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Pre-service Secondary Mathematics Teachers' Behaviors in the Proving Process

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Pre-service secondary mathematics teachers' (PSMTs) understanding and ability of constructing a proof is not only important for their own learning process, but also important for these PSMTs to help their future students learn how to do proofs. Therefore, this study is focused on and explains PSMTs' behaviors that they revealed throughout the proving process of a proposition. In this qualitative case study, the participants were fifteen volunteer PSMTs from a public university in Turkey. The participants were given a proposition which they were asked to think aloud and prove it on the blackboard. The findings of this study show that PSMTs' behaviors and thoughts regarding the given proposition were limited. In detail, PSMTs had difficulties in application of mathematical language and notations, understanding the meaning of the given proposition, knowing where to get started on a proof, using examples efficiently, using appropriate and efficient methods to construct the proof, and defining logical structures of the proposition to construct the proof.

Keywords: proof, proving, pre-service secondary mathematics teachers, teacher education

INTRODUCTION

When considered in a scholarly sense, mathematics can be defined as the science of proving in its essence. Heinze and Reiss (2003) suggest that theorems and their proofs are one of the fundamental components of mathematics while Tall (2002) states that proof is at the center of the modern mathematical thought. Proving is a process based on a group of mental habits such as determining structures and variables, defining hypotheses and organizing logical arguments (Ball et al., 2002) and has many important roles and functions in terms of mathematics and mathematics education. According to Hanna (1990), what makes the proof valuable for mathematicians is that the proof not only shows the accuracy of results, but also reveals necessary mathematical bonds. Hemmi (2010) considered various research (Bell, 1976; De Villiers, 2002; Hanna, 2000; Hemmi & Löfwall, 2009) to compile functions of the proof as follows:

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- Verification/conviction (concerned with the truth of a statement)
- Explanation (providing insight into why it is true)
- Systematization (the organization of various results into a deductive system of axioms, major concepts and theorems)
- Communication (the negotiation of meaning and "transmission" of mathematical knowledge)
- Aesthetic
- Intellectual challenge
- Transfer (proofs can offer techniques to attack other problems or offer understanding for something different from the original context; see Hanna & Barbeau, 2008; Hemmi, 2006; Hemmi & Löfwall, 2009). (Hemmi, 2010, p. 273)

In addition, based on Resnik's (1992) statement, it is possible to add two new functions to the purposes of use of proof. The first function is that mathematicians aim to show not only the new results by using proof, but also alternative notations of previous results (sometimes with a more simple or economic notations or by the help of information obtained from a different field of mathematics). As a second function, proof enables derivation of axiomatic notations for previous result that were obtained in a non-systematic way or re-theorization of an axiomatic system that exists until that moment (Resnik, 1992).

State of the literature

- Proof is extremely important in understanding mathematical structures, their characteristics and relations.
- Proving is a process based on a group of mental habits such as determining structures and variables, defining hypotheses and organizing logical arguments and has many important roles and functions in mathematics and mathematics education.
- Although proof and proving are considered important at all stages of mathematics education, existing studies have illuminated that students have had many difficulties in the proving process and there is wide range of reasons for such difficulties.

Contribution of this paper to the literature

- This qualitative study explores behaviors that pre-service secondary mathematics teachers revealed throughout the proving process of a proposition.
- This study initially examined what kind of behaviors pre-service teachers presented when constructing proof of a proposition, and then identified the difficulties experienced by the pre-service teachers as part of these behaviors.
- This study contributes significant insights and recommendations to the existing literature by identifying and analyzing the behaviors that pre-service secondary mathematics teachers revealed when they are thought aloud on the blackboard.

As it is seen, proof has a very wide area of influence in terms of function of it. Both the magnitude of meaning attributed to the proof and very large functional area of the proof make the proof very important not only for professional mathematicians, but also for students, teachers, educators and program developers who are interested in mathematics. In fact, we can observe that there is an increasing interest in proof and proving in the last 20 years (Chin & Lin, 2009), and this interest is not only at the level of academic researches. It is also about how proof and proving can be taught better in primary and secondary education within the scope of curriculums. It is also important to note that researchers suggest that the proof should be at the center of mathematics education at all levels (primary, secondary and higher education) (Ball et al., 2002; Common Core State Standards for Mathematics (CCSSM), National Council of Teachers of Mathematics (NCTM), 2000; Schoenfeld, 1994). According the NCTM Standards (2000), the place and importance of proof has increased considerably compared to previous standards (Knuth, 2002). There are two groups in NCTM Standards under the names of content (6 standards) and process (4 standards). One of the process standards is related to the '*reasoning and proof*'. As it is seen, proof is one of the 4 main process standards and is considered necessary for K-12 education. In NCTM Standards (2000), detailed descriptions regarding the reasoning and proof are summarized as follows:

