MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2022, 18(12), em2192
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/12656

Pre-algebraic aspects in arithmetic strategies – The generalization and conceptual understanding of the 'Auxiliary Task'

Taha Ertuğrul Kuzu [1*]

[1] University of Münster, Münster, GERMANY

Received 05 October 2022 ▪ Accepted 10 November 2022

Abstract

In the last decades, a broad international reform approach was visible in support of mental calculation strategies: Instead of being solely 'transition strategies' for learning the standard algorithms, the understanding of numerical relations is essential for mental calculation strategies, making them highly important for a viable understanding of arithmetics. Yet, mental calculation strategies are not only important for understanding arithmetics, but highly relational strategies such as the 'Auxiliary Task' might have an important role in the emergence of a pre-algebraic understanding of numerical relations. In this qualitative study from Germany, 4th and 5th grade learners' (n=18) processes of interpreting the 'Auxiliary Task' are examined by conducting linguistic and epistemological analyzes of their conceptual understanding of the 'Auxiliary Task' utilizing a design-based research framework. Insights are given into specific, language-related forms of pre-algebraic generalizations of the 'Auxiliary Task' as well as into developmental processes within the designed learning-environment.

Keywords: Auxiliary Task, cognitive gap, design-based research, pre-algebraic thinking

INTRODUCTION: BETWEEN ARITHMETIC AND ALGEBRA

In the transition from primary to secondary school, arithmetic and algebra are two highly important topics. Although both topics are modelized distinctly in curriculum – for example in the German curriculum –, educational research in mathematics shows repeatedly that these topics are highly connected and cannot be assigned to school forms (i.e., assuming that arithmetic is only for primary school and that algebra is only for secondary school) (for an overview, see Kieran et al., 2016). The interrelation as well as transitional processes between these topics comprise the so-called 'cognitive gap' (Herscovics & Linchevski, 1994) and it is still highly important to gain deeper insights into these processes.

In recent years, there has been much research into the 'cognitive gap' and concepts like 'early algebra' or 'pre-algebra' emerged[1], trying to explain the 'cognitive gap' from a more theoretical perspective. There are various new frameworks and concepts – for example from Radford (2010) or Kieran (2018) –, but what is still missing is a broader empirical fundament: For example, insights into students' interpretations and pre-algebraical thinking processes in various contexts, going beyond pre-algebraic thinking in much-examined contexts like figural numbers, have yet to be given (Kieran, 2018). Most studies about pre-algebraic thinking are conducted in the context of figural problems, word problems or regarding functional thinking, and studies focusing more arithmetical contexts – like mental calculation strategies – are rare, but since *"arithmetical knowledge in primary classes already includes abilities of conversion that ultimately harbor algebraic potential* […] *without relying on formal algebraic tools such as elaborated representations and terms"* (Schwarzkopf et al., 2018, p. 195), an important research gap appears: Insights into students pre-algebraic thinking when generalizing arithmetic operations in highly relational arithmetic contexts like the 'Auxiliary Task'. Utilizing a design-based research approach and an interpretative analysis method, *languaging* processes of learners with regard to their rule-generalization are examined in an explorative qualitative approach to gain insights into the complex

[1] Normally, both terms are used synonymously or similarly. In this article, I prefer to use the term pre-algebra.

✉ tkuzu@uni-muenster.de **(*Correspondence)**

Contribution to the literature

- Recent studies show that pre-algebraic thinking – meaning the generalized understanding of mathematical structures in concrete numbers or terms through a profound conceptual reflection - is an important aspect between arithmetics and algebra.
- Most studies about the pre-algebraic thinking show its relevance in the context of figural numbers, functional thinking or word problems and only first insights are given into its relevance in the context of more arithmetic tasks. Especially for highly relational arithmetic formats like the 'Auxiliary Task' (the compensation strategy), detailed insights into students possible pre-algebraic thinking processes are still missing, although a highly relational, flexible view on numbers is necessary for utilizing the strategy.
- This study contributes to this research gap by giving insights into students' generalization of the 'Auxiliary Task' by reconstructing pre-algebraic thinking processes when thinking and reflecting upon the strategy, its conceptual side and the rule behind it. In a qualitative design-based-research study, n = 18 11 to 14 year old students' languaging processes are analyzed to show the emergence of a first notion of numbers-as-indeterminate when conceptually reflecting the 'Auxiliary Task', thus showing a) its high relevance for fostering pre-algebraic thinking processes and b) giving insights into the meaning-related languaging-processes when explaining the rule behind the strategy.

interrelation between arithmetical and algebraic thinking when generalizing the 'Auxiliary Task'.

THEORETICAL BACKGROUND: PRE-ALGEBRAIC THINKING IN THE CONTEXT OF THE 'AUXILIARY TASK'

Pre-Algebraic Thinking in the Context of Figural, Functional, and Word Problems

In the last years, different approaches and models were formulated to grasp this transition from arithmetic to algebraic thinking by describing so-called 'pre-algebraic thinking', which can be defined as the conceptual reflection and generalization of arithmetic rules, relations and structures in numbers, terms, and equations with concrete examples, but not yet with alphanumerical symbols (Kieran, 2011, 2018; Kieran et al., 2016; Mayer, 2019; Radford, 2018; Schwarzkopf et al., 2018; Steinweg, 2013). Steinweg (2013) explicits that in pre-algebra, *"relations, patters, and structures of concrete numbers, mathematical equations and terms are focused, stimulated/ explained and fostered"* (p. 12-13, translation from author) and Nührenbörger similarly highlights that

> "Pre-algebra […] aims thus at a structurally viable understanding of elementary arithmetic rules and relationships, which is on the one side linked to concrete numbers and terms and on the other side to the generalizability of those in an exemplary way" (Mayer, 2019, p. V, translation from author).

Still, insights are missing into learners' pre-algebraic thinking processes in different topics and into the way language, manipulatives, and more explicit approaches to the conceptual understanding of arithmetical content, like mental calculation strategies, affect pre-algebraic interpretations of learners (Steinweg et al., 2018). In this article, especially the 'Auxiliary Task' will be focused since it is a highly rule-based and relational mental calculation strategy and will be analyzed by utilizing an interpretative approach.

Pre-algebraic aspects in the context of figural problems are a research topic of high interest and a lot of research studies were conducted in this context in the last years (for an overview, see Kieran et al., 2016). Being highly influential, Radford (2010, 2014, 2018) for example examined 13-15 year old learners' interactions in small group sessions in a longitudinal study, where tasks about figural patterns were given to solve and discuss, and describes a model of three components for the development of algebraic thinking in these tasks: He describes a first component of predominantly arithmetic thinking, where learners identify arithmetic processes such as adding a specific number from one concrete figure to another. The second component consists of first pre-algebraic processes, where the students start to generalize their identifications, but yet on a factual or contextual level: They generalize through the (verbal or gesture-based) description of a rule about a *"noticed commonality to all the terms of the sequence"* (Radford, 2010, p. 55), thus detaching their thinking successively from concrete numbers. For the third component, the learners should be able to generalize more abstractly by further schematizing the noticed phenomena, which means that they formulate a *"rule providing one with an expression of whatever term of the sequence"*, stressing out that arithmetic generalizations would fail to meet this third component (Radford, 2010, p. 55). In his later works, Radford (2018) also emphasizes that the emergence of pre-algebraic thinking is not abrupt but through repeated and reflected arithmetic operations (Radford, 2018), but his approach has a strong emphasis on figural patterns and word problems. Yet, it also refers to arithmetical operations as a possible starting point of pre-algebraic thinking and he highlights the generalized rule-formulation after a phase of repeated and reflected arithmetic calculations as an important 'trigger' of pre-algebraic thinking. For the latter, three 'key aspects' are described:

- *Indeterminacy:* The problem involves not-known or non-determined numbers or quantities like variables, parameters, equations, etc.,
- *Denotation*: The indeterminate numbers involved in the problem have to be named or symbolized, i.e., through alphanumerical signs, natural language, gestures, unconventional signs, or a mixture of these, and
- *Analyticity:* The indeterminate quantities are treated as if they were known numbers by operating on them starting from the indeterminate quantities and operating on them through adding, subtracting, multiplying, dividing them, etc., as if they were known numbers (Radford, 2014).

Warren et al. (2006) conducted a study about functional thinking in elementary classrooms. They examined the development of primary school students' functional thinking during a teaching experiment (n=45) and their first algebraic explanations of functional relations. The study was designed to enable students to build mental representations in order to explore the use of function tables by focusing on the relationship between input and output numbers with the intention of extracting the algebraic nature of the arithmetic involved. The results indicate that elementary students are not only capable of developing functional thinking but also of communicating their (pre-)algebraic thinking both verbally and symbolically by giving explanations in (first) abstract and generic forms (Warren et al., 2006). The first aspect, the verbal explanations of the learners' generalizations are described with first case studies and show the possibility of early algebraic explanations, but a general description of the linguistic realization is not elaborated, leading to a specific question for further investigation being indicated by Warren and Cooper (2008): How do students generalize their explanations of (functional) relations verbally and how are these condensed (i.e., in written form)?

This *generalization of patterns, rules, and relations*, ranging from a first noticing of commonalities in patterns or arithmetic operations to the justification and objectification of those, seems to be a key aspect in all approaches, being also emphased in further studies showing generalizations of arithmetic processes through verbalization prior to the use of alphanumerical symbols (Schwarzkopf et al., 2018; Steinweg et al., 2018). Accordingly, Kieran (2011) points out that *"thinking relationally about operations, number and numerical operations"* is an important facet of pre-algebraic aspects in arithmetic strategies, meaning an *"algebraic thinking in form of generalizing relationships for operations with emphasis on relational and compensating operations"* (p. 584).

