Options of Discovering and Verifying Mathematical Theorems – Task-design from a Philosophic-logical Point of View

Michael Meyer [1*]

[1] University of Cologne, GERMANY

ABSTRACT

Mathematical theorems can be discovered and verified in different ways. The philosophical logic provides a conceptual understanding of these processes and helps to distinguish different options of initiating them. These options are established by analyzing different textbooks and describe the possibilities of the discovery and (empirical) verification of mathematical knowledge.

Keywords: abduction, discoveries, induction, options of task-design, philosophical logic, verifications

INTRODUCTION

From an organisational point of view, a task can be defined as an obligation, to execute an activity (cf. Nordsiek & Nordsieck-Schroeer 1969, pp. 193). Regarding design principles like discovery learning (Bruner 1961), or more special the guided reinvention principle (cf. Freudenthal, 1991) these activities require an active student who gains mathematical theorems on his own. Other authors want to activate the students' "engagement and sense making" (Barzel et al., 2013, p. 284, also Kieran et al., 2015). All of these authors accentuate the independent student who recognizes mathematical relations on his or her own. The creation of new ideas by setting up and establishing hypotheses is central. From a more logical point of view the discovered (hypothetical) knowledge can not be certain and should be verified. Together, two questions arise: How can students gain new knowledge? How do we get a 'hypothesis'?

Based on a conceptual understanding of the processes of discovering and verifying mathematical theorems I am going to present a system of options, in which the plurality of ways of constructing and establishing theorems is reduced to a few basic elements. The options of task-design have been found by analysing tasks in German textbooks.

The conceptual understanding of discoveries and verifications presented in this paper is based on the theory of Charles S. Peirce. His elaboration of the three inferences abduction, deduction and induction can be used as a tool for differentiating and designing mathematical tasks (for other tools see e.g. Leung & Bolite-Frant 2015). In the course of his philosophical thinking, Peirce offered several different forms and descriptions of the inferences abduction and induction (cf. Fann, 1970; Magnani, 2001). At the beginning of his work, he named the inference from an observed fact to an explanatory case "hypothesis" and afterwards "retroduction" and "abduction". The shift in his writings did not only refer to his theory of abduction but also to his theory of induction. In his previous writings, he assigned induction as being the inference of generating a general rule from a sample of data. This concept of induction is well known and widely used in mathematics education (e.g. Olander & Robertson, 1973; Pedemonte, 2007[1]; Winch, 1913). In his later writings, Peirce recognized that "in almost everything I printed before the beginning of this century I more or less mixed up Hypothesis and Induction" (Peirce, CP 2.227). At this stage, he describes abduction in a more narrow sense as "the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea [...]" (Peirce, CP 5.171).

---

[1] This perspective contrasts to those in Pedemonte & Reid (2011), in which the first author changed her view and combined the elementary inference to the argumentation pattern of Toulmin. As this comparison implies some logical problems (cf. Meyer, 2007) I am going to use the patterns of the inferences here (Meyer, 2010).

Contribution of this paper to the literature

- From a philosophic-logical point of view abduction resp. induction is going to be regarded as the decisive inferences of generating new (mathematical) rules resp. verifying these rules.
- The contribution of this study lies in presenting different options of designing tasks for elaborating mathematical rules. Here the elaboration comprises the generation and the (empirical) verification.
- The reconstruction of empirical learning processes shows the empirical usability of one special option.

In mathematics education research, the theory of abduction and the corresponding pattern (cf. chapter 2) have been used in several studies in order to reconstruct processes of discovering and verifying knowledge. Hoffmann (1999) described discoveries using the pattern of abduction from a theoretical point of view. Meyer (2010) reconstructed explanations and verifications of students in order to describe processes of discovering knowledge and the coherences to proving processes – also in order to gain a definition of discoveries. Meyer and Voigt (2010) analyzed processes of mathematical modelling. Kunsteller (e.g., 2018) used the theory of abduction in order to sharpen the concept of acquiring knowledge throughout family resemblances and especially analogies and metaphors. Krumsdorf (2015) used the inferences in order to analyze example-bound proving between inductive verification and formal proof. Reconstructions of problem solving processes have been done by Söhling (2016).

ABDUCTION AND INDUCTION

In his later writings Peirce defines the three inferences abduction, induction and deduction as three different steps in the process of inquiry:

> *… there are but three elementary kinds of reasoning. The first, which I call abduction … consists in examining a mass of facts and in allowing these facts to suggest a theory. In this way we gain new ideas; but there is no force in the reasoning. The second kind of reasoning is deduction, or necessary reasoning. It is applicable only to an ideal state of things, or to a state of things in so far as it may conform to an ideal. It merely gives a new aspect to the premises. … The third way of reasoning is induction, or experimental research. Its procedure is this. Abduction having suggested a theory, we employ deduction to deduce from that ideal theory a promiscuous variety of consequences to the effect that if we perform certain acts, we shall find ourselves confronted with certain experiences. We then proceed to try these experiments, and if the predictions of the theory are verified, we have a proportionate confidence that the experiments that remain to be tried will confirm the theory. I say that these three are the only elementary modes of reasoning there are. (CP 8.209)*

Thus, abduction appears as the inference from observed facts to new ideas. Induction appears as the inference from given ideas to the empirical confirmation of these ideas. Peirce's conceptualization differs from other concepts of induction as a path from observed facts towards new rules. In the following course of this paper, this distinction will shortly be confirmed by a closer look at the structural patterns of abduction and induction. As deduction is the well-known 'typical' mathematical inference, it is not discussed in detail here.