Instructional programs from prekindergarten through grade 12 should enable all students to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; [and]
- select and use various types of reasoning and methods of proof. (NCTM, 2000, p. 56)

In recent years, 48 states in the United States were adopted CCSSM which presents higher standards in mathematics. Like the NCTM Standards, these standards emphasized the importance of proof and proving for mathematical practice. For example, in high school mathematics standards, it is indicated that students formalize and improve their geometry knowledge that they learn in middle school by developing careful proofs. Similar to NCTM and CCSSM standards, reasoning and proof is one of the basic skills in both previous (Ministry of National Education (MoNE), 2011) and existing (MoNE, 2013) secondary mathematics curriculums in the MoNE Teaching Programs in our country, Turkey.

Undoubtedly, teachers play the most important role in putting into practice the matters set out in the CCSSM and NCTM Standards and our Teaching Programs. The knowledge, opinion, attitude and skills of teachers related to the proof and proving will directly influence their ways of teaching proof and therefore the quality of teaching (Uğurel & Moralı, 2010). If teachers' understanding of proof is limited, it is likely that misconceptions of many students about proof will persist (Stylianides & Stylianides, 2009). Therefore, it is necessary to improve both pre-service and in service mathematics teachers' knowledge and skills regarding the understanding, perceiving and making a proof. Pre-service secondary mathematics teachers (PSMTs) who have a limited experience related to proof and proving in high school years encounter the actual proofs during their undergraduate education in mathematics. Therefore, it can be inferred that undergraduate education is essential in the development of PSMTs' knowledge and skills related to the proof. Thus, in recent years, researches are the priority and important in determining the existing situation regarding mathematics teachers' knowledge of proof and how to improve their skills in proving. In the existing literature, there are many studies that shed light on undergraduate students' approaches in understanding and making proof (Jones, 1997; Martin & Harel, 1989; Moore, 1994; Recio & Godino, 2001; Stylianides & Stylianides, 2007; Weber, 2001). However, in our country, there is still limited number of research in this field (especially on in-depth review of the proving process). In this study, our main goals are to identify PSMTs' behaviors while they were given a proposition to prove and to determine the challenges that emerge as part of these behaviors. In other words, it's aimed to depict the relationships between exhibited behaviors and experienced challenges. PSMTs' behaviors that they revealed in the process of proving are discussed based on detailed observation, while challenges they faced are addressed based on the existing literature. Accordingly, a summary is provided below regarding the studies that focus on difficulties/ challenges encountered/ experienced in proving by individuals who studied mathematics or mathematics education at a university.

Difficulties experienced by students in the proving process

Although proof and proving are considered important at all stages of education, existing studies have indicated that students have had some difficulties in the proving process (Haverhals, 2011, İpek, 2010; Moore, 1994; Sarı Uzun & Bülbül, 2013). As we aimed in this study, it is important to identify PSMTs' difficulties in proving as it establishes a ground for conducting studies to eliminate those difficulties.

Finlow-Bates, Lerman and Morgan (1993) suggested that many first-year undergraduate students had difficulties following chains of reasoning. In his dissertation, Moore (1990) also worked with 16 undergraduate students who attended a course which was designed to teach the proof, and studied those students' understanding of proof. The author listed the difficulties encountered by students in the proving process under seven headings: 1- cannot state the definitions; 2- lack intuitive understanding of the concepts; 3- cannot use concept images to write a proof; 4- fall to generate and use examples; 5- do not know how to structure a proof from a definition; 6- cannot understand and use language and notation; and 7- do not know how to begin a proof. In his other study, Moore (1994) examined the reasons for why university students (who studied mathematics or mathematics education) had difficulties in constructing and completing a proof, and listed the major reasons that he found as follows: *1- conceptual understanding, 2- mathematical language and notation, and 3- getting started on a proof*. In his dissertation, İpek (2010) found that pre-service elementary mathematics teachers' problems in the process of proving arise out of the lack of proving methods and concept knowledge that they have had. Selden and Selden (1995) also conducted a study with 61 undergraduate students who were enrolled in a course that was designed to introduce mathematical reasoning and proof, and they found that the participants were not able to determine logical structure of the expressions in the theorems.