The Interrelation between Generalizations, Conceptual Understanding, and Its Languaging

'Generalization' is understood as a sign-related *comprehension of the relations and properties* of a mathematical object, being elaborated in a specific situation or reference context but going beyond it, meaning a transferability *"from concrete numbers to general* [numbers] […] *an independence from the chosen concrete numbers"* (Steinbring, 2006, p. 160). Such a process of 'generalization' comprises a 'deep' form of conceptual understanding that goes beyond factual or merely procedural knowledge and for its emergence, learners have to understand *how* a mathematical object is constructed by operating on it in a specific reference context – in other words by *"operating with mathematical knowledge and generalizing it"* (Steinbring, 2006, p. 134) – and thus, a learner

> "requires signs and symbols for mathematical knowledge, but […] these signs and symbols themselves are not the knowledge […]. In the ongoing development of the knowledge, the interpretations of the sign systems and the chosen according to reference contexts will be modified or generalized by the epistemological subject or the learner" (Steinbring, 2005, p. 92).

Steinbring (2006) also emphasizes the crucial role of 'language means' when understanding and generalizing the relations and proportions of mathematical objects by referring to the Vygotskian paradigm of the interrelatedness of thinking-and-speaking: Mathematical signs *"are used in communication with other persons in order to develop mathematical knowledge"* (Steinbring, 2006, p. 134). A viable generalization thus requires language means with a communicative function as well as language means with a cognitive/epistemic function (Wessel, 2020).

Since 'language means' play an important role in the generalization processes of learners, it is important to consider *languaging*-processes as an integral part of the conceptual development. 'Languaging' is a term coined by Swain (2006) to describe the interrelation between thinking-and-speaking when *"making meaning and shaping knowledge and experience through language"* (p. 98). When a

> "person is producing language, what he or she is engaging in is a cognitive activity; an activity of the mind […] language to mediate cognition (thinking) […] it is too simplistic to think of language as being only a conveyer of meaning. Rather we need to think of language as being an agent in the making of meaning" (Swain, 2006, p. 95).

In the context of mathematical generalizations and its languaging, Warren and Cooper (2008) stress out how challenging it might be to express generalization in language: In their study about pattern-recognition, *"many students could not express the pattern in general language, and when using the language confusion occurred"* (Warren & Cooper, 2008, p. 182). Steinweg et al. (2018) further examined this specific languaging-problem by analyzing the language means used by learners as 'agents' of pre-algebraic generalizations in the context of figural numbers and functions. They reconstruct language means like "For example, …", "… and so on" and "I always do …" as markers for the thinking of pre-algebraic aspects and furthermore, although students do not (yet) use variables in form of alphanumerical symbols in the primary school, their language means show a similar function, for example by using words or signs with a variable character (for example "the first number") or by using concrete numbers combined with a generalizing expression, so-called 'quasi-variables' (for example "I always calculate three times three") (Steinweg et al., 2018). Hence, from an epistemological viewpoint, learners produce new signs in form of *"verbal formulations, own words with exemplary descriptions* [and] *means of showing and referring (deictic)"* when generalizing relations or proportions of mathematical objects (Steinbring, 2006, p. 145).

Mental Calculation Strategies and 'Auxiliary Task'

For reflecting the conceptual meaning of arithmetical processes, rules and relations, especially mental calculation strategies are appropriate (Britt & Irwin, 2008, 2011; Serrazina & Rodrigues, 2021). There are at least three main mental calculation strategies: The HTU-strategy, meaning a successive calculation of the hundreds, tens, and units in both numbers, the stepwise-strategy, meaning the partitioning and successive calculation of the second number and the compensation-strategies, meaning strategies utilizing the compensation rule (Blöte et al., 2001; Selter et al., 2012). In the German context, the so-called 'Auxiliary Task' is a special case of the compensation strategies: it means the generation of a new task through the modification of either the first or second number in the first step and the compensation of the modification in the last step (**Figure 1**).

In **Figure 1**, the task 332-118 is solved by using the 'Auxiliary Task' with an ordinal representation. For fostering the *analytical noticing* before calculation, a thinking bubble is given and an ordinal visualization of the 'Auxiliary Task' is offered for relating the procedural steps with conceptual facets (Prediger et al., 2016). In successive steps, the modification of the second number has to be added back (+2) since it was taken away too much in the second step (332-120). Thus, the compensation is not just a local modification, it is a highly relational processes with specific determinants: The first numerical term *can be* manipulated through adding a specific number (x) in the identified rounding gap but then the last numerical term *has to be* compensated through adding or taking away x.

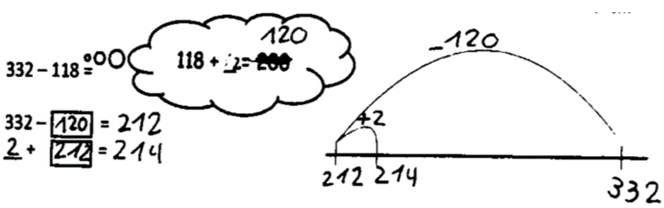


Figure 1. The 'Auxiliary Task' with its steps and an ordinal non-numbered line (Kuzu, 2022a)

The stepwise process is emphasized when using the 'Auxiliary Task', whereas in another variant of the compensation strategies, the so-called 'Simplifying'-strategy, learners compensate directly in the first step (Selter et al., 2012). The 'Auxiliary Task' as well as the 'Simplifying'-strategy require the learners to be able to think about the numbers beforehand, by doing a so-called *analytical noticing* and in comparison to the other strategies, they are less algorithmically – or rather technical – and induce the learners to use *numerical relations*: After seeing *beforehand* the possibility of changing the task by rounding up or down a number near to the next tens, hundreds, etc., they have to calculate the modified term and compensate it afterwards (Threlfall, 2002). Thus, the compensation strategies are highly relational mental calculation strategies utilizing *numerical relations* through termic manipulations. For fostering the use and conceptual understanding of the compensation rule, the 'Auxiliary Task' seems to be slightly easier than the 'simplifying'-strategy since it is conducted in two steps instead of one big step (Rathgeb-Schnierer & Rechtsteiner, 2018). The stepwise modification and compensation of 'Auxiliary Task' depends on the three complexity dimensions *arithmetic* (addition, subtraction, multiplication, division), *form* (single objects or groups of objects) and *direction* (rounding-up or rounding-down) of modification, being explained in the cardinally situated taking-away-logic:

- **In subtraction tasks,** 'rounding-up' the first or second number means I have to compensate in the last step through *adding* what was taken away too much.
- **In subtraction tasks,** 'rounding-down' the first or second number means I have to compensate in the last step through *taking away* what is still 'missing'.
- **In addition tasks,** 'rounding-up' the first or second number means I have to compensate in the last step through *taking away* what was added too much.
- **In addition tasks,** 'rounding-down' the first or second number means I have to compensate in the last step through *adding* what still has to be added.

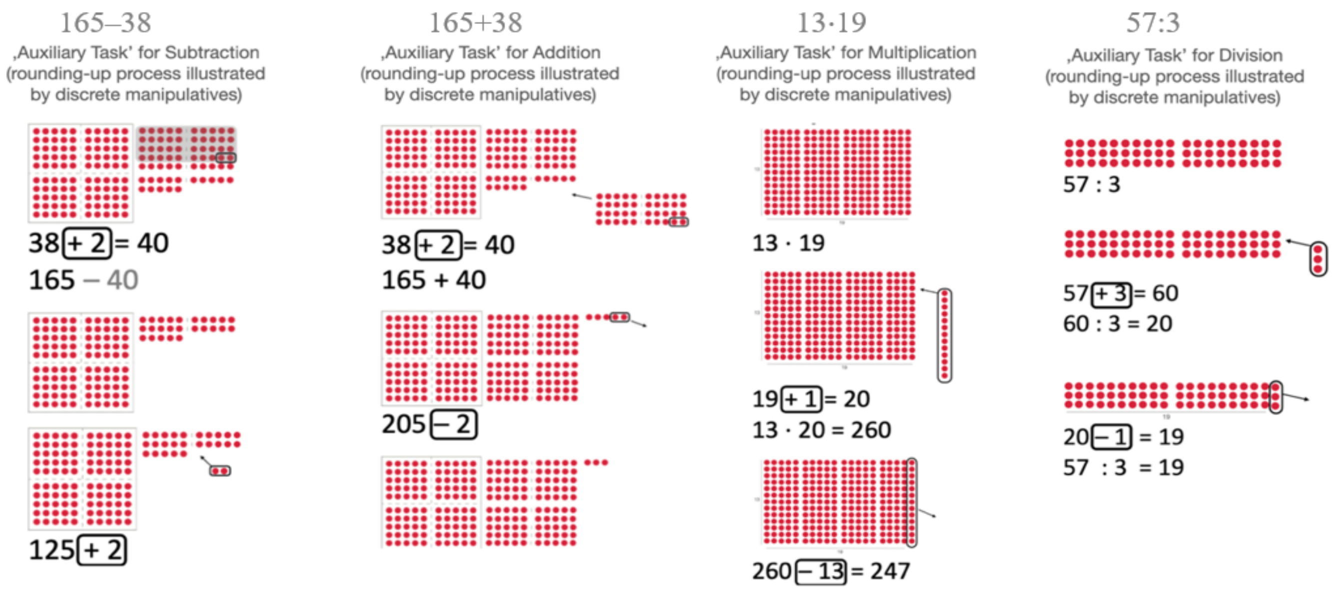


Figure 2. The 'Auxiliary Task' illustrated with discrete objects for all four arithmetic (rounding-up) (Source: Author's own elaboration)