Peirce defined the "perfectly definitive logical form" (CP 5.189) of abduction as follows:

> *"The surprising fact, C, is observed;*
> *But if A were true, C would be a matter of course,* *Hence, there is reason to*
> *suspect that A is true." (CP 5.189, 1903)*

Hempel and Oppenheim claimed conditions every (scientific) explanation has to fulfil: First, it has to contain at least one general rule (Peirce also used a general rule in his former writings, cf. CP 2.622). Second, the observation has to be inferable from the explanation deductively (cf. Stegmueller, 1976, p. 452). Thus, we get a rule as mediator between the observed fact ("R" in **Figure 1**, "C" in the quotation of Peirce) and the explanatory case ("C" in **Figure 1**, "A" in the quotation of Peirce). Together, we get the pattern of abduction as presented in **Figure 2**, which represents the cognitive way of finding an explanation for an observed fact (for a longer consideration cf. Meyer 2010).

$$
\begin{array}{ll}
\underline{\text{fact (result):}} & R(x_0) \\
\text{rule:} & \forall i: C(x_i) \Longrightarrow R(x_i) \\
\text{case:} & C(x_0)
\end{array}
$$

Figure 1. One pattern of abduction[2]

Let us consider an example: Sherlock Holmes notices a (surprising) fact: flower soil has been left next to a dead body of a person. One possible explanation for this given fact is: The gardener killed the person after work (the explanatory case). The fact can explain the observed fact as we know that if a careless gardener kills a person after work, he would leave traces of the gardening at the scene. Thus, we get the following abduction:

<u>fact (result):</u>	Flower soil is left next to the dead body of a person.
rule:	If a gardener murders a person after work, he leaves traces of the gardening at the scene.
case:	The gardener could have murdered the person after his work.

Figure 2. Sherlock Holmes and the careless gardener

It is not unusual in detective stories that a gardener is the suspect – especially, when flower soil (a special result of traces of gardening) has been found next to the dead body. Obviously, also another rule might have been causal for the observed fact. The murdered person might have done work in the garden before or another murderer might have tried to avert suspicion. If this rule is true and causal for the observed fact, then the case has to be true, as the predicate (the gardener is the murderer) is given by the rule and the subject (the dead person) is given by the result (the former fact).

Let us therefore consider the following rule as a mathematical example: *When multiplying two powers with the same base a^b und a^c (a, b, c $\in \mathbb{N}$) then the product equals a^{b+c}.* In order to make the students able to discover this rule, the teacher (resp. the textbook) can request them to determine $3^2 \cdot 3^5$ and 3^7 by using a calculator. By considering the equality of the outcomes the students should set up a general rule. The following abduction would enable the recognition of the multiplication law:

<u>fact (result):</u>	$3^2 \cdot 3^5 = 9 \cdot 243 = 2187$ and $2187 = 3^7$
rule:	*When multiplying two powers with the same base a^b und a^c (a, b, c $\in \mathbb{N}$) then the product equals a^{b+c}.*
case:	The bases are the same and 2+5=7.

Figure 3. An abduction for recognizing a mathematical rule

By doing an abduction the observed fact appears as a "result" of the rule, if we are aware of the rule – as it thereby appears as one concrete element of the consequence of the rule. Thus, the observed fact gains the logical status as a (concrete) result of the rule within the abduction. Thus, an abduction starts with only one given premise (the observed fact). The recognition of rule and case takes place simultaneously in order to explain the observed fact: We can not infer rule and case step by step, because the case is entirely contained in the antecedence of the rule. Thus, if the rule is present, the case is (implicitly) present, too. On the other hand, we need the rule to infer the case. Eco writes: "[…] the real problem is not whether to find first the Case or the Rule, but rather how to figure out both the Rule and the Case *at the same time*, since they are inversely related, tied together by a sort of chiasmus" (Eco, 1983, p. 204). The existence of only one given premise also indicates that an abduction can not provide secure knowledge: It is a hypothetical inference insofar that also other rules could have been causal for the observed fact. Another rule would also bring in another case with it. In mathematics, it is not common that different mathematical

---

[2] This is one pattern which has proven to be useful for reconstructing students' comments and mathematical textbooks. In the literature, also other patterns of abduction exist (cf. Schurz, 2008).

rules can be causal for the observed fact – contrarily to the situation of Sherlock Holmes. Certainly, in educational settings the student might also consider the concrete chosen numbers of the teacher as causal for his observation.

If the observed fact can not be explained by using known rules, a new rule (and thus a new case) has to be created. Eco speaks of "creative abductions" (ibid. 1983, p. 206) to name these creative moments. If known rules are used in order to explain the fact, Eco (ibid.) differentiates between "undercoded" (the rule has been elected under different rules) and "overcoded" (it is nearly obvious to explain the observed fact by one concrete rule). Thus, the type of the abduction and, thus, the form of creativity is due to the cognitive horizon of the involved person. E.g., the abduction in **Figure 3** is not challenging for an expert in mathematics (and, thus, not a creative abduction), contrarily to a student who did not know the rule before.

Thus, if a student has to discover a new rule, we have at least two options: First, the task has to provide a concrete fact which is a result of the rule, the student has to discover. It would make sense, if this fact can not be explained by known rules in order to avoid conflicts. Second, the student has to be able to elaborate a fact which can be explain by this rule. Different options of providing facts due to this philosophical-logical point of view are given later on.

The inference from a given case and a given rule is called *deduction* (cf. **Figure 4**).

case:	$C(x_0)$
rule:	$\forall i: C(x_i) \Longrightarrow R(x_i)$
result:	$R(x_0)$

Figure 4. Pattern of deduction

Deduction is the most decisive inference in mathematics as this is the only inference which can guarantee the truth of the conclusion of this inference (the result) – presupposed the premises (case and rule) are correct. Thus, the truth of the conclusion depends on the truth of the rule. This is going to be important later on.