In other study, Baker and Campbell (2004) studied undergraduate students' general deficiencies in the process of proving of mathematical propositions, and they indicated that those students were unable to understand concepts correctly, had lack of knowledge on the methods of proving the statements, failed to allocate sufficient discussion time to understand the meanings of concepts in the proving process, and used mathematical terminology incorrectly. Similar to Baker and Campbell's (2004) study, Sarı Uzun and Bülbül (2013) examined PSMTs' difficulties in constructing proofs, and investigated the impacts on the development of their proving skills after five week classroom activities based on group discussions were carried out. During the study, it was determined that pre-service teachers had difficulties in using mathematical language and notation, constructing a proof, selecting the proving method and focusing on that method, getting started on a proof, explaining the proof, understanding the concepts of proof, determining the validity of proof, and writing down the proof and expressing what they thought about the proof. These researchers suggest that teaching method and activities which could be designed similar to their teaching method and activities could be helpful to solve pre-service teachers' difficulties in proving.

Köğce (2013) worked with 99 first grade pre-service elementary mathematics teachers in his study, and he presented a possible answer of a student to the proof of the statement "The sum of any three consecutive numbers is three times of the middle number". Köğce (2013) attempted to determine these pre-service teachers' proving levels based on their opinions about that statement. As a result of the study, it is found that most of the pre-service teachers thought that verifying the given statement by numerical values is sufficient to prove that statement. In her thesis study, Demiray (2013) investigated pre-service elementary mathematics teachers' achievement levels in refutation, proof by contrapositive and proof by contradiction. She reported that the pre-service teachers failed to verify the proof and thus their proofs were considered invalid because of 'using the numbers to prove the statement' and 'direct restatement of the given statement'. In another study, Güler, Kar, Öçal and Çiltaş (2011) revealed that pre-service teachers experience difficulties in using examples when they were given a statement or a theorem to prove.

Weber (2001) worked with seven undergraduate students and examined the proving of some statements including group homomorphisms and isomorphisms in an abstract algebra course. As a result of his analysis, Weber (2001) reported that the students failed to construct the proof and experienced difficulties in the process even though they had necessary knowledge required to construct the proof. In particular, Weber (2001) indicated that it is not enough for students to know or remember a theorem or concept to construct the proof. Doruk and Kaplan (2013) examined pre-service elementary teachers' interpretation and evaluation skills of a proof on the convergence of sequences, and similar to Weber's (2001) study, they demonstrated that pre-service teachers' familiarity with theorems is not effective in their interpretation and construction of proof.

In another study, Weber (2006) graded student's difficulties in mathematical proof in three categories: inadequate conceptual knowledge of mathematical proof, misunderstanding of a theorem, and inadequacy in developing strategies to prove a statement or a theorem. In this study, Knapp (2005) concluded that students' difficulties in constructing proof fit into two main categories. First, students do not know how to use the language and logic of proof. Second, students lack the domain specific knowledge, such as definitions, theorems, and the ability to generate examples. In their study on, Knuth and Elliot (1997) studied with nine pre-service mathematics teachers, and examined their understanding of mathematical proof and expectations from the proof. They reported that pre-service teachers were incompetent and experience difficulties in understanding mathematical proof. In their study on a wider sample group, Selden and Selden (1995) revealed that the participants were not able to determine logical structure of the statements in a theorem. There are more studies which have similar findings with all these existing studies. But it would be more useful to present an integrated framework rather than discussing these studies individually. At this point, we can use the list of difficulties presented by Karahan (2013) based on the literature as regards to the difficulties experienced by students in mathematical proof. This list is as follows:

- Many students have a limited awareness of what proof is about (Hoyles, 1997).
- Students do not know what to do in the process of proving (Weber, 2001).
- Students do not understand the purpose of the proof and appreciate its role in mathematics (Hoyles & Jones, 1998).
- Students fail to comprehend the models for proof (in terms of format) (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012)
- Students lack the knowledge of content and strategy regarding the proof (Knapp, 2005)
- Students are unaware of the importance of definitions (Alcock, 2008)
- Students are unfamiliar with notations, do not understand the notations or know how to start constructing a proof. (Segal, 1999)
- Students lack experience to prove a statement or a theorem (in the process of comprehending the meaning of proof without focusing on the proof or constructing their own proof) (Hemmi, 2008).
- Students lack understanding of a theorem or a concept or misuse theorems or concepts regularly (Weber, 2001)
- Students are unaware of the logical reasoning and aspects of rigor which govern the proving process (Knapp, 2005)
- Students have problems in the method, concept and communication to prove a statement of a theorem (Remillard, 2010). (Karahan, 2013, p. 23)