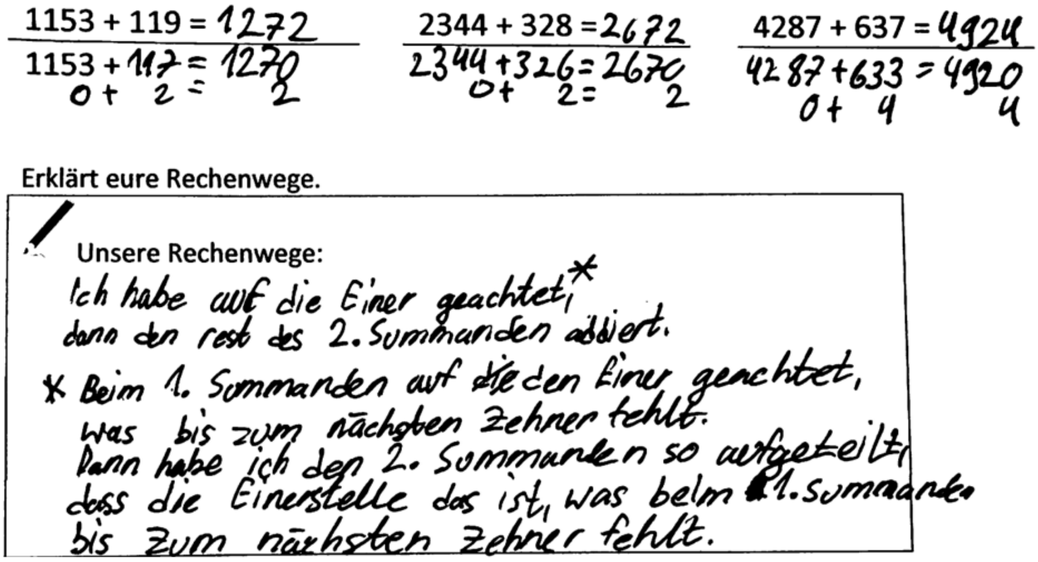
- **In multiplication tasks,** 'rounding-up' the first or second number means I have to compensate in the last step through *taking away* the group of objects I have added too much.
- **In multiplication tasks,** 'rounding-down' the first or second number means I have to compensate in the last step through *adding* the group of objects that still have to be added.
- **In division tasks,** 'rounding-up' the first or second number means I have to compensate in the last step through *taking away* the group of objects I have added too much.
- **In division tasks,** 'rounding-down' the first or second number means I have to compensate in the last step through *adding* the group of objects I that still have to be added.

All eight cases are similar in terms of the complexity dimensions, but different in their realizations: The compensation always depends on the conceptual meaning of the rounding-up or rounding-down process with regard to the arithmetic. A special case is the rounding-down process since it is similar to the Stepwise-strategy, but it utilizes a different way of thinking: Not the number partition (in tens and ones) is focused but the modification and compensation (Kuzu, 2022b).

The interplay of stepwise manipulation and compensation while using the 'Auxiliary Task' for all four arithmetic can be illustrated conceptually by using different forms of manipulatives, but especially discrete-cardinal manipulates (**Figure 2**) seem to stimulate more explicit conceptual thinking and explanation processes because of the visibility of the amount being added when rounding up or down, whereas ordinal representations (**Figure 2**) only implicitly illustrate the modification as being part of the first big jump (Kuzu, 2022a).

In **Figure 2**, the conceptual facet of the 'Auxiliary Task' is visualized with discrete objects (for rounding-up processes): What is added and taken away (the modification) is illustrated through dots in black frames and in case of subtraction by using the black frames in a transparently grey box since the modification is taken away *within* the taken away amount. The intention is to make visible the modification and – after using, explaining, and relating the numerical and iconic representations several times (Wessel, 2020) – to enable the learners in seeing the numbers, manipulatives, or iconic representations of the task as *non-determined, flexible elements,* as parameters they are allowed to temporarily change to make the task easier to calculate. Yet, for the primary school, it is a complex thinking process to interpret specific parameters of numerical tasks as 'gaps' that can be manipulated and compensated in all four arithmetic operations. Furthermore, the cases presented in **Figure 2** are not all possible cases (but the cases being based on seeing 'gaps'): Students could also double or half specific numbers or use neighbor-tasks–like 6×6 for calculating 6×7–to generate 'Auxiliary Tasks', where no 'gap' to the next tens has to be interpreted but a proximity to other tasks. In all of these 'Auxiliary Tasks', the conceptual meaning of the compensation process has to be reflected in an appropriate learning-environment and is an important pre-step to the interpretation of the equality-sigh as a *balance-sign* (Mayer, 2019; Schwarzkopf et al., 2018): For understanding the 'Auxiliary Task', students have to recognize that the compensation need comes from an occurring inequality after rounding up or down.

From a further conceptual perspective, a reflected and generalized understanding of the 'Auxiliary Task' may lead to a notion of numbers and terms as highly flexible objects one can modify *always*, if compensated adequately, thus a notion of numbers-as-indeterminate



Translation of the task: *"Explain your way of calculation."* and *"Our calculation ways:"*
Translation of the students' answer: *"I looked at the ones*, then the rest of the 2. summand added. *Looking at the ones of the 1. summand, what was missing to the next tens. Then I split up the 2. summand in a way that the position of the ones is that which is missing until the next tens at the 1. summand"*

Figure 3. Marcel's individual variant of an 'Auxiliary Task' (looking for the tens-complements) (Kuzu, 2022b)

being similar to variables-as-indeterminate might occur (Akgun & Ozdemir, 2006; Korntreff & Prediger, 2022): The first or second number would then be regarded as a parameter, which can be modified in plural, non-determined ways (rounding-up, rounding-down, adding or taking away a specific amount, doubling it, halving it, etc.). Normally, students are told to not change the numbers and terms they are given, but in the case of the 'Auxiliary Task', they have to break with this norm purposefully and might do this in a systematic, highly creative way as the solution of the student Marcel shows (**Figure 3**).

In **Figure 3**, Marcel uses an 'Auxiliary Task,' which aims at modifying the term by adding the tens-complements: He modifies the term by adding what is missing until the next tens in the ones of the original term. In his explanation, he generalizes the numbers as well as the modification-and-compensation-process in the 'Auxiliary Task' by referring to the *"1./2. Summand"*, *"the ones"*, *"what is missing …"* and by describing his idea of looking at the tens-complements (*"… in a way that the position of the ones is that which is missing until the next tens"*) (Kuzu, 2022b).

To summarize the theoretical and subject-related insights, the 'Auxiliary Task' might be highly relevant with regard to the 'cognitive gap' between the arithmetic and algebraic thinking because of the analytical, manipulative, and non-determined view on the numbers in the task, but further research is needed, especially through reconstructing learners' individual notions concerning possible pre-algebraic generalizations (Kuzu & Nührenbörger, 2021).

Research Question

Recent studies show a high *interrelation* between arithmetic and algebraic thinking and mostly do this in the context of word problems or figural patterns, but in terms of highly relational arithmetic strategies such as the 'Auxiliary Task', only first insights are described. It has yet to be described, in which specific forms learners generalize their conceptual understanding in the context of highly relational arithmetic strategies. Following this research gap, the general research aim being focused in this study is to reconstruct learners' language means when generalizing their conceptual as well as procedural understanding of the 'Auxiliary Task', with a strong emphasis on the generalization of the compensation rules after a phase of conceptually reflecting the strategy. Since recent research also showed an important role of language means for generalizations of the conceptual understanding (Akinwunmi, 2012; Steinweg et al., 2018), a reconstructivistic approach including interdisciplinary perspectives about thinking-related linguistic aspects is important.

Being derived from this necessity to conduct an interdisciplinary and explorative research about the learners' use and understanding as well as generalization of the 'Auxiliary Task', following research question will be addressed in the empirical analyzes of this article:

Q1: In which linguistic forms do 11-14 year old students generalize their conceptual thinking of the arithmetic strategy 'Auxiliary Task' and which language means do they use to realize these generalizations?