Contrarily to an abduction, in which different rules can be used to explain an observed fact, a deduction is a necessary inference: Given the specific case and rule, only one result can be inferred, because the subject (x_0) is given by the case and the predicate (R) to this subject is determined by the rule.

case:	$C(x_0)$
result:	$R(x_0)$
rule:	$\forall i: C(x_i) \Longrightarrow R(x_i)$

Figure 5. Pattern of induction

Figure 5 shows the pattern of induction. Induction has often been used to describe the generation of new rules (see above). The underlying concept of this point of view can be described as follows: 'What can be observed a couple of times is always valid' – also known as the "principle of induction". However, the pattern of induction shows already one decisive problem: How do we get the idea of combining this fact and this result? The common understanding of induction leaves an important question unanswered: How do we get the presumption that the case has something to do with the result; what leads us to connect this case and this result as premises of one inference? As this connection can not be inferred by an induction, it has to be noticed by another inference before. Furthermore, if the connection of case and result has been recognized, the general rule, as a mediator between case and result, must be (implicitly) present, too.

In his later writings at the beginning of the 19th century also Peirce recognized that an induction can not lead to new rules. From this moment, he did not consider induction as being the inference of creating new rules any more: "in almost everything I printed before the beginning of this century I more or less mixed up Hypothesis and Induction" (Peirce, CP. 2.227; afterwards he called hypothesis abduction). Induction now appears as the inference of the (empirical) confirmation of ideas and abduction appears as the inference from observed facts to new ideas. Thus, his new conceptualization differs from the common concept of induction and he defines the three inferences as different steps in the process of inquiry (cf. the quotation above).

There are at least two ways of how to use the logical combination of abduction, deduction and induction for an empirical confirmation of a hypothesis: The first way, the Bootstrap-Model (Carrier, 2000, p. 44), points out that the subjects $(x_{0,...,i})$, which motivated the creation or association of the general rule, are specific. Other similar subjects $(x_{i+1,...,m})$ can be used to confirm or to refute the rule or its coherence to the result (**Figure 6**).

1. Abduction	2. Deduction	3. Induction
result 1	case 2	case 2
rule 1	rule 1	result 2
case 1	result 2	rule 1

Figure 6. The structure of verification by the Bootstrap-Model (the same rule occurs in every step)

Starting with an observed fact (result 1) we cognitively suggest a general rule. In the second step this rule is used to deduce a prediction: If the rule is correct, a similar case (case 2) must have a necessary consequence (the result of the deduction). The predicted result is now going to be tested. If the test confirms the prediction, the rule gets confirmed by another example.

The empirical confirmation of the rule in **Figure 3** can be developed as follows: If the given rule (*When multiplying two powers with the same base a^b und a^c (a, b, c ∈ ℕ) then the product equals a^{b+c}.*) is correct, then $3^4 \cdot 3^8$ (which have the same base, case 2) has to equal 3^{12} (result 2). This prediction can be drawn by the deduction in **Figure 7**.

case:	In $3^4 \cdot 3^8$ the bases are the same and 4+8=12.
rule:	*When multiplying two powers with the same base a^b und a^c (a, b, c ∈ ℕ) then the product equals a^{b+c}.*
result:	$3^4 \cdot 3^8$ has to equal 3^{12}.

Figure 7. A deduction for the necessary prediction for a new case under the presupposition of a correct rule

The truth of the result of a deduction depends on the truth of the rule. If the rule is valid and the case fits to the rule, the result has to be valid, too. Thus, if the result is valid, we have an empirical evidence that the rule could also be valid.

Now a calculator can be used again in order to test the prediction. After the calculator has confirmed the predicted outcome, we can confirm our rule:

case:	In $3^4 \cdot 3^8$ the bases are the same and 4+8=12.
result:	$3^4 \cdot 3^8 = 3^{12}$ (confirmed by calculator)
rule:	*When multiplying two powers with the same base a^b und a^c (a, b, c ∈ ℕ) then the product equals a^{b+c}. (rule gets confirmed)*

Figure 8. An induction for the empirical comfirmation of a mathematical rule

Within the induction in **Figure 8** the result is not a prediction as it already has been tested (e.g., by using the calculator). It can be named as a result, as it has to be a result of the presupposed rule in order to confirm this rule. The new case and the new result confirm the previously generated rule by another example.

Within the second way of verification, the hypothetico-deductive approach (Carrier, 2000, p. 44), another (known) rule is used in order to deduce necessary consequences of the abductively suggested rule, which can be tested afterwards (also the hypothetic case of the abduction could be tested, **Figure 9**). As this kind of verification could only be reconstructed a couple of times, it will not be discussed in every detail here (except for on example in section 4).

1. Abduction	2. Deduction
result 1	case 2 (= rule 1 or case 1)
rule 1	rule 2
case 1	result 2

Figure 9. The general structure of hypothetico-deductive verification (without the inductive step)

METHODOLOGICAL REMARKS

During a collaborative work Prof. Dr. Joerg Voigt (University of Munster, Germany) and I used the three inferences abduction, deduction and induction to analyse the introduction of mathematical theorems in mathematics textbooks. An abduction, the characteristic inference for discovering mathematical rules (resp. theorems), gets initiated by only one given premise: the surprising fact which appears as a result of a rule, if we are aware of this rule. If a task in a textbook is supposed to lead the students to a theorem, the task have to provide concrete results of this theorems or (at least) the task has to enable the students to gather them. This implies the following consequences for the reconstruction of tasks (The method of reconstruction is described shortly here. For a more detailed characterisation please consider Meyer and Voigt, 2009): First, the scientist can look for mathematical rules the students should discover. These rules are often described in the following course of the book or in the reference book for the teacher. Second, the task has to be considered in an interpretative way in order to establish a result of the rule. Sometimes these results are presented directly in the task. If not, possible ways of elaborating results have to be considered by working in different thinkable ways on the mathematical content.