As it is mentioned, students experience many difficulties in the proving process and there is wide range of reasons for such difficulties. This situation can be taken naturally when we consider that reading and constructing the proofs is a complex process (Blanton, Stylianou, & David, 2003). What is really surprising is that students from all levels (elementary, secondary and higher education) have such and similar difficulties. This general view about the difficulties shows us the importance of teachers' competencies in proof and proving as well as their undergraduate education. In that case, we should defend the idea of prioritization and dissemination of researches which focus on pre-service teachers' knowledge and skills to prove a proposition or a theorem as we aim to serve that idea in this study. It was determined that most of the researchers worked with the first and the second grade pre-service mathematics teachers, especially in Turkey (Altıparmak, 2014; Ceylan, 2012; Çiltaş & Yılmaz, 2013; Özkaya, Işık, & Konyalıoğlu, 2014). However, pre-service teacher who participated in this study were chosen from different grades. Another common aspect of existing studies in Turkey is that the proof of a theorem or a statement is presented in writing (on a paper) and pre-service teachers construct their proofs in writing (Ceylan, 2012; Çiltaş & Yılmaz, 2013; Demiray, 2013; İmamoğlu, 2010; Köğce, 2013; Özkaya, Işık, & Konyalıoğlu, 2014). Similar to Sarı, Altun, and Aşkar's (2007) and Sarı Uzun and Bülbül's (2013) studies, we asked each pre-service teacher to think aloud on the blackboard and construct the proof by explaining his/ her behaviors. But, Sarı Uzun and Bülbül (2013) did not interact with the participants; they just reminded participants to think aloud and to explain what they wrote. In this study, the researchers encouraged PSMTs to explain their thoughts and behaviors and identified their difficulties by asking them questions. In most of the existing studies related to the proof and proving, researchers worked with pre-service elementary mathematics teachers (Ceylan, 2012; Çiltaş & Yılmaz, 2013; Demiray, 2013; Doruk & Kaplan, 2013; Güler, Özdemir, & Dikici, 2012; İpek, 2010; İskenderoğlu, Baki & İskenderoğlu, 2010; Köğce, 2013; Özkaya, Işık, & Konyalıoğlu, 2014). In this study, the sample group consists of PSMTs.

We do not directly focus on the difficulties encountered by PSMTs in the proving process. First of all, we examined what kind of behaviors the PSMTs presented when constructing proof of a proposition. Then, we attempted to identify the difficulties experienced by the pre-service teachers as part of these behaviors. This study contributes significant insights and recommendations to the existing literature by identifying and analyzing the behaviors that PSMTs revealed when they proved a proposition. In detail, this paper might help mathematicians (who teach mathematics courses in undergraduate level) and mathematics educators understand and explore what kinds of behaviors, difficulties, and reasoning PSMTs could face in and what kinds of proof mechanisms they could follow while they are given a proof. By considering the results of the study, mathematicians and mathematics educators might understand their students' thoughts, behaviors and abilities regarding the proof and proving process in detail.

METHODOLOGY

Qualitative research methods are used in this study. Qualitative research is a research method that can be examined in a systematic way based on participants' experiences (Ekiz, 2003; Patton, 2002).

Participants and settings

Participants of this study consisted of 15 volunteer PSMTs (5 men, 10 women) from a large public university in the Aegean Region in Turkey. Five of the participants were in the second grade, while 3 were in the third grade and 5 were fourth grade and 2 were in the fifth grade in their program. Homogeneous sampling is adopted to determine the sample. This method includes examination of a homogeneous sub-group or state in the population. Our sample consisted of PSMTs with different genders, grade levels and academic standing in a manner consistent with the general student profile in a department of secondary mathematics teaching. Two criteria were considered when selecting the participants. These were being in second or a higher grade and participating in the study voluntarily. When selecting the participants, we especially chose them from the second or higher grades, because first graders were already taking theoretical mathematics courses, so it was difficult for them to reach a certain competency level in terms of proving. On the other hand, it was anticipated that PSMTs took the courses Analysis I & II, Abstract Mathematics I & II, and Linear Algebra I & II in the first year of their education, so the second or higher graders already had experience to construct a proof, so they might reach a higher competency.