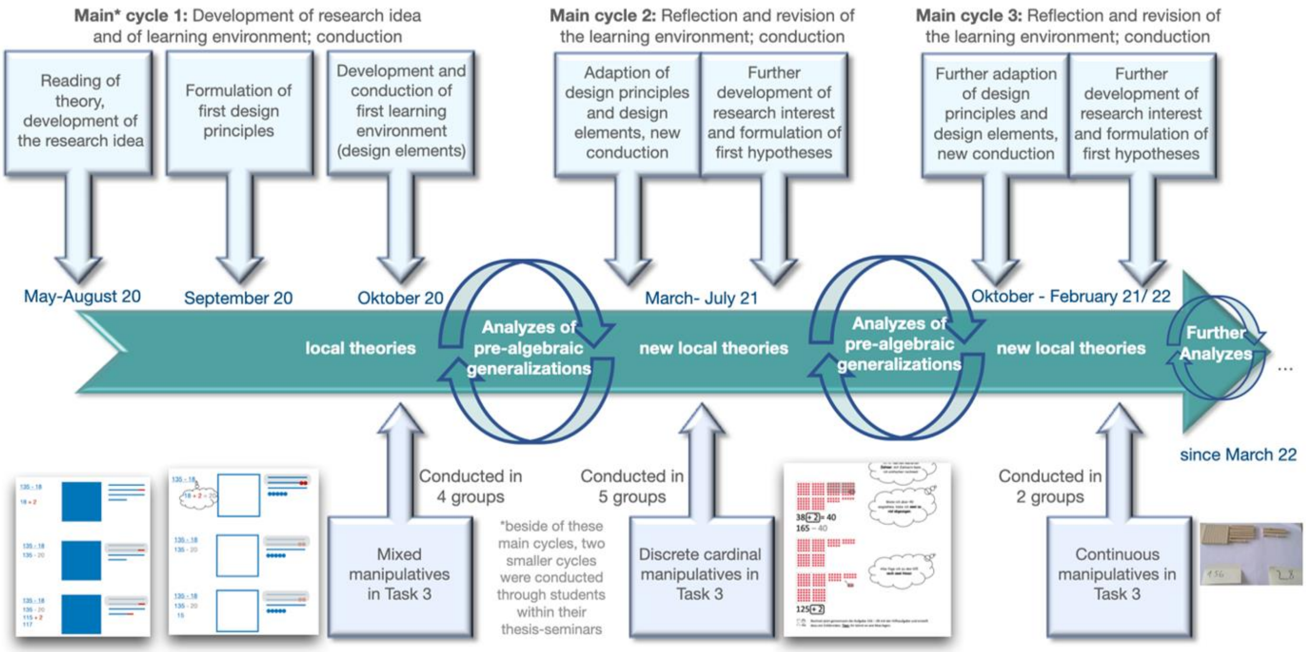


Figure 4. The design-cycles of the study (Source: Author's own elaboration)

METHOD OF THE STUDY

Framework: Design-Based Research

For the research aim outlined before, a design-based research approach (Prediger et al., 2015) with three iterations was developed because of the necessity for gaining explorative, *local insights* about learners' processes of interpreting the 'Auxiliary Task' and the compensation strategy (Prediger et al., 2015; Wittmann, 2021) and because of a specific design-problem concerning the conceptual understanding of the 'Auxiliary Task': Normally, a symbolic-procedural or in some cases an ordinal representation is chosen for explaining the compensation process and only very few task designs have a cardinal way of representing the compensation process, thus there is a need to compare different approaches and their (possible) effects on the interpretational process (Kuzu & Nührenbörrger, 2021). These three cycles were based on the three steps *analysis, preparation*, and *conduction* (**Figure 4**).

The *design elements* in the learning environment, meaning the tasks, graphical representations, the sequencing of the tasks etc., were designed according to three *design principles,* meaning general maximizes being based on empirical and theoretical insights (Kuzu, 2022a; Van den Akker, 1999):

1. **Fostering of a richly entwined conceptual understanding preceding procedural calculation:** This design principle aims at fostering the learners' understanding of the 'Auxiliary Task' through fostering a *"transition from informal thinking and conceptual understanding to the procedural rules"* (Glade & Prediger, 2017, p. 185). In this sense, the learners of the study had to solve a conceptual task about the meaning of the 'Auxiliary Task', through register relation with an ordinal as well as cardinal representation, before a transition to solving tasks with the calculation algorithm were given.
2. **Content-and-language-integration through register relation:** All tasks were (re-)designed after pre-analyzing possible meaning-related language means from a prescriptive as well as descriptive perspective (meaning students' own language means, analyzed after a piloting session). These language means were used in the learning environment for the purpose of scaffolding processes of verbalization and thinking (Hammond & Gibbons, 2005) and after each design-cycle, the language means used in the learning environment were adapted and optimized (Pöhler & Prediger, 2015; Wessel, 2020).
3. **Sequencing tasks with the aim of fostering generalization processes:** According to the theoretical and empirical insights illustrated before, a process of generalization is important for the emergence of pre-algebraic thinking (Radford, 2010, 2018) and this was the main aim of the third design principle. It was realized through a task structure with four phases, in which step-wise processes of generalization were fostered:
    a. Phase 1: Understanding the idea/concept with register relation and calculating with it.
    b. Phase 2: Formulation and explanation of the rule (verbally).
    c. Phase 3: Formulation of a short rule (through using graphical representations).

d. Phase 4: Confrontation/transfer to a new situation/task (e.g., a more termic representation of the 'Auxiliary Task') (Schwarzkopf et al., 2018).

Method of Analysis: 'Interaction Analysis' (Krummheuer & Naujok, 1999)

To reconstruct, analyze and systematize learners' individual notions about the 'Auxiliary Task', understood as viable, partially-viable or non-viable 'mental models' being attached to situations and actions (Bauersfeld, 1980; Fischbein, 1989; Kuzu, 2019; Prediger, 2019; Vergnaud, 2009), an interpretative approach following the framework of the so-called 'interaction analysis' (Krummheuer & Naujok, 1999) was utilized: In a careful process of sequentially analyzing 'interaction units' (here: Tasks and subtasks regarding the generalization of the 'Auxiliary Task') turn-by-turn, explaining hypothesis with regard to the research interest were formulated abductively and discussed carefully in groups of researchers. After that, the initially abductively formulated hypotheses were linked to possible theoretical aspects deductively and then tested in further transcript sequences inductively (Meyer, 2009; Schütte et al., 2019). After going through this interpretational process, frequently abducted and intersubjectively plausible explanation hypothesis were seen as categories, meaning non-singular, similar explaining hypothesis which are linked through resemblances: When *"abductions' results, cases or rules are similar to each other one can speak of (family) resemblances in an inferential way."* (Kunsteller, 2018, p. 373).

With the aim of gaining insights into learners' interpretational processes in dependence to the designed learning environment, two main steps were followed: In a first open step, teachers' questions and students' answers were analyzed and discussed in groups of researchers by using an interpretative, turn-by-turn method aiming at the reconstruction of individual notions regarding the 'Auxiliary Task' within interactional processes (Nührenbörger & Steinbring, 2009; Steinbring & Nührenbörger, 2010). Especially the linguistic processes of articulating the thinking about and the interpretation of the 'Auxiliary Task' as well as the resulting generalization in form of a rule-formulation were focused. Thus, *languaging*-processes (Swain, 2006) – meaning the use and learning of language means with relation to cognitive processes and in this sense as an agent for understanding conceptual aspects (Swain, 2006, p. 95) – were reconstructed: The language means used to describe the rule about the compensation process and the numerical relations were analyzed by coding and comparing the language means for each learner in the highly interactional transcript sequences. Not every utterance was (naively) seen as *languaging*, only those with an indication were interpreted as such. In a second step, the identified languaging processes from step 1 were examined further by specifically focusing epistemological processes of interpreting the 'Auxiliary Task' by analyzing detailed and sign-related processes of meaning-negotiation (Nührenbörger & Steinbring, 2009) related to the compensation rule as well as ways of interpreting the 'rounding gap' by using so-called *epistemological triangles* (Steinbring, 2006) (**Figure 5**).

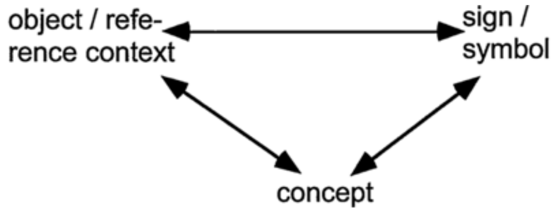


Figure 5. Epistemological triangle (Steinbring, 2006, p. 135)

In these epistemological triangles, mainly individual ways of languaging were reconstructed, but co-constructional processes were considered as an integral aspect. The 'signs/symbols' stand for mathematical objects, whose interpretation is necessary in an interactional situation and which can consist of manipulatives, symbols or utterances, 'objects/reference contexts' are aspects of knowledge explicitly or implicitly recurred to for explaining the signs being explained and 'concept(ual nuance)s' are either full concepts (like the 'part-of-whole'-concept) or smaller conceptual nuances (like ‚whole as the total of equally sized pieces of one object/ of "1"') (Steinbring, 2005). The focus tasks were task 4a and 4b since these two tasks covered a self-contained unit with regard to the research interest: The learners had to recapitulate and explain the 'Auxiliary Task' in their own words (4a) and had to formulate a general rule regarding the strategy (4b).

Sample of the Study

In total, 520 min. of small group interview data was recorded and transcripted according to utterance based transcription norms being developed at the Institute of Research and Development in Mathematics Education (Kuzu, 2019) and n=18 learners from grade 3 to 6 (age group 11-14) were analyzed in case studies since the broader research question of the study is to examine the arithmetic and algebraic understanding of the 'Auxiliary Task' in the transition from primary school to secondary school.

The small groups consisted of medium-achiever learners from the same classes, being based on the teacher-evaluation, and since interactions were important, peer-groups were preferred. The classes being chosen were state school classes for gaining explorative insights under standard or rather non-particular conditions with regard to student- or school-performance. Since the interviews took place under pandemic conditions, a higher distance between the learners, the permanent use of masks and interruptions due to the need of disinfection, air conditioning etc. have to be considered as possible interaction-affecting factors.