If there is no mathematical rule explicitly mentioned in the reference book or in the textbook, a task could lead to different rules. Thus, sometimes different ways of discovering could be reconstructed. The reconstruction of analysing different ways of empirical verifications of a mathematical theorem can be done analogically. Examples will be presented in the following course of this paper. In the concerning project, about 50 textbooks have been analysed. For the reconstruction of different latent meanings of possible and real (cf. chapter 5 of this article) student solutions, the method of "objective hermeneutics" (Oevermann et al., 1987) has been used: Different versions of reading a task resp. a document of a student have been considered in order to identify latent structures of meaning, which do not contradict to the theoretical foundations.

Analyzing the tasks, different logical structures for discovering and verifying theorems could be observed and resulted in a system of options a teacher or a school-book author can use for constructing tasks for the discovery and the empirical verification of mathematical rules. As a lot of research has been done on different forms of deductions, options of the process of proving mathematical rules are not going to be presented. The options are exemplified in consideration of the rule which has been already regarded above, the *multiplication law* for powers with the same base: *When multiplying two powers with the same base a^b und a^c ($a, b, c \in \mathbb{N}$) then the product equals a^{b+c}.*

OPTIONS OF TASK-DESIGN

In this section, different options of task-design are going to be presented. These options concern the processes of discovering mathematical rules and processes of empirical verification. Certainly, the presented options should not exclude the existence of others. The following table includes an advanced organizer of the following options.

Table 1. List of options of task-design

process	option
discovering I	*discovery by a special fact (result)*
discovering II	*discovery by a typical fact (result)*
discovering III	*discovery by a couple of facts (results)*
discovering IV	*discovery by a class of facts (results)*
discovering V	*discovery with a latent idea of proof*
empirical verification I	*verification by a special case*
empirical verification II	*verification by a specific other case*
empirical verification III	*verification by a couple of cases*
empirical verification IV	*verification by a class of cases*
empirical verification V	*verification with a latent idea of proof*
empirical verification VI	*verifying by networking*

Options of Task-design: Discovering Mathematical Theorems

Option 1: Discovery by a special fact (result)[3]. In order to discover the multiplication law a student could rewrite 100 000 · 100 = 10 000 000 by using powers with the base 10. Therefor, he only has to use the definition of

---

[3] As shown in the theoretical part of this article, the observed fact gets the logical status of a result of a rule, if we are aware of the rule. In order to emphasize this coherence, I am going to use both words.

powers deductively. The new equation can be used in order to discover a general rule. The discovery of the multiplication law can be made by the abduction in **Figure 10**:

<u>fact (result):</u>	$10^5 \cdot 10^2 = 10^7$ (by using the rule of attaching zeros)
rule:	Multiplication law
case:	The bases are the same and 5+2=7.

Figure 10. A creative abduction made by a special fact (result)

Concerning the example, the result of the abduction might be so familiar for a student that the rule of attaching zeros could be enough for him to explain the result. We speak of a "discovery by a special result", if known rules can be used to explain the result and the student is not in need of finding a new rule. Then the abduction would not be a creative one.

In other words: By using a special fact (result) it is not necessary to discover a new rule in order to explain the observed fact. Also, an explanation of results which have been deduced by other rules before, is not necessary. Thus, if we want to develop a *need for explanation* of our students, this option is not the best method. Nevertheless, this option shows how mathematical rules in school can interact – and this is also a benefit of education.

Option 2: Discovery by a typical fact (result). The abduction by a typical result increases the chance to discover a new rule (instead of using known rules). Regarding the example above: The student determines $3^5 \cdot 3^2$ and 3^7 by using a calculator. Afterwards he should – in consideration of the equality – discover a general rule. The necessary abduction has been presented in **Figure 3**. The result is a "typical result" for the theorem, because the students have not been in contact with rules yet, which compete with the requested one in order to explain the result abductively. Therefore, they have to generate a new rule.

Option 3: Discovery by a couple of facts (results)[4]. A mathematical rule can gain more plausibility, if it is discovered by more than one result. The risk decreases that the rule might only be valid for particular kinds of numbers and not for all. The students, for example, calculate $243 \cdot 9, 16 \cdot 4$ and $25 \cdot 5$ and determine the outcomes by using powers with the lowest base. Contrarily to **Figure 10** not only the numbers (the basis and the exponents) but also the quantity of terms in the result and the case of the abduction would change.

Option 4: Discovery by a class of facts (results). If the results form a whole class, the discovery of a rule can get a particular plausibility. Concerning the example, the students could be required to remember the rule for attaching zeros in order to multiply powers with the base 10. They should write down this (known) rule by using powers with the base 10 and recognize a more general (!) rule which is also valid for other bases. The abduction in **Figure 11** describes this option.

<u>fact (result):</u>	$10^a \cdot 10^b = 10^{a+b}$ (by using the rule of attaching zeros)
rule:	Multiplication law
case:	The bases are the same.