Academic standing was not a criterion when selecting the participants, but we made sure that all or a substantial part of PSMTs were at various levels of academic standing (low, medium or high). In detail, the first and the second authors of the paper taught various theoretical and education courses to the participants throughout their five year education program, so they were familiar with the academic standing of the volunteer participants. After a group of PSMTs from different grade levels were informed about the study, volunteered participants were selected. Although the researchers planned to study with a total of 20 PSMTs that includes 5 PSMTs from each grade (2nd, 3rd, 4th and 5th), it was concluded that the study can be conducted with 15 PSMTs in order to maintain the diversity in terms of volunteering, gender and academic standing.

Each pre-service teacher's available times were determined to conduct the interview. They were requested to complete the interview in one day. Therefore, we selected one day that was available for majority of participants; all PSMTs were interviewed one after another on that day. All of the interviews were recorded using a video camera.

Researchers had a conversation with each pre-service teacher in the first 5-7 (approximately) minutes of the interview. At the same time, researchers informed PSMTs about the purpose and content of the study. In this processes, researchers tried to set the PSMTs at ease and answer their questions, if any. PSMTs were not given certain time to complete the proof. The interview was ended based on PSMTs' own requests or their failure to present an opinion or to proceed further regarding researchers' observation.

Data collection

In our study, pre-service teachers were given a mathematical proposition, and each pre-service teacher was asked to prove it alone on the blackboard by thinking aloud. Statement of the proposition is as follows:

If An= {p prime number: p|n} ve d = m.n, show that Ad = Am U An.

We purposefully chose the proposition since it requires a little preliminary knowledge and concept to prove, and the given statement does not have a complex structure to make the understanding difficult. Thus, it is aimed to prevent PSMTs from showing their lack of knowledge or advanced topics as a reason for their failure to prove the proposition. When we chose the proposition, the authors especially reviewed the sources regarding abstract mathematics and created a list of propositions which are the most and understandable statement (including 9 propositions). Afterwards, as a result of individual and group discussions of authors as regards to that list, they unanimously decided to use this proposition. Another reason for the selection of that proposition is that it is not commonly used in the resources. Students face with some propositions too often (for example, $\sqrt{2}$ is irrational or the sum of two odd numbers is even), it becomes easier for them to memorize the proving of such propositions. This leads them to try to remember the proof in their memories and present the proof exactly or similarly rather than making their own reasoning. Therefore, we selected proposition that is not a frequently used one. Nevertheless, a pre-service teacher (PT) (PT-9) indicated that he saw that proposition before.

At least two researchers were in the room while each pre-service teacher was proving the proposition. The researchers encouraged the PSMTs to explain their thoughts and behaviors and identified their difficulties by asking them questions. When we consider the existing literature, in some studies which has a similar purpose with the present study (e.g. Blanton, Stylianou, & David, 2003; Köğce, 2013), data is collected with a single proof problem in order to conduct a comprehensive research. Therefore, it is accepted that data collection with a single proposition which allows making an in-depth research is not constitute a constraint.

Data analysis

Before the data analysis, the researchers carefully transcribed the video-recorded interviews. The analysis was conducted based on these transcription texts, and the content of these transcription texts was analyzed. At this stage, each researcher made their own examinations first and determined under which themes the emerging behaviors can be examined. Afterwards, the researchers met and discussed on the themes they developed and decided that the existing data could be analyzed under three themes. Moreover, they created descriptive codes based on the content of these themes and the analysis was conducted under these codes. While researchers created the themes and the codes under these themes, they also considered the results of the similar existing studies (e.g. Moore (1994); Sarı, Altun, & Aşkar, 2007) to substantiate validity and reliability of the codes. The themes and codes identified by the authors are as follows:

T1: Behaviors that are presented before the proving and when they start to construct the proof:
- C1. Reading and expressing the given proposition aloud
- C2. First ideas that are produced on the given proposition
- C3. Thoughts about the way(s) to be followed to construct the proof

T2: Expressions and mathematical representations regarding the proving:
- C4. Making verbal statements mainly when proving
- C5. Behaviors based on mathematical representations

T3: Using examples throughout the proving process:
- C6. Examples that are produced without guidance of researchers
- C7. Examples that are produced with guidance of researchers

RESULTS

The findings that are obtained from the analysis are presented by considering basic transitions, pattern-related characteristics and breaking points in the process of proving the proposition that is presented to PSMTs.