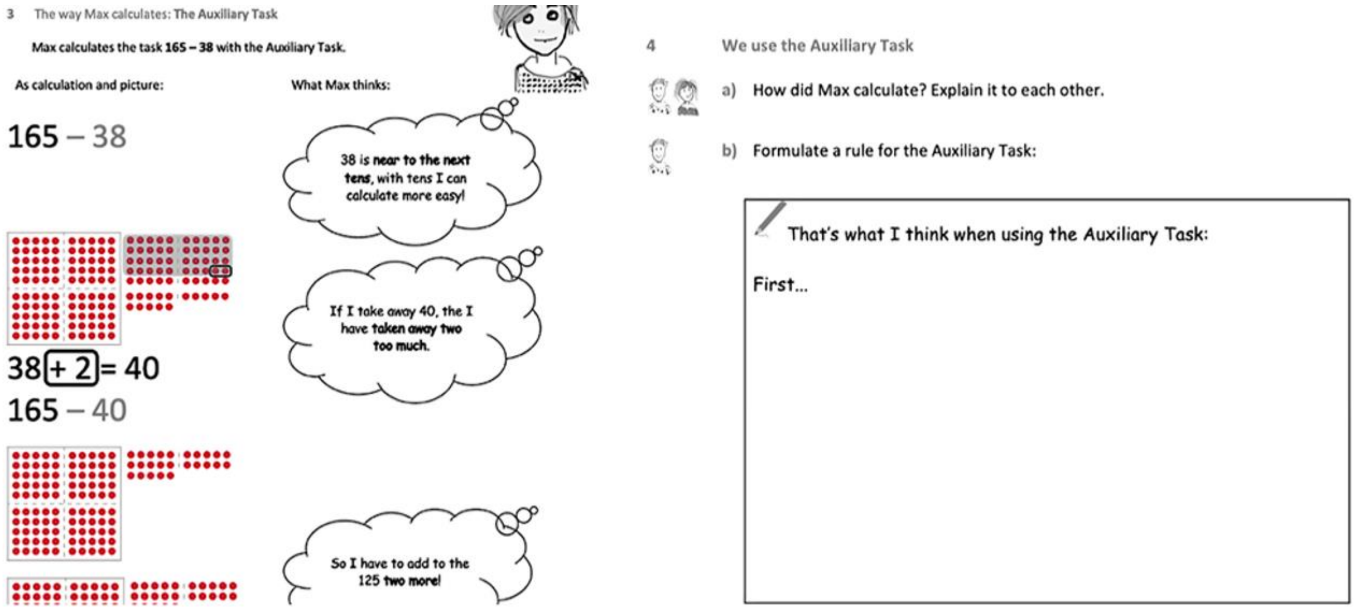


Figure 6. The cardinal representation and scaffolding in task 3 and the explanation of the rule in task 4 (Source: Author's own elaboration)

For the case studies discussed in this article, task 4b was chosen, in which the learners were asked to formulate and explain their rule about the 'Auxiliary Task'. Task 4 is an explanatory task following after the task about the conceptual understanding of the compensation process.

EMPIRICAL INSIGHTS: LEARNERS' GENERALIZATIONS OF THE SUBTRACTION-'AUXILIARY TASK'

Martin's and Behiye's Generalized Explanation of the 'Auxiliary Task'

The sequence from the case study is about the formulation of a rule about the 'Auxiliary Task': The task aiming at the conceptual understanding through a *cardinal representation of the compensation process* was solved before and now, the students had to recapitulate what Max did (task 4a) and to formulate a rule in their own words (task 4b) (**Figure 6**). In **Figure 6**, the learners have to interpret the 'Auxiliary Task' conceptually with a discrete-cardinal representation (Kuzu, 2022a).

The explanation is not only based on the iconic picture but is accompanied by enactive operations: On **Table 1**, cardinal material (ones-dots, stripes of tens-dots, and squares of hundreds-dots) is given and has to be laid down simultaneously to the explanation of Max according to the task design. The transcript sequence starts after the teacher asked what to write down on the work sheet as an answer to task 4b.

In the transcript sequence in **Table 1**, Martin and Behiye formulate the rule to the 'Auxiliary Task' for the first time. It starts with Martin's utterance in turn 73, where he gives a first impulse about when to use the 'Auxiliary Task'. First generalized aspects regarding the 'Auxiliary Task' are visible here: He does not only mention 38 (the number of the second summand in task 3), but also other numbers with similar characteristics (28 or 8 as numbers with "8" in the position of the ones). He stresses out that it is *always* plus 2 calculated, indicating a generalized view on the numbers Max (the fictive student of the task) chose. What is also visible is a thinking of a sequence of actions, which Martin verbalizes through logically connecting language means like *"That is … then … and then …"*. By sequencing his thoughts according to the steps to be made, he describes the compensation process on the termic level: For the 'Auxiliary Task', the manipulation of the first term and the last term has to be anticipated before solving the task. Behiye does not disagree with him, instead she agrees with a high emphasis (through articulating her approval three times, in turn 74, 76, and 78). After the question of the teacher aiming at a formulation of the rule from Behiye (probably because he wanted to check if she really has understood the rule), Behiye repeats a similar explanation in turn 80, not in an identical way but through adding further important aspects: First, she shows a highly variable-like word use when speaking of the *"starting number"* (a language mean not given in the task), and then she seems to think this starting number as a kind of indeterminate number when emphasizing that *"it does not matter* [which number one chooses]*"*. Furthermore, she gives new number examples with different characteristics if compared to Martins number example from turn 73: She thinks up numbers with a "6" and "9" in the ones (186 for the first summand and 29 for the second summand), not only with "8" as Martin did, thus detaching her rule-formulation from the numbers given in the task (165 and 38), and when describing the sequence of action on termic level, she does not only describe the steps through similar language means as

Table 1. Original and translated transcripts-1

Turn	Name	
Original transcript		
73	Martin	Also, Max hat immer 28 oder 38 oder 8 und ja.
74	Behiye	Ja *[lacht]*.
75	Martin	plus 2 gerechnet. Das sind 40. Dann ja 164 minus 40 oderso gerechnet.
76	Behiye	Mhm *[bestätigend]*.
77	Martin	Und dann noch plus 2.
78	Behiye	Ja *[nickt]*.
79	Teacher	Willst du es auch nochmal in eigenen Worten erklären oder würdest du es wirklich genauso erklären? *[schaut auf Behiye]*.
80	Behiye	Also, er hat halt eine Anfangszahl und dann zum, die lautet dann zum Beispiel 186. Ist ja egal welche. Äh. Dann minus zum Beispiel 29. Dann ehm damit es einfacher im Kopf zu rechnen ist, zieht man dann die 30 ab. Das sind dann 156. Äh und dann, weil man ja eben äh 30 abgezogen hat und von der 186 und nicht 29 wie es in der Aufgabe stand. Dann ja.
81	Martin	Noch plus 1.
82	Behiye	Ja plus 1.
Translated transcript		
73	Martin	Well Max, he has always 28 or 38 or 8 and that is always.
74	Behiye	Yes *[laughs]*.
75	Martin	plus 2 calculated. That is 40. Then it is 164 minus 40 or so calculated.
76	Behiye	Mhm [affirmative].
77	Martin	And then also plus 2.
78	Behiye	Yes *[nods]*.
79	Teacher	Would you like to explain again in your own words, or would you explain it in the exact same way? *[looking at Behiye]*.
80	Behiye	Well, he has a starting number and then, let us say for example 186. It doesn't matter what. Eh. Then minus 29 for example. Then ehm that is for calculating it easier in the head, therefore you subtract the 30. That is then 156. Eh and then, because one has eh subtracted 30 just now and from the 186 and not 29 as it was given in the task. Then yes.
81	Martin	Plus one more.
82	Behiye	Yes plus 1.

Martin (by repeating *"then … and then …"*), she also gives an explanation *why* one has compensate, *"because one has eh subtracted 30 just now and from the 186 and not 29"*, indicating that she links the compensation need to the priorly modified number. Interestingly, her language use (and also Martins) seems to have an underlying, imagined chronological structure: She describes the first modification on the termic level as a first conditional step, where different possibilities are thinkable (*"let us say for example …, then …"*), leading to an interim result and referring to it for the second step with the language mean *"just now"*, which is *"then"* leading to a compensative modification again (by adding "1").

Thus, with regard to the research question, different pre-algebraic aspects are reconstructable here in this short sequence: The students talk about the first and second numbers as changeable objects that have to be chosen according to the rule behind the strategy. A detailed epistemological analysis shows that on the level of the sign-interpretation, the complex process behind the 'Auxiliary Task' is explained by using references to the compensation need emerging after modifying the first term (see **Figure 7**).

Figure 7 shows an epistemological triangle mainly with regard to Martins and Behiyes co-constructed explanation of the rule, which seems to be linked to a reference context on two levels, firstly on a concrete level and secondly on a more abstract level, although a (fully) developed abstract understanding cannot be inferred yet. Regarding the first level, Behiye's utterance *"for calculating it easier"* seems to be a direct hint at the same sentence being visible in task 3 (see **Figure 6**), which may have also affected the sequencing of the steps (being a predominant design aspect in task 3). Regarding the abstract level, a first *structural view* on the 'Auxiliary Task' seems to be observable in the explanation of Behiye because of a (first) generalized understanding: Her examples or rather the signs she constructs, consisting of similar but slightly modified numbers in comparison to the task numbers, as well as her explanation of the termic process of modification and compensation, seem to indicate a so-called *generic example*, meaning an explanation through a carefully chosen representative example/object, where the calculations and reasoning are presented in a manner that it allows to see that the explanation would work for similar objects, thus containing first generalized aspects (Lew et al., 2020). Behiye indicates this representativeness in Turn 80 by using language means like *"let us say for example"* and *"it doesn't matter what"* within her explanation of her calculation procedure. Interestingly, by using these language means, she also implicates which number has to have a specific property: The first number can be every number, the 'Auxiliary Task' can be used in any case.

