pathways or levels of achievement. Instead, our results provide hints of what is possible to achieve with a whole class in terms of understanding a relevant topic such as measurement of 3D objects with the aid of IVR. Besides, since the experience was carried out with a single class, we do not attempt to give the results any degree of generality.

Finally, the study also opens some lines of inquiry. One is the adaptation of the approach to other 3D software of augmented or virtual reality. Another line would be to analyze the role of manipulative materials to support virtual reality experiences, according to educational levels. Also the possibility of adapting these activities to the multiplayer mode, now available in NeoTrie, where students can manipulate virtual objects and have access to different representation systems in a single scenario at the same time. This would lead to explore ways of managing collaborative dynamics in a multiplayer scenario. Addressing the balance between affective factors and meaningful, substantial learning in this type of environment is also important, especially with students of younger ages.

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: This study was partially funded by the ERDF (European Regional Development Fund) research project from the FEDER-Andalusian Regional Government grant UAL2020-SEJ-B2086, and by the Spanish Ministry of Science and Innovation grant PID2020-117971GB-C22.

Ethical statement: The authors stated that ethical review and approval were waived for this study, due to its anonymized nature. The activities were revised and authorized by the school. The parents were informed before their children participated in the study.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Armstrong, M., Dopp, C., & Welsh, J. (2022). Design-based research: What is DBR, why might one do it, and how does one do it well? In R. Kimmons (Ed.), *Education research*. BYU Open Textbook Network.

Battista, M. (2003). Understanding students' thinking about area and volume measurement. In D. H. Clements, & G. Bright (Eds.), *Learning and teaching measurement* (pp. 122-142). National Council of Teachers of Mathematics.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education*, 27, 258-292. https://doi.org/10.2307/749365

Ben-Haim, D., Lappan, G., & Houang, R. T. (1985). Visualizing rectangular solids made of small cubes: Analyzing and affecting students' performance. *Educational Studies in Mathematics, 16*(4), 389-409. https://doi.org/10.1007/bf00417194

Cangas, D., Morga, G., & Rodríguez, J. L. (2019). Geometry teaching experience in virtual reality with NeoTrie VR. *Psychology, Society & Education, 11*(3), 355-366. https://ojs.ual.es/ojs/index.php/psye/article/view/2270

Carbonell-Carrera, C., & Saorin, J. L. (2017). Virtual learning environments to enhance spatial orientation. *EURASIA Journal of Mathematics, Science and Technology Education, 14*(3), 709-719. https://doi.org/10.12973/ejmste/79171

Cevikbas, M., Bulut, N., & Kaiser, G. (2023). Exploring the benefits and drawbacks of AR and VR technologies for learners of mathematics: Recent developments. *Systems, 11*(5), 244. https://doi.org/10.3390/systems11050244

Clements, D. H., & Sarama, J. (2011). Early childhood mathematics intervention. *Science, 333*(6045), 968-970. https://doi.org/10.1126/science.1204537

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *Journal of the Learning Sciences, 13*(1), 15-42. https://doi.org/10.1207/s15327809jls1301_2

Demitriadou, E., Stavroulia, K. E., & Lanitis, A. (2020). Comparative evaluation of virtual and augmented reality for teaching mathematics in primary education. *Education and Information Technologies, 25*, 381-401. https://doi.org/10.1007/s10639-019-09973-5

Drickey, N. A. (2000). *A comparison of virtual and physical manipulatives in teaching visualization and spatial reasoning to middle school mathematics students* [Doctoral dissertation, Utah State University].

Fuchs, P., Moreau, G., & Guitton, P. (2011). Introduction to virtual reality. In p. Fuchs, G. Moreau, & P. Guitton (Eds.), V*irtual reality: Concepts and technologies* (pp. 3-10). CRC Press. https://doi.org/10.1201/b11612-3

Guss, S. S., Clements, D. H., & Sarama, J. H. (2022). High-quality early math: Learning and teaching with trajectories and technologies. In A. L. Betts, & K. P. Thai (Eds.), *Handbook of research on innovative approaches to early childhood development and school readiness* (pp. 349-373). IGI Global. https://doi.org/10.4018/978-1-7998-8649-5.ch015

Kotranza, A., Lind, D. S., Pugh, C. M., & Lok, B. *(2009). Real-time in-situ visual feedback of task performance in mixed environments for learning joint psycho-motor-cognitive tasks* [Paper presentation]. The 8th IEEE International Symposium on Mixed and Augmented Reality. https://doi.org/10.1109/ISMAR.2009.5336485

Lowrie, T. & Logan, T. (2018). The interaction between spatial reasoning constructs and mathematics understandings in elementary classrooms. In K. S. Mix, & M. T. Battista (Eds.), *Visualizing mathematics* (pp. 253-276). Springer. https://doi.org/10.1007/978-3-319-98767-5_12