T1. Behaviors that are presented before the proving and when they start to construct the proof

This stage covers the process from the moment the pre-service teachers saw the proposition to the moment they started to construct the proof, and focuses on behaviors of pre-service teachers that they presented at this stage. The behaviors that were presented before the proving and at the start of construction of proof are examined under 3 main headings: C1 - *Reading and expressing the given proposition aloud*; C2- *First ideas that are produced on the given proposition*; C3- *Thoughts about the way(s) to be followed to construct the proof.*

C1. Reading and expressing the given proposition aloud

Eight of the PSMTs read the proposition aloud as soon as they first saw the given proposition (See Table 1). Some of the pre-service teachers (PT-2, 8) read the proposition as it is written, while others expressed the proposition with their own sentences [PT-1, 3, 10, 12, 13, 15]. As it is shown in Table 1, pre-service teachers expressed the proposition with their own sentences and used mathematical language considerably.

Some PSMTs did not know how to continue the proving after reading the proposition or expressing it with their own sentences. For example, 4 pre-service teachers [PT-2, PT-3, PT-4, PT-15] emphasized that they did not know where to start and how to continue the proposition as follows:

> (PT-2): I initially couldn't figure out where to start the proposition. I don't know. Maybe it is because I'm not relaxed right now. You need to set the way, with its all stages, including the beginning. You'll set the process yourself but after you find out where to start, it's easier to do the rest of it.
> (PT-3): I mean I gave a specific example but I can't generalize it right now. Where should I start to generalize it? Does the union of these sets give this? How so? I need to know that.
> (PT-4): I guess I don't know where to start. But even if you tell me where to start, I couldn't do it, because I don't know how to proceed.
> (PT-15): I can't see where to start in a theorem.

C2. First ideas that are produced on the given proposition

As explained in the previous section, some of the PSMTs read the proposition aloud, while others attempted directly to make comments and produce ideas aimed at comprehension of what was given and asked in the proposition without reading the given proposition or re-expressing it with their own words. At this stage, it is found out that pre-service teachers had different approaches towards the proposition. Their statements are given in detail in Table 2.

It is observed that when PT-4 first saw the proposition, PT-4 experienced difficulties in explaining the meaning of statements in the proposition and reading mathematical language and notations, and attributed a false meaning to the division sign in the statement "p divides n" and made an incorrect reasoning as a result. In the reasoning process, PT-4 perceived the statement division as divided by and this

Table 1. C1. Reading and expressing the given proposition aloud *

PT NO	STATEMENTS
PT-1	We have a set that consists of prime numbers, p divides n. Then, a d number is defined. It is m times n. Prove that A_d is equal to A_m union A_n.
PT-2	If A_n is equal to prime number p, p divides n. If d is equal to m times n, then prove that A_d is equal to A_m union A_n.
PT-3	It is given that p is a prime number and p divides n. And if d is equal to m times n, prove that A_d is equal to A_m times A_n.
PT-8	The set A_n consists of prime number p, and if p divides n. d is equal to m times n. prove that A_d is equal to $A_m \cup A_n$.
PT-10	...and p divides n which is a prime number
PT-12	d is equal to m times n and p divides n. Prove that A_d is equal to A_m times A_n.
PT-13	We have a set and p consists of prime numbers and p divides n. We're given this [showed d=m.n], d is equal to m times n... and is this union? And this set [showed A_d] consists of d's. It is equal to union of these two sets. If d is equal to m times n, then our set A_d is equal to union of A_m and A_n. And we'll show that. Firstly, we look at A_n. It is already given. Likewise, A_m is also given and p is a prime number.
PT-15	Now, we have a set A_n. This A_n consists of p prime numbers. And p divides n numbers. We have a number d and it consists of m times n's. A_d is equal to A_m union A_n.

** The statements written in italics in square brackets are additional statements of researchers. PT is the acronym for Pre-service Teacher, and the numbers indicate the number of each pre-service teacher.*

Table 2. C2. First ideas that are produced on the given proposition

PT NO	STATEMENTS
PT-4	If p is a prime number, I thought that p divided n is a rational statement. The set A_n consists of rational numbers.
PT-5	Is it p divides n or p difference of n? Firstly, I prove that based on this data [showed A_n= {p prime: p|n}]. p is a prime, and p divides n. If p is a prime number, I think of the divisors of a prime number. Sorry, I consider the prime numbers which will divide n. The question first gives me that. Then it says that d is equal to m times n. The number d is equal to m times n. And considering that A_d is equal to the union of A_m and A_n, when we put the set A_d in place of [showed A_n= {p prime: p|n}], and p is a prime, d should be here [showed p|n]. We can consider it as p divides d.
PT-7	Now, we can add n divides p here... For example, A_d … If d is equal to m times n, then A_d will be equal to $A_{m.n}$ so...
PT-9	The set A_n consists of rational numbers, the set consists of numbers as p is divided by n…. p is a prime number, p is divided by n. Since it is a prime number, the statement must be fractional.
PT-11	[is there] anything about m? "Considering the numbers in A_d and it can be divided by a prime number again, it is a multiple of a prime number. Then if both of them are a statement of this..." [Continued by giving an example]
PT-14	If p is a prime number, p divides n... So... n has to be the multiples of prime number or the prime number itself. If p is a prime number, p divides n... İnitially, I try to concretize it. I give examples of p and try to find the sets of d". [Afterwards, PT-14 read the given proposition aloud].

mistake led PT-4 to express the set A_n as a set that includes rational numbers.