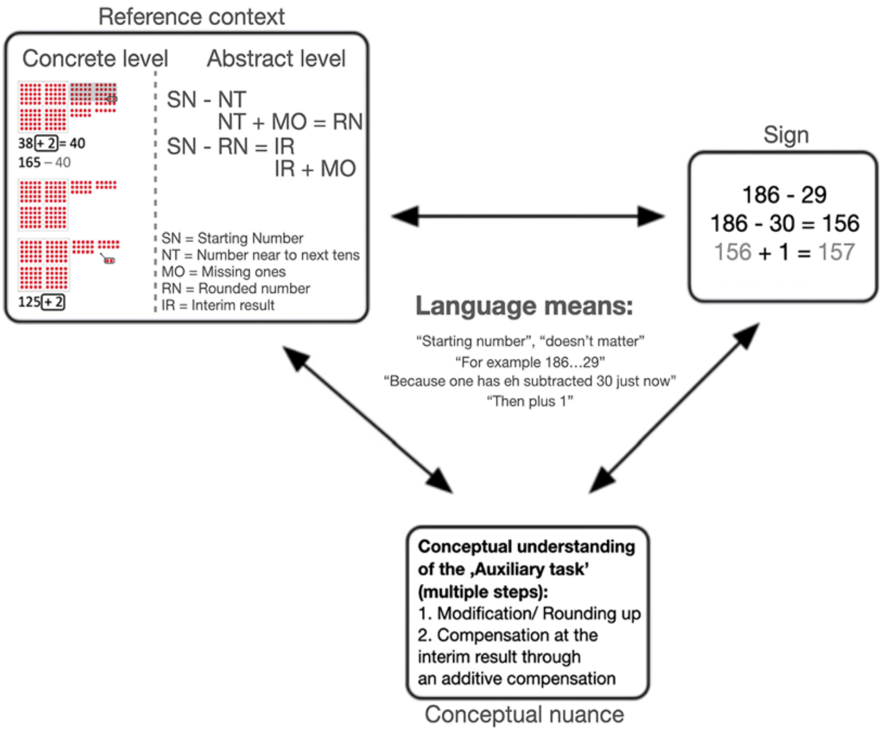


Figure 7. Epistemological triangle illustrating Behiye's interpretation in turn 73 and 80 (implicit facets in grey) (Source: Author's own elaboration)

With regard to the task construction, it is an indeterminate number. The second number has to be a number near to the next tens, which she chooses as a number being easily modifiable by being near to the next tens (*"for calculating it easier in the head"*). It is at least partially determined, what has to be given: It can be any number, which is near to the next tens, and then one has to add the 'unknown', being the number missing the next tens, and add it back again (in case of a subtraction task). Thus, she generically highlights what is important for the 'analytical noticing' when using the 'Auxiliary Task': One has to look especially at the second number whilst knowing that the first number is irrelevant or could be any number. It is noteworthy how she switches from speaking of "him" (the fictive student Max, whose strategy she explains) to what "one" has do since "one" is an indefinite pronoun standing for a generality of persons, not only Max. Thus, not only Max but everybody could use the strategy.

Stephan's and Mara's Generalized Explanation of the 'Auxiliary Task'

Two other learners, Stephan and Mara, discuss their own rule in a similar way as Martin and Behiye when solving task 4b (see **Figure 6**). Their discussion is about the question, what one has to pay attention to when using the 'Auxiliary Task' (see **Table 2**).

Here, in this sequence, a further generalization of the 'Auxiliary Task' is observable: In turn 282-285, the rounding rule is expanded in both directions, which is rounding up (*"if one has 115 now, then one goes to 120"*) or rounding down (*"if one has 114 one goes to 110"*), and the modification range is widened to the hundreds if compared to the first sequence (there, only a rounding process with regard to the tens and only the process of rounding *up* was described). In turn 291, Stephan is adding an important facet: He emphasizes the "2" being added in the example from the learning-environment (*"to put again the plus two"*), but he then seems to notice that no number with "8" in the ones or non-concrete numbers are discussed since he corrects himself in the same utterance, changing his statement into the more general utterance *"or the plus, well, to take it plus"*, which is indeterminate at the point which number exactly has to be taken plus. One can assume that he may have realized that now not a specific number, but a general number, or an indeterminate number makes more sense for explaining the rule. His utterances show an epistemological process, where he at first explains the rounding-rule and then adds the need for compensation (see **Figure 8**).

In **Figure 8**, Stephan's process of interpreting and generalizing the 'Auxiliary Task' is illustrated: Although only the numbers 114 and 115 are mentioned by Stephan in the related turns 282, 284, and 286, with turn 284 and 286 being continuations of the utterance he starts in 282 (Mara 'completes' his utterances in-between, but Stephan seems not to hear her or rather speaks on without being disturbed), one may assume that Stephan probably thought of a known rule from his mathematics lessons prior to the study: Rounding up for numbers bigger than five and rounding down for numbers smaller than four. He thus seems to identify a flexible process of rounding up or down as a starting point for

Table 2. Original and translated transcripts-2

Turn	Name	
Original transcript		
282	Stephan	Also, man muss drauf achten, dass man immer zum nächsten Zehner und wenn man halt.
283	Mara	Hunderter auch.
284	Stephan	Also, wenn man jetzt 115 hat, geht man äh.
285	Mara	Zu 120.
286	Stephan	und wenn man aber 114 hat geht man zu 110 [nickt zu Mara] das stimmt.
287	Mara	Oh, ja stimmt.
289	Stephan	Und darauf muss man achten.
290	Mara	Stimmt.
291	Stephan	Und man muss am Ende drauf achten wieder die plus zwei zu legen oder die plus- also das plus zu nehmen. Ok das schreiben wir jetzt auf.
Translated transcript		
282	Stephan	Well, one has to pay attention that one goes always to the next tens and if one has.
283	Mara	Hundreds also.
284	Stephan	Well, if one has 115 now, then one goes eh.
285	Mara	To 120.
286	Stephan	and if one but if one has 114 one goes to 110 *[nods to Mara]* that is right.
287	Mara	Ah yes, it's right.
289	Stephan	And that is what one has to pay attention to.
290	Mara	True.
291	Stephan	And at the end one has to pay attention to put again the plus two or the plus- well, to take it plus. Ok, let us write that down.

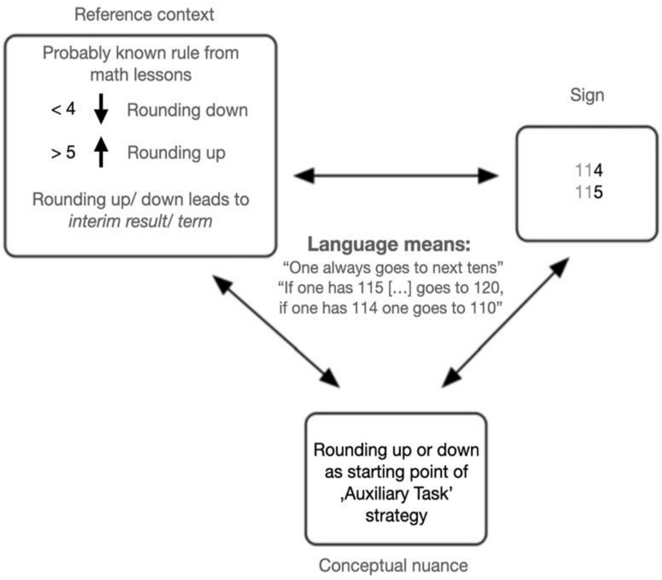


Figure 8. Epistemological triangle for turn 282, 284, and 286 (Source: Author's own elaboration)

the 'Auxiliary Task', although this was not part of the designed learning-environment: There, only a process of rounding-up was described and the language mean "rounding-up" was not used, but interestingly, he uses the term 'next tens' like an indeterminate element or variable standing for a plurality of possibilities, which he then concretizes through the self-chosen numbers "114" and "115". He seems to interpret these signs ("114" and "115") as exemplary numbers near to the next tens, as numbers going beyond their numerical value and standing for a property (numbers being roundable up or down to the next tens), and explains, that *"if one has 115 […] one goes to 120"* and *"if one has 114, one goes to 110"*. Linguistically, he utilizes an if-clause and combines it with an indefinite pronoun ("one") – in a similar way as Behiye used it – before giving concrete, but new numbers as examples (not being used in the task), thus he indicates a decontextualized, general rule, which goes beyond the numbers given in the task. With this generalization, he explains how the interim result, resulting from the rounding-up or rounding-down process, is obtained when using the 'Auxiliary Task'. At this point, Stephan does not refer to the necessary modification after obtaining the interim result, but he adds that missing aspect later in turn 291 (see **Figure 9**).

Figure 9 shows an epistemological triangle for turn 291, where the prior reference context of the rounding-rules seems to become a new object: An interwovenness, which is typical for epistemological processes (Steinbring, 2006, p. 159). Stephan now links the rounding-process to the compensation process, but only mentions an additive compensation as was given in the learning-environment. He now recurs to the same example from the learning environment (compensation through "+2") so that at this point, no fully viable interpretation can be stated, but at least the partially-viable interpretation (modifications in a term are compensated) seems to be reconstructable. It is only partially viable because the necessity for compensation seems to be understood, but only for one case: The compensation through *adding back, what was taken*, which does only work when rounding up, not when rounding down (then one would have to take away "2" again). Furthermore, he seems to refer to the contextual manipulatives when he mentions "to put again two" (see turn 291), indicating a contextual reactivation of the cardinal material being used priorly.