Martín-Gutiérrez, J., Mora, C. E., Añorbe-Díaz, B., & González-Marrero, A. (2017). Virtual technologies trends in education. *EURASIA Journal of Mathematics, Science and Technology Education, 13*(2), 469-486. https://doi.org/10.12973/eurasia.2017.00626a

Moral-Sánchez, S. N., Sánchez-Compaña, M. T., & Romero-Albaladejo, I. (2023). Uso de realidad virtual en Geometría para el desarrollo de habilidades espaciales [Use of virtual reality in Geometry for the development of spatial skills]. *Enseñanza de las Ciencias. Revista de investigación y experiencias didácticas, 41*(1), 125-147. https://doi.org/10.5565/rev/ensciencias.5442

Novak, J. D. (2009). Foreword. In K. Afamasaga-Fuata'i (Ed.), *Concept mapping in mathematics: Research into practice*. Springer. https://doi.org/10.1007/978-0-387-89194-1

Olmos-Raya, E., Ferreira-Cavalcanti, J., Contero, M., Castellanos, M. C., Giglioli, I. A. C., & Alcañiz, M. (2018). Mobile virtual reality as an educational platform: A pilot study on the impact of immersion and positive emotion induction in the learning process. *EURASIA Journal of Mathematics, Science and Technology Education, 14*(6), 2045-2057. https://doi.org/10.29333/ejmste/85874

Pedraz, P. (2017). Warning: Play before gamifying. *A la luz de una bombilla*. http://www.alaluzdeunabombilla.com/2017/02/14/aviso-jugar-antes-de-gamificar/

Pittalis, M., & Christou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics, 75*(2), 191-212. https://doi.org/10.1007/s10649-010-9251-8

Pitta-Pantazi, D., & Christou, C. (2010). Spatial versus object visualization: The case of mathematical understanding in three-dimensional arrays of cubes and nets. *International Journal of Educational Research, 49*(2-3), 102-114. https://doi.org/10.1016/j.ijer.2010.10.001

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM, 47*(6), 877-891. http://doi.org/10.1007/s11858-015-0722-3

Rodríguez, J. L. (2022). Exploring dynamic geometry through immersive virtual reality and distance teaching. In P. R. Richard, M. P. Vélez, & S. Van Vaerenbergh (Eds.), *Mathematics education in the age of artificial intelligence. Mathematics education in the digital era* (vol 17). Springer, Cham. http://doi.org/10.1007/978-3-030-86909-0_15

Rodríguez, J. L., Romero, I., & Codina, A. (2021). The influence of NeoTrie VR's immersive virtual reality on the teaching and learning of geometry. *Mathematics, 9*, 2411. https://doi.org/10.3390/math9192411

Rupnow, T. J., O'Dell, J. R., Barrett, J. E., Cullen, C. J., Clements, D. H., Sarama, J., & Rutherford, G. (2022). Children's construction of a volume calculation algorithm for a rectangular prism with a dynamic virtual manipulative. *The Journal of Mathematical Behavior,* 67, 100998. https://doi.org/10.1016/j.jmathb.2022.100998

Ryan, J., & Williams, J. (2007). *Children's mathematics 4-15: Learning from errors and misconceptions*. Open University Press.

Smith, J. P., van den Heuvel-Panhuizen, M., &Teppo, A. (2011). Learning, teaching, and using measurement: Introduction to the issue. *ZDM Mathematics Education, 43*, 667-820. https://doi.org/10.1007/s11858-011-0369-7

Swan, M. (2020). Design research in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 192-195). Springer. https://doi.org/10.1007/978-94-007-4978-8_180

Tan Sisman, G., & Aksu, M. (2016). A study on sixth grade students' misconceptions and errors in spatial measurement: Length, area, and volume. *International Journal of Science and Mathematics Education*, *14*(7), 1293-1319. https://doi.org/10.1007/s10763-015-9642-5

Tang, Y. M., Au, K. M., Lau, H. C. W., Ho, G. T. S., & Wu, C. H. (2020). Evaluating the effectiveness of learning design with mixed reality (MR) in higher education. *Virtual Reality, 24*, 797-807. https://doi.org/10.1007/s10055-020-00427-9

Tumová, V., & Vondrová, N. (2017). Links between success between non-measurement and calculation tasks in area and volume measurement and pupils' problems. *Scientia in Education, 8*(2), 100-129. https://doi.org/10.14712/18047106.1011

UNESCO. (2011). UNESCO ICT competency framework for teachers. *UNESCO*. https://unesdoc.unesco.org/ark:/48223/pf0000213475

Voulgaris, S., & Evangelidou, A. (2004). Volume conception in late primary school children in Cyprus. *Quaderni di Ricerca in Didattica* [*Research Papers in Didactics*], *14*, 1-31.