PT-5 tried to understand the division sign in the proposition by asking questions to the researchers. Similarly to PT-4, PT-5 failed to use the mathematical language correctly and started to express opinions with an incorrect reasoning regarding the division sign. But then PT-5 realized the mistake in his/her opinions and started to interpret and produce ideas on what kind of set the set A_d will be based on the set A_n. As indicated in Table 2, PT-7, differently from other pre-service, preferred to follow the way of using notation ($A_{m.n}$) for the statement A_d based on what is given in the proposition.

Similar to pre-service teachers PT-4 and PT-5, PT-9 had difficulties to identify what kind of elements the given set includes because of the lack of mathematical notation and language. Also, PT-9 showed that s/he had some misconceptions by saying "*it must be fractional because it is a prime number...*", thinking that elements of the set consists of rational numbers, the numerator of which is the prime number and denominator consists of n's.

After seeing the proposition, PT-11 firstly questioned whether any information is given regarding the number m. After a researcher said that m is a number like n, PT-11 thought for a while and then started to understand the elements of the set A_n just like other pre-service teachers. At this point, PT-11 said that the elements of the set A_d consist of the multiples of prime numbers, attempted to verify the proposition by giving numerical values and presented an example in that regard. In that example, PT-11 said that if 7 is taken as prime number, the numbers multiplied with 7, for example 21 or 70, can be assumed as m and n.

When PT-14 was given proposition, s/he made comments on how it should be constructed. Based on the statement "p divides n", PT-14 thought that the number n should be the multiples of any prime number or the prime number itself. This indicates that the pre-service teacher, who did not think enough whether the multiples of prime numbers can be a prime number, failed to reason about the rule regarding being an element of a set. After that, PT-14 preferred to verify by giving numerical values to the proposition, just like PT-11, to obtain a concrete example; s/he mentioned that we can find d by using example of numbers that can be given to p.

It is important to note that both tables, Table 1 and Table 2, did not include PT-6, because PT-6 presented a behavior directly proceeding to the construction of proof without any behavior under C1 and C2. Also, another different aspect of this stage is that PT-9 emphasized that s/he saw this proposition and its proof before. After this statement researchers asked this pre-service teacher to construct the proof as s/he remembered. Although PT-9 said s/he remembers the proposition after reading it in silence, PT-9 had problems to interpret the division sign and stated that s/he did not remember how it was proved.

Considering the findings that are presented until now, the main interesting point is the difficulties experienced by PSMTs in understanding the notations in the proposition. Some pre-service teachers had difficulties to translate the given statements from mathematical language to Turkish. Specifically, pre-service teachers indicated:

Is it p divides n or p difference of n? (PT-5)
But I don't understand something here. What is p divides n? (PT-8)
Mr., is that p divides n? [Shows the division sign] (PT-9)
Anything about m? [tries to make sense of m] (PT-11)
Is d a number? Is m a number? What is A_d? (PT-14)

It is interesting that pre-service teachers was familiar with the division sign (They used the sign in many courses: abstract mathematics, algebra and linear algebra), one-third of the pre-service teachers had difficulties to name the division sign. PSMTs' these behaviors indicate that they misunderstood the division sign and had difficulties in using mathematical language and notation because they did not understand the concepts enough.

C3. Thoughts about the way(s) to be followed to construct the proof

In this section, some PSMTs revealed various ideas regarding how the proof can be constructed based on their interpretation of the meaning of given proposition. Thus, this section explains the ways that the PSMTs used to construct the proof. For instance, PT-1 is the only participant who correctly interpreted the proposition. When we look at PT-1's statement in Table 3, it is examined that s/he started the proof with a correct reasoning in the beginning. PT-1 explained that s/he can find the sets A_n and A_m, and then s/he thought that s/he can automatically find the set A_d under the relationship of d=m.n. Then, PT-1 showed that the union set and the set A_d are equal by directly finding A_d rather than finding A_d with the union of the sets A_m and A_n. PT-1 started to explain the proof with his/her own words, and then made two different estimations, and applied his opinions into practice in the following stages.

On the other hand, the PT-3 stated that the proposition has a structure of '*if then*' and therefore s/he can construct the proof by reaching one side on the basis of other side (unilaterally). PT-3's first opinion as part of this perspective includes making trials on what is given based on the components of the set A_n. However, PT-3 failed to concrete this perspective; s/he could not give an example and his/her approach towards identifying and applying the mathematical signs and logical structures in the proposition was not enough to prove the statement.

Table 3. C3. Thoughts about the ways to be followed to construct the proof

PT NO	STATEMENTS
PT-1	I can find $A_m$and A_n. I think that we can then find A_d automatically or find A_d and If I can see that both are equal to the same statement, then I'd say that they're equal.
PT-3	It says prove that A_d is equal to the union of A_m and A_n. Firstly, I'd look how to get this result from among the given statements [showed $A_d = A_m \cup A_n$]. Since it [the statement] has "if", it'd be enough for me to prove one-way. Then I'd try to test the given statements on this [showed A_n= {p prime: p|n}]. If A_n is equal to p prime number, then p divides n.
PT-6	[After reading the proof mumblingly] Then I could state n as p times k. Also here, d is taken likewise. [when writing A_d= {m prime: m|d} ve A_d= {n prime: n|d}] This way, I could state d. Actually, A_d could be the union of these two. Because..." [Then deleted the sets A_d that s/he wrote].
PT-13	Writing the set A_m with the set A_n... Then, if this is given [shows d=m.n], we need to find this [showed $A_d = A_m \cup A_n$] by using this [shows d = m.n].
PT-15	Now... We need to get out of here, m times n.... I'm thinking to get here [showed $A_d = A_m \cup A_n$] by using this [showed d=m.n]. I need to be careful about A_n and A_m. I want to think something about them. I want to find A_m and A_n.

PT-6 firstly interpreted the rule that enables being a part of the set based on the way the set A_n is introduced, and considered expressing n as the multiplication of p and k. From this point of view, PT-6 predicted that the set A_d can be indicated as the union of the sets A_n and A_m, by thinking on how d can be expressed. Although PT-6's reasoning is correct, his subsequent examples and opinions were not sufficient to lead him to reach the result. In other words, although PT-6 started the proof correctly, he has a limited reasoning to construct and complete the proof.

PT-13 started to construct the proof with his own words and continued producing ideas in a more narrow scope compared to other participants. PT-13 wrote down the sets A_n and A_m and stated that the set A_d should be found under the equality d=m.n and interpreted the proposition in a sense. Similarly to PT-15, PT-13 stated that the way to be followed should be in a manner to focus on the union set based on d=m.n, and especially focused on identifying the sets A_m and A_n (See Table 3).

At this stage, PSMTs managed to make predictions by presenting behaviors that did not include any apparent mistake or deficiency. Some of the PSMTs reasoned on the set A_n or the number d only, while others produced ideas on the mechanism of proof as well (such as there is 'if', therefore it is a one-way to prove the statement).

At the same time, PSMTs mentioned some negative issues that they felt or experienced before and during the proving process. These are as follows:

> I'm prejudiced regarding the proving:
> (PT-4): [Do you have any specific prejudice? "Yes, I mean it's like proof is frightening".
> (PT-14): Normally, I can construct proof, it's very easy. As you said, it requires basic knowledge, not complicated, but I'm biased, so..." "Maybe it's because I'm a little biased, maybe I was afraid of it in the past."
> I don't feel relaxed
> (PT-2): Maybe it's because I'm not relaxed right now...
> (PT-4): [What if you did this on a paper. You're doing this here in front of us, does this have any impact on that?] "I don't think so but a little, slightly."
> (PT-14): I'm not relaxed right now. I don't want to solve this. I don't know, I'm a bit nervous. Normally if I solve this at home...

T2. Expressions and mathematical representations regarding the proving

Under this theme, PSMTs' approaches to the proof are discussed. In order to better understand the PSMTs' reasoning regarding to the proof, it is considered that it would be useful to look at their verbal expressions and mathematical representations. This theme is examined under two codes regarding PSMTs' verbal expressions and mathematical representations in the proving process. This two dimensional perspective offers a wider framework on identifying how PSMTs thought and how they expressed what they thought.

C4. Making verbal statements mainly when proving

The first code (C4) under this theme includes verbal expressions. Although the code C4 is stated as verbal expressions, it also often includes mathematical representations that PSMTs used. However, such mathematical representations used by PSMTs were short