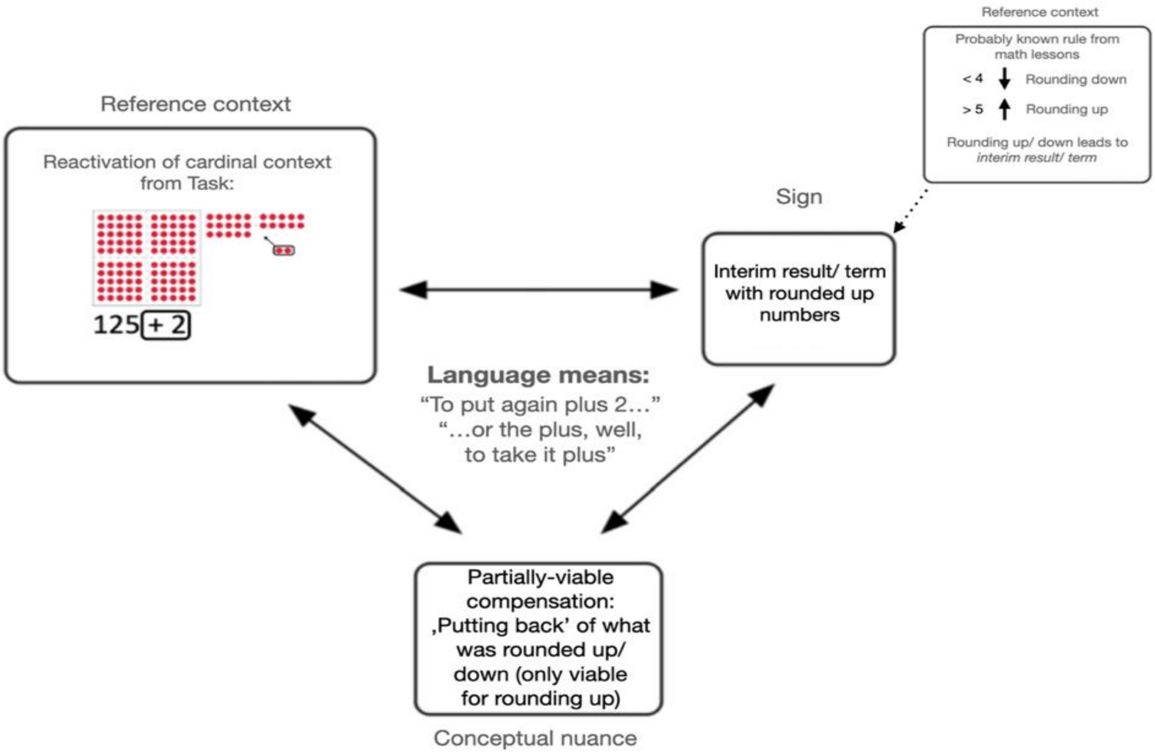


Figure 9. Epistemological triangle for turn 291 (Source: Author's own elaboration)

Similar to Martin's and Behiye's interaction and as mentioned earlier, Stephan's explanation is not only his own explanation, but it also results from a co-constructional process with Mara: In the sequence from turn 282-291, both learners seem to be agreeing to each other's explanation and make fast additions to each other's sentences (see for example Mara's comments in turn 283 and 285).

RESULTS, DISCUSSION, & LIMITATIONS

For answering the research question Q1 regarding the forms of generalization in the context of the 'Auxiliary Task', the analyzes show that a first form of pre-algebraic thinking occurs with regard to the three key aspects of algebraic thinking: the *indeterminacy*, *denotation*, and *analyticity*. An emergence of the first aspect, the *indeterminacy*, could be inferred because learners linked the modification in the first step of the 'Auxiliary Task', which was seen as a *possibility*, with the then occurring *necessity* to compensate it, and interpreted this process in a detached way by using language means like 'variables'. This becomes visible when Stephan and Mara generalize the 'Auxiliary Task' by verbalizing the process in a non-concretized or rather detached way. In turn 282 for example (*"Well, one has to pay attention that one goes always to the next tens …"*), Stephan starts with a generalized explanation of what one has to pay attention to (*"that one goes always to the next tens"*), where the language mean 'next tens' is used like a variable standing for the process of rounding-up, and concretizes his explanation afterwards by adding the if-clause in turn 284 and 286 (*"If one has, 115 now […] if one has 114 […]"*) so that a process of thinking about generalized aspects and concretizing them in form of self-chosen examples follows upon the generalization – a pattern occurring in further case studies also (Kuzu, 2022a; Kuzu & Nührenbörger, 2021). The *denotational aspect* became visible when learners spoke of the parameters of the 'Auxiliary Task' with a variable-like word use as it was described in Steinweg et al. (2018): They used words like *"the rounded-up number"*, *"take away what was rounded-up"* or *"what's missing to the next tens,"* referring to 'Auxiliary Task'-specific non-concretized and detached aspects. A further comparison with the forms of pre-algebraic verbalization (Steinweg et al., 2018) shows that the verbalization of relations has similarities, but also important differences in comparison to the verbalizations from the context of figural numbers, especially in terms of conditional sentences: Instead of articulating *"if …, then …"*-relations, a more flexible way of thinking about conditions related to the termic structure became visible when learners stated that one *may* choose to manipulate the second number and that *"it does not matter"* which number is chosen since the modification works always since one then *has to* compensate that modification. Thus, it could be reconstructed that language means were also important for pre-algebraic thinking processes in the context of the 'Auxiliary Task', but important differences were reconstructable on the termic level through different forms of conditional sentences. Regarding the third aspect, the *analyticity*, the learners started to show a structural understanding of the way the compensation process worked when indicating that one can always think about how to modify the termic elements (either the first or second number) with regard to 'gaps', which can either be rounded-up or rounded-down. A thinking of possible modifications *beforehand* thus became visible in the data, matching the idea of analyticity in form of the *analytical noticing* Threllfall (2002) already mentions from a theoretical perspective. What also became visible in the analyzes was the strong connection between generalization and the cardinal or ordinal context: Stephan and Mara generalized their thinking by referring to what has to be added or taken away (see turn 291: "to put again the plus two"). Prior analyzes in Kuzu (2022a, 2022b, see also Kuzu & Nührenbörger 2021) confirm this hypothesis insofar as that learners tend to generalize the compensation process by referring to *"what was taken away"* or *"the jump being made earlier."* Especially the cardinal representation leads to a higher amount of context-related generalizations, although important differences in the effect of using discrete-cardinal versus continuous-cardinal objects in the learning-environment could be reconstructed: The continuous-cardinal representation (with tens-lines and hundreds-squares instead of tens-dots and hundreds-dots, see **Figure 4**) led to non- or only partially-viable notions because of the ones-dots being put *onto* the tens-lines, hiding the part under it and leading to the non-viable notion of double-subtraction (learners interpreted the representation as *"one has to take away two and then 10"* instead of thinking of two as being an integral part of 10 already, see Kuzu, 2022a).

To summarize the empirical findings, an arithmetic format like the 'Auxiliary Task' may have the potential to stimulate first pre-algebraic aspects going beyond (solely) arithmetic aspects, *if* the learners are motivated to think about the strategy more generally after discussing the conceptual meaning of the strategy and if they are asked to formulate a (general) rule or explanation afterwards. The verbalization process bears a high importance with regard to pre-algebraic generalizations and this study contributes to these findings by showing that relations and structures of arithmetic strategies, especially of highly relational strategies such as the 'Auxiliary Task', may lead to pre-algebraic generalizations: The students showed a highly indeterminate way of thinking of specific parameters like *"the starting number"* (turn 80), of the numerical examples and range (by making up own tasks with the same numbers at first and giving examples of similar numbers afterwards, see turn 80 and 282) and of sequencing their procedure of modifying the first term and compensating the modification by adding the value to the interim term with chronological language means (see turn 73, 80, 282, and 291). In the case studies presented before, the learners thus verbalized their way of thinking the 'Auxiliary Task' by successively explaining and generalizing more aspects (like the indetermination of the compensation value, see turn 291).

Limitations arise with regard to the size of the sample group: The explorative insights are gained through an in-depths analysis of the interpretation processes of n=18 students, thus they have to be validated and differentiated in further case studies. The explorative insights seem to indicate that the 'Auxiliary Task' is highly complex in terms of a relational understanding of the termic modifications and the compensation process, and a further question is, if there are specific age-dependent patterns between the learners: Since learners with an age-difference of three years (11-14 years) were focused, more detailed analyzes might show differences between the age groups. This emerging assumption has to be examined in further analyzes.

Funding: This study is funded by RUHR-Futur/the Mercator Foundation and is part of the broader MaG-study ("mathematics from a single mould").

Acknowledgements: The author would like to thank to the colleagues for the productive and helpful discussions, especially Prof. Dr. Marcus Nührenbörger for his insightful comments.

Ethical statement: The author stated that the University of Münster did not have specific protocols for this type of studies when this research was carried out. Informed consents were obtained from the participants. The data was treated as confidential information used exclusively for research purposes. It is impossible to identify the participants from the data.

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

REFERENCES

Akgun, L., & Ozdemir, M. E. (2006). Students' understanding of the variable as general number and unknown: A case study. *The Teaching of Mathematics, 9*(1), 45-52.