Figure 11. A creative abduction made by a class of facts (results)

Option 5: Discovery with a latent idea of proof: This option does not only lead to a mathematical rule, but possibly also to an idea of proof of this rule. In order to discover the multiplication law with a latent idea of proof, the following tasks could be solved: Firstly, write 3^5 and 3^2 as products ($3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ and $3^2 = 3 \cdot 3$). Secondly, form the product of these products and write it down by using powers with the base 3 [$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^7$].
Thirdly, discover a general rule and proof it.

By solving the first and second part of the task, the observed fact (the result of the upcoming abduction) is getting inferred by a chain of deductions. In these deductions, the definition of powers is going to be used three times: in order to gain the two products and in order to form the final power.

$3^5 \cdot 3^2 \stackrel{\text{def}}{=} 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \stackrel{\text{def}}{=} 3^7$

Afterwards, the multiplication rule can be discovered by the abduction in **Figure 12**.

---

[4] Here the plural form "facts (results)" is used. Surely, there is only one fact resp. one result (after discovering an explanation) in an abduction, but the elements within the fact resp. the result increase. Regarding the number of elements, I am using the plural form here.

<u>fact (result):</u>	$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^7$
rule:	Multiplication law
case:	The bases of the multiplied powers are the same and $5 + 2 = 7$.

Figure 12. Discovery with a latent idea of proof

If the deductions for setting up the fact are generalized (by regarding 3, 5 and 2 as variables) afterwards, it is possible to prove the rule of the later abduction:

$$x^a \cdot x^b \stackrel{\text{def}}{=} \underbrace{x \cdot \ldots \cdot x}_{a \; times} \cdot \underbrace{x \cdot \ldots \cdot x}_{b \; times}$$

$$\underbrace{x \cdot \ldots \cdot x}_{a \; times} \cdot \underbrace{x \cdot \ldots \cdot x}_{b \; times} = \underbrace{x \cdot \ldots \cdot x}_{a+b \; times} \stackrel{\text{def}}{=} x^{a+b}$$

In other words: In order to elaborate the fact (the result of the prospective abduction) the students are confronted with the main steps of the proof of the rule before they become aware of the rule. The elaboration of the fact requires the use of the definition of powers for concrete numbers. The later proof is in need of the same steps – only that the concrete numbers have to be changed to variables. The idea of proof is latent, because the students have to recognize it and this is (of course) not self-evident. From a mathematics educational point of view, discoveries with a latent idea of proof contain a special chance. The hypothetical moment of generating new knowledge is combined with the rigour of a later proof: The proof of the knowledge is rooted in the process of discovering the knowledge. In the words of Reichenbach (1938): The "context of discovery" gets combined with the "context of justification". Therefore, the benefit of this option compared to the pure discovery of knowledge is even greater than the benefit of discovering knowledge compared to more reproducing approaches. Later on, I am going to show, that students of different ages seem to be able to recognize an idea of proof.

Options of Task-design: Verifying Mathematical Theorems Empirically

Option 1: Verification by a special case. After the student has discovered the multiplication law by one or more results an empirical verification is in need of a new case. Let us consider the following example: the rule has been discovered by using $3^5 \cdot 3^2$ and 3^7. Now the students are asked to verify it by using the calculator. Thus, the students have to deduce a prediction: If the discovered theorem is correct, the multiplication of other concrete powers would result in an addition of the exponents. Let us imagine, a student uses $3^6 \cdot 3^7$ which has to equal 3^{13}, if the rule is correct. If the verification is successful (e.g., the calculator matches the predicted product), the induction in **Figure 13** can be used to confirm the rule of the abduction:

case:	In $3^6 \cdot 3^7$ the bases are the same and $6+7=13$.
<u>result:</u>	$3^6 \cdot 3^7 = 3^{13}$ (by the calculator)
rule:	Multiplication law (gets confirmed)

Figure 13. An induction made by a special case

The problem of this way of verification is that the basis 3 has already been used to discover the multiplication law. The verification does not confirm the rule as being a general one. Thus, the students could recognize a general rule which is only valid for the basis 3. This problem can also occur in a comparable way if the used exponents for the verification equal those of the former abduction.

Option 2: Verification by a specific other case. If the verification has to be done by numbers which are completely different to those which had been used to discover the theorem, the verification gets a specific plausibility. Thus, the fear decreases of verifying a rule which is only valid for a few numbers or a particular number.

Options 3 and 4: Verification by a couple of cases (3) and by a class of cases (4). If the number of the cases increases, for which a suggested rule is valid (and not always with the same basis or exponents), the rule can gain high plausibility:

> *"[...] having verified the theorem in several particular cases, we gathered strong inductive evidence for it. [...] Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job." (Polya, 1954, pp. 83)*

A verification by a couple of cases is pretty simple: Compared to the induction in **Figure 13** the number of products in the case and result is increased – ideally, by different numbers as bases and exponents.

If the discovered rule should be verified by a class of cases, the student needs to know other rules to specify the deduced prediction and, thus, to infer the result of the induction. Considering the example, he could use the law of attaching zeros. Then the induction in **Figure 14** describes an induction for confirming the rule:

case:	In $10^b \cdot 10^c$ the bases of the multiplied powers are the same.
result:	$10^b \cdot 10^c = 10^{b+c}$ (by using the law of attaching zeros)
rule:	*Multiplication law (gets confirmed)*

Figure 14. An induction made by a class of cases to confirm the multiplication law

Option 5: Verification with a latent idea of proof. Nearly analogous to the option 5 for discovering mathematical theorems, the verification of a rule can also imply a latent idea of proof. Now the deductive steps, which can be used structurally for the proof, need to be taken to infer the result of the deduction. Let us assume the students had discovered the rule by the result $3^5 \cdot 3^2 = 3^7$ which had been determined by the calculator. Now they should verify it by $7^4 \cdot 7^5$ without using the calculator. The amount of the numbers causes the students not to solve the task by calculating and, thus, using the definition of powers. The generalization of the deductive steps within the test in order to confirm the deduced prediction can be used afterwards to proof the multiplication law.

Comparable to the option "discovery with a latent idea of proof", this option implies the potential of combining the aspects of empirical approaches and more theoretical ones. Thus, using this potential in mathematics education can be of great importance.

Option 6: Verifying by networking. The former options for verifying rules depend on the Bootstrap-Model: A rule gains more plausibility, if it is valid for more examples. The option "verifying by networking" bases on the hypothetic-deductive approach. Unfortunately, this option could only be reconstructed a few times in mathematics textbooks.

Let us consider an example of this option: The student assumes the multiplication law to be correct. Now, he infers that this rule can also be verified by geometry. In geometry, powers can be described by areas and volumes. In the three-dimensional space, a successful verification of the multiplication law can happen, if the student realises that the volume of a cube can be determined by "base area times height" (cf. **Figure 15**).

case:	$a^1 \cdot a^2$ are the amount of the height resp. the base area of a cube. In the terms for height and base area, a^1 and a^2, the bases of the powers are the same, and $1 + 2 = 3$.
result:	$a^1 \cdot a^2 = a^3$ (by using the law of attaching zeros)
rule:	*Multiplication law (gets confirmed)*

Figure 15. Verifying by networking

The potential of this option is given due to the combination of different aspects of mathematics (here the different fields: arithmetic and geometry). Such an approach can be used not only to show the links between the different aspects, but also to enable students to use knowledge from one aspect to the other (potentially in order to overcome problems in one aspect).

EMPIRICAL FINDINGS

Regarded from the point of not only forcing discoveries or empirical verifications but also proofs, the options of discovering and verifying mathematical rules with a latent idea of proof seem to be the best ones, as these processes get directly combined to the (following) process of proving the discovered or verified rule. Surely, taking a more educational point of view also the other methods of discovering and verifying theorems have benefits. For example, one aim might be to give the students more opportunities to discover the theorem or to make them be aware of theorems in a short time span.

In a further step, an empirical study has been executed in order to test the applicability of the option "discovery with a latent idea of proof". Thus, I tried to analyze whether students are able to recognize a latent idea of proof. The empirical data for the text emerged from different classrooms in Germany. The first example has been

videotaped in a ninth-grade classroom (students aged from 14 to 15 years), the second one from a grade four (students aged from 9 to 10 years). Classroom communication has been videotaped and transcribed. The rules of transcription can be found in the appendix. The solutions of the students are going to be reconstructed following an interpretative research paradigm (cf. Voigt, 1984): For the reconstruction of latent meanings of real student solutions, the method of "objective hermeneutics" (Oevermann et al., 1987) has been used (see above) in order to reconstruct the applicability of the theoretical constructs.

In a ninth grade classroom, the multiplication law was introduced by the following task (translated):

a) Rearrange the term $7^6 \cdot 7^9$ in order to get only one power.

b) Formulate an assumption for all $x^a \cdot x^b$.

c) Prove your assumption.

By solving the first part, the students elaborate a phenomenon. Spoken more generally from an expert point of view: This phenomenon can be considered as a concrete result of the multiplication law and thus, it can be the starting point of the abduction for discovering this law. The elaboration of the phenomenon is not arbitrary, as the procedure of elaboration plays an important role in part c. In part b the students have to explain it – in other words: The students have to execute an abduction and formulate the general rule. At the end, the proof for the general theorem is questioned. Therefore, the former steps of part a have to be generalized. The following example of a student's solution is going to illustrate this procedure:

The task has been given to the students without any comment to the content. **Figure 16** shows the solution of Petra:
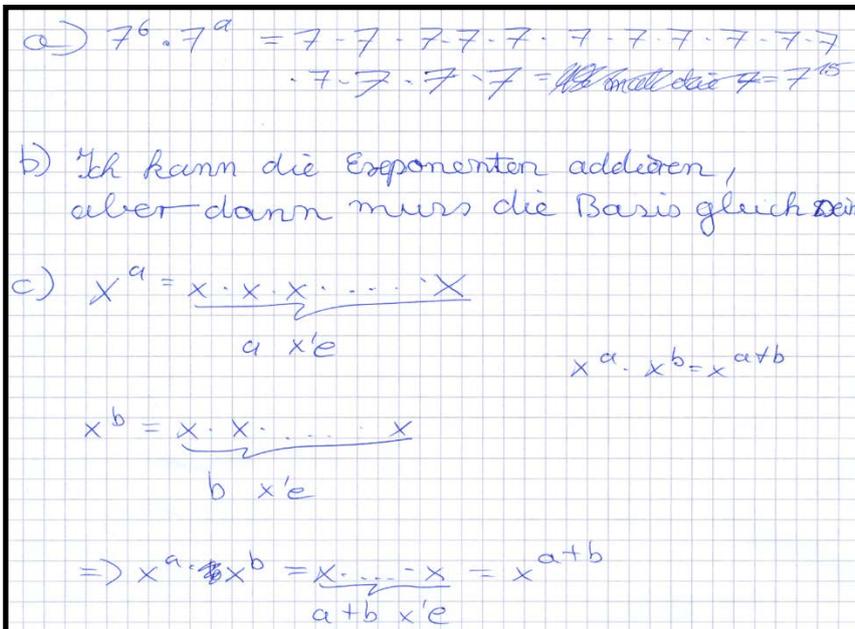

Figure 16. The solution of Petra (translation of b: I can add the exponents, but than the bases have to be the same)

As claimed by the task, Petra solves part a by considering the powers as a chain of products. Afterwards she combines the whole product in order to gain one power. For both steps, the definition of powers has to be used deductively:

$$7^6 \cdot 7^9 \xrightarrow{Def.of\ power} 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$$

$$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \xrightarrow{Def.of\ power} 7^{15}$$

For solving part b, Petra speaks of exponents which can be added, if the bases are the same. This can be regarded as the rule of her abduction, which has to be proven in part c. The solution in part c is just slightly more explicit, but the only modification concerns the use of variables instead of the number 7.

Thus, we can say that Petra seems to have realized the idea of proof (the repeated use of the definition of powers within the same structure) by its manifestation in the later proof (part c).

The described options of task-design are certainly not reduced to the multiplication law. They can be used concerning nearly every mathematical theorem: This will be shown by the following example (used in grade 5, students 10-11 years old):

a) Determine the results of the following tasks.

$125+126+127 = \boxed{}$ $\qquad$ $\boxed{} : 3 =$

$414+415+416 = \boxed{}$ $\qquad$ $\boxed{} : 3 =$

$71814+71815+71816 = \boxed{}$ $\qquad$ $\boxed{} : 3 =$

b) Make a conjecture about the relations of three consecutive summands.

c) Justify your conjecture.

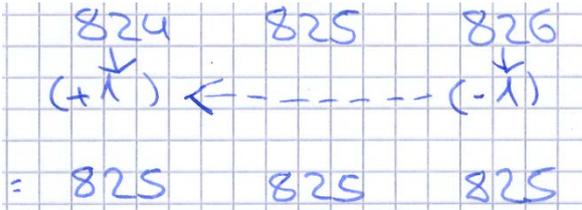Timo (10 years) also uses other summands and explains the possibility of an inverse modification of the numbers.

Figure 17. The inverse modification of the summands by Timo

Within his notes in **Figure 17**, Timo showed that he did not add the summands. Moreover, he made an inverse modification of the summands in order to get three equal summands. Thus, as the addition of three similar numbers can be regarded as a multiplication by 3, the division by 3 "cancels" the multiplication. Within picture 17 Timo did not use the special characteristics of the numbers (except for their difference). In the classroom discussion Timo justified his conjecture: "After the division by 3 you get the number in the middle as the former summands can be made equal." Thus, this example again shows a student gaining a new rule ("If x is the quotient of the sum and quantity of the summands, then the sum of 3 numbers can be inversely modified to x+x+x.") with an idea of proof. The former latent idea seems to get manifested by his later statement for proofing the rule.

FINAL REMARKS

As described in this paper, abduction is the only inference which can be used to gain new rules. Abductive processes start by only one given premise: the (surprising) fact. Thus, in order to enable students to discover mathematical rules (resp. theorems), we have to give them those facts or at least make them able to elaborate the facts. The structure of the fact is simple. It has to be a concrete element of the consequence of the (new) rule, which has to be conjectured or discovered by the abduction. The abduction is the process of explaining the facts by seeing them as a result of a (new) rule. As we can never exclude that not another rule has been causal for the observed fact, an abduction does not guarantee certainty and we are also not able to methodize the process of making abductions.

In order to verify a hypothesis, we might be able to prove it. Other ways of verification are the Bootstrap-Model and the hypothetic-deductive approach, which show how abductively inferred knowledge can be empirically verified. Both approaches can be used to confirm a rule, but not to prove it. Only if the deduced predictions are not true, we can get certainty in that way that the hypothesis has to be rejected (or at least be modified). The analyses of textbooks resulted in different options for task-design in order to discover or to verify a theorem. The presented options for discovering mathematical theorems show how tasks can be created to enable students to recognize a theorem by a creative abduction. They also show that it is possible

- to increase the chance to discover a new rule,
- to increase the plausibility of the discovered rule and
- to enable the students to recognize the idea of proof for the theorem while discovering the theorem.

The presented options for verifying mathematical theorems show how tasks can be created to verify an already discovered theorem. These options make it possible to increase the plausibility of the discovered theorem and to enable students

- to recognize the idea of proof for the theorem while verifying the theorem and
- to make students aware that a possible hard deductive proof can be successful.

It is not useful to reduce the options to those with a latent idea of proof, and to let students work on the task in small steps in order to acquire the solution. This would reduce the students' freedom of solving the tasks, as this sometimes is in need of highly structured instructions.

As the understanding of the inferences and their use for analysing tasks is hard, we have to ask for their benefits. The scientist might already notice a benefit in reducing the variety of possible ways of solving a task to three elementary inferences (cf. the periodic table of elements in chemistry). The authors of textbooks can use the analyses by the inferences and the different options in order to reflect on their own ideas or to gain new ideas for the creation of tasks for discovering and verifying mathematical rules. Furthermore, in teacher education and in in-service training, the knowledge of the inference can help to prepare worksheets and to understand the (expressed) solutions of the students. It is the variety of possibilities which can and should be used to develop different elaborations of and to a mathematical rule. The application of this variety is the productive benefit of the options.

The presented options should not exclude that there are also other ones. In further work, my working group is going to reconstruct other processes. Those options belong e.g. to the use of analogies in order to discover new rules (Kunsteller 2018).

REFERENCES

Barzel, B., Leuders, T., Prediger, S., & Hußmann, S. (2013). Designing Tasks for Engaging Students in Active Knowledge Organiszation. In C. Margolinas et al. (Ed.), *Task Design in Mathematics Education*. Proceedings of ICMI Study 22. Oxford.

Bruner, J. S. (1961). The act of discovery. *Havard Educational Review*, *31*, 21-32.

Carrier, M. (2000). Empirische Hypothesenpruefung ohne Felsengrund. In F. Stadler (Ed.), *Elemente moderner Wissenschaftstheorie* (pp. 43-56). Berlin: Springer.

Eco, U. (1983). Horns, Hooves, Insteps. In U. Eco & T. A. Sebeok (Eds.), *The sign of the three. Dupin, Holmes, Peirce* (pp. 198-220). Bloomington: Indiana University Press.

Fann, K. T. (1970). *Peirce's theory of abduction*. The Hague: Nijhoff. https://doi.org/10.1007/978-94-010-3163-9

Freudenthal, H. (1991). *Revisiting mathematics education*. Dodrecht: Kluwer.

Hoffmann, M. (1999). Problems with Peirce's concept of abduction. *Foundations of Science*, *4*(3), 271-305. https://doi.org/10.1023/A:1009675824079

Kieran, C., Knott, L., & Yang, Y. (2015). Frameworks and Principles for Task Design. In A. Watson & M. Ohtani, M. (Eds.), *Task Design in Mathematics Education. An ICMI Study 22.* Cham: Springer. https://doi.org/10.1007/978-3-319-09629-2_2

Krumsdorf, J. (2015). *Beispielgebundenes Beweisen* (Dissertation). University of Munster, https://core.ac.uk/download/pdf/56470239.pdf [10.04.2018]

Kunsteller, J. (2018). Developing Rules Due to the Use of Family Resemblances in Classroom Communication. In P. Ernest (Ed.), ICME-13 Monographs. The Philosophy of Mathematics Education Today. Dodrecht: Springer. https://doi.org/10.1007/978-3-319-77760-3_24

Leung, A., & Bolite-Frant (2015). Theme A: Tools and Representations. In C. Margolinas (Ed.), Task Design in Mathematics Education. Proceedings of ICMI Study 22. 2013. Oxford.

Magnani, L. (2001). *Abduction, Reason and Science*. New York: Kluwer. https://doi.org/10.1007/978-1-4419-8562-0

Meyer, M. (2007). Entdecken und Begruenden im Mathematikunterricht. Von der Abduktion zum Argument. Dissertation. Hildesheim: Franzbecker.

Meyer, M. (2010). Abduction – A logical view of processes of discovering and verifying knowledge in mathematics. *Educational Studies in Mathematics*, *74*, 185-205. https://doi.org/10.1007/s10649-010-9233-x

Meyer, M., & Voigt, J. (2009). Entdecken, Pruefen und Begruenden. Gestaltung von Aufgaben zur Erarbeitung mathematischer Saetze. *Mathematica didactica*, *32*, 31-66.

Meyer, M., & Voigt, J. (2010). Rationale Modellierungsprozesse. In B. Brandt, M. Fetzer & M. Schuette (Eds.), Auf den Spuren Interpretativer Unterrichtsforschung in der Mathematikdidaktik. Munster: Waxmann, 117-148.

Nordsiek, F., & Nordsieck-Schroeer, H. (1969). Aufgabe. In E. Grochla (Ed.), Handwoerterbuch der Organisation. Stuttgart: Poeschel, 191-199.

Oevermann, U. et al. (1987): Structures of meaning and objective Hermeneutics. In: Meha, V. et al. (Eds.), *Modern German sociology*. (European Perspectives: a Series in Social Thought and Cultural Cticicism) (pp. 436–447). New York: Columbia University Press.

Olander, H. T., & Robertson, H. C. (1973). The effectiveness of discovery and expository methods in the teaching of fourth-grade mathematics. *Journal for Research in Mathematics Education*, *4*(1), 33-44. https://doi.org/10.2307/749022

Pedemonte, B. (2007): How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*, 23-41. https://doi.org/10.1007/s10649-006-9057-x

Pedemonte, B., & Reid, D. (2011): The role of abduction in proving processes. *Educational Studies in Mathematics*, *76*(3), 281-303. https://doi.org/10.1007/s10649-010-9275-0

Peirce, Ch. S. CP. (n.d.). *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by C. Hartshorne & P. Weiss, 1931-1935; Volumes VII-VIII, ed. by A.W. Burks, 1958 – quotations according to volume and paragraph). Cambridge: Harvard UP.

Polya, G. (1954). *Mathematics and Plausible Reasoning*. Vol. 1. Princeton: UP.

Reichenbach, H. (1938). *Experience and prediction*. Chicago: UP.

Schurz, G. (2008). Patterns of abduction. *Synthese, 164*, 201–234. https://doi.org/10.1007/s11229-007-9223-4

Söhling, A.-C. (2016). The learning of mathematics via problem solving. In T. Fritzlar (Ed.), *Problem solving in mathematics education. Proceedings of the 2015 joint 17th conference of ProMath and the 2nd GDM working group on problem solving* (pp. 215-225). Munster: WTM.

Stegmueller, W. (1976). *Hauptstroemungen der Gegenwartsphilosophie*. Vol. 1. Stuttgart: Koerner.

Voigt, J. (1984). Interaktionsmuster und Routinen im Mathematikunterricht. Theoretische Grundlagen und mikroethnographische Falluntersuchungen. Weihnheim: Beltz.

Watson, A., & Ohtani, M. (2015, Eds.). *Task Design in Mathematics Education. An ICMI Study 22.* Cham: Springer.

Winch, W. A. (1913). *Inductive versus deductive methods of teaching*. Baltimore: Warwick.

http://www.ejmste.com