Akinwunmi, K. (2012). *Zur Entwicklung von Variablenkonzepten beim Verallgemeinern mathematischer Muster* [*Developing concepts of variables when generalizing mathematical patterns*]. Springer. https://doi.org/10.1007/978-3-8348-2545-2

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11*, 23-41. https://doi.org/10.1007/BF00369158

Blöte, A. W., Van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology, 93*, 627-638. https://doi.org/10.1037/0022-0663.93.3.627

Britt, M. S., & Irwin, K. C. (2008). Algebraic thinking with and without algebraic representation: A three-year longitudinal study. *ZDM, 40*, 39-53. https://doi.org/10.1007/s11858-007-0064-x

Britt, M. S., & Irwin, K. C. (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In J. Cai, & E. Knuth (Eds.), *Early algebraization* (pp. 137-159). Springer. https://doi.org/10.1007/978-3-642-17735-4_10

Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the Learning of Mathematics, 9*(2), 9-14.

Glade, M., & Prediger, S. (2017). Students' individual schematization pathways-empirical reconstructions for the case of part-of-part determination for fractions. *Educational Studies in Mathematics, 94*(2), 185-203. ttps://doi.org/10.1007/s10649-016-9716-5

Hammond, J., & Gibbons, P. (2005). Putting scaffolding to work: The contribution of scaffolding in articulating ESL education. *Prospect, 20*(1), 6-30.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59-78. https://doi.org/10.1007/BF01284528

Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. In J. Cai, & E. Knuth (Eds.), *Early algebraization* (pp. 579-593). Springer. https://doi.org/10.1007/978-3-642-17735-4_29

Kieran, C. (2018). *Teaching and learning algebraic thinking with 5- to 12-year-olds*. Springer. https://doi.org/10.1007/978-3-319-68351-5

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teaching*. Springer. https://doi.org/10.1007/978-3-319-32258-2

Korntreff, S., & Prediger, S. (2022). Students' ideas about variables as generalizers and unknowns: Design research calling for more explicit comparisons of purposes. In J. Hodgen, & G. Bolondi (Eds.), *Proceedings of 12th Congress of the European Society for Research in Mathematics Education* (pp. 1-8). ERME

Kunsteller, J. (2018). Developing rules due to the use of family resemblances in classroom communication. In P. Ernest (Ed.), *The philosophy of mathematics education today* (pp. 365-377). Springer. https://doi.org/10.1007/978-3-319-77760-3_24

Kuzu, T. (2019). *Mehrsprachige Vorstellungsentwicklungsprozesse–Lernprozessstudie zum Anteilskonzept bei deutsch-türkischen Lernenden* [*Multilingual concept development processes-learning process study on the concept of participation in German-Turkish learners*]. Springer. https://doi.org/10.1007/978-3-658-25761-3

Kuzu, T. (2022a). Understanding the 'Auxiliary Task' conceptually – Discrete versus continuous cardinal objects. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th PME-Conference* (Vol. 3, pp. 99-106). Alicante University Press (PME).

Kuzu, T. (2022b). Kreative Denkwege oder umständliches Denken? Einblicke in alternative Vorgehensweisen zur, Hilfsaufgabe'. In Gesellschaft für Didaktik der Mathematik (Eds.), *Beiträge zum Mathematikunterricht 2022* (pp. 1-4). WTM. Paper accepted.

Kuzu, T., & Nührenbörger, M. (2021). The conceptual understanding of mental calculation strategies: Meaning-making in the case of the 'Auxiliary task'. In J. Novotná & H. Moraová (Eds.), *Proceedings of the International Symposium on Elementary Maths Teaching* (pp. 270-280). Charles University, Faculty of Education.

Lew, K., Weber, K., & Mejía Ramos, J. P. (2020). Do generic proofs improve proof comprehension? *Journal of Educational Research in Mathematics, 30*, 229-248. https://doi.org/10.29275/jerm.2020.08.sp.1.229

Mayer, C. (2019). *Zum algebraischen Gleichheitsverständnis von Grundschulkindern. Konstruktive und rekonstruktive Erforschung von Lernchancen* [*On the algebraic understanding of equality by elementary school children. Constructive and reconstructive exploration of learning opportunities*]. Springer. https://doi.org/10.1007/978-3-658-23662-5

Meyer, M. (2009). Abduction, induction–confusion. Notes on the logic of interpretative social research. *Zeitschrift für Erziehungswissenschaft* [*Journal of Educational Science*], *12*, 302-320. https://doi.org/10.1007/s11618-009-0067-1

Nührenbörger, M., & Steinbring, H. (2009). Forms of mathematical interaction in different social settings: Examples from students', teachers' and teacher-students' communication about mathematics. *Journal of Mathematics Teacher Education, 12*, 111-132. https://doi.org/10.1007/s10857-009-9100-9

Pöhler, B., & Prediger, S. (2015). Intertwining lexical and conceptual learning trajectories–A design research study on dual macro-scaffolding towards percentages. *EURASIA Journal of Mathematics, 11*(6), 1697-1722. https://doi.org/10.12973/eurasia.2015.1497a

Prediger, S. (2019). Theorizing in design research: Methodological reflections on developing and connecting theory elements for language-responsive mathematics classrooms. *Avances de Investigación en Educación Matemática* [*Research Advances in Mathematics Education*], *15*, 5-27. https://doi.org/10.35763/aiem.v0i15.265

Prediger, S., Clarkson, P., & Bose, A. (2016). Purposefully relating multilingual registers: Building theory and teaching strategies for bilingual learners based on an integration of three traditions. In R. Barwell, P. Clarkson, A. Halai, M. Kazima, J. N. Moschkovich, N. Planas, M. Phakeng, P. Valero, & M. Villavicencio-Ubillús (Eds.), *Mathematics education and language diversity* (pp. 193-215). Springer. https://doi.org/10.1007/978-3-319-14511-2_11

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes–an overview on achievements and challenges. *ZDM Mathematics Education, 47*(6), 877-891. https://doi.org/10.1007/s11858-015-0722-3

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA, 4*(2), 37-62.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal, 26*, 257-277. https://doi.org/10.1007/s13394-013-0087-2

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 3-27). Springer. https://doi.org/10.1007/978-3-319-68351-5_1

Rathgeb-Schnierer, E., & Rechtsteiner, C. (2018). *Rechnen lernen und Flexibilität entwickeln: Grundlagen–Förderung–Beispiele* [*Learning arithmetic and developing flexibility: Basics-support-examples*]. Springer. https://doi.org/10.1007/978-3-662-57477-5

Schütte, M., Friesen, R. A., & Jung, J. (2019). Interactional analysis: A method for analyzing mathematical learning processes in interactions. In G. Kaiser, & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 101-129). Springer. https://doi.org/10.1007/978-3-030-15636-7_5

Schwarzkopf, R., Nührenbörger, M., & Mayer, C. (2018). Algebraic understanding of equalities in primary classes. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 195-212). Springer. https://doi.org/10.1007/978-3-319-68351-5_8

Selter, C., Prediger, S., Nührenbörger, M., & Hußmann, S. (2012). Taking away and determining the difference—a longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79*(3), 389-408. https://doi.org/10.1007/s10649-011-9305-6

Serrazina, L., & Rodrigues, M. (2021). Number sense and flexibility of calculation: A common focus on number relations. In A. G. Spinillo, S. L. Lautert, & R. E. S. R. Borba (Eds.), *Mathematical reasoning of children and adults* (pp. 19-40). Springer. https://doi.org/10.1007/978-3-030-69657-3_2

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction. An epistemological perspective.* Springer.

Steinbring, H. (2006). What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61*, 133-162. https://doi.org/10.1007/s10649-006-5892-z

Steinbring, H., & Nührenbörger, M. (2010). Mathematisches Wissen als Gegenstand von Lehr-/Lerninteraktionen. Eigenständige Schülerinteraktionen in Differenz zu Lehrerinterventionen [Mathematical knowledge as a subject of teaching/learning interactions. Independent school interactions in difference to teacher interventions]. In U. Dausendschön-Gay, C. Domke, & S. Ohlhus (Eds.), *Wissen in (Inter-)Aktion. Verfahren der Wissensgenerierung in unterschiedlichen Praxisfeldern* [*Knowledge in (inter)action. Methods of knowledge generation in different fields of practice*] (pp. 161-188). De Gruyter.

Steinweg, A. S. (2013). *Algebra in der Grundschule. Muster und Strukturen–Gleichungen–funktionale Beziehungen* [*Algebra in elementary school. Patterns and structures–equations–functional relationships*]. Springer. https://doi.org/10.1007/978-3-8274-2738-0

Steinweg, A. S., Akinwunmi, K., & Lenz, D. (2018). Making implicit algebraic thinking explicit: Exploiting national characteristics of German approaches. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds* (pp. 283-307). Springer. https://doi.org/10.1007/978-3-319-68351-5_12

Swain, M. (2006). Languaging, agency and collaboration in advanced second language proficiency. In H. Byrnes (Ed.), *Advanced language learning* (pp. 95-108). Continuum.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50*, 29-47. https://doi.org/10.1023/A:1020572803437

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83-94. https://doi.org/10.1159/000202727

Warren, E., & Cooper, T. (2008). Generalizing the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*, 171-185. https://doi.org/10.1007/s10649-007-9092-2

Warren, E., Cooper, T., & Lamb, J. (2006). Investigating functional thinking in the elementary classroom: Foundations of early algebra reasoning. *The Journal of Mathematical Behavior, 25*(3), 208-223. https://doi.org/10.1016/j.jmathb.2006.09.006

Wessel, L. (2020). Vocabulary in learning processes towards conceptual understanding of equivalent fractions—specifying students' language demands on the basis of lexical trace analyses. *Mathematics Education Research Journal, 32*, 653-681. https://doi.org/10.1007/s13394-019-00284-z

Wittmann, E. C. (2021). *Connecting mathematics and mathematics education. Collected papers on mathematics education as a design science*. Springer. https://doi.org/10.1007/978-3-030-61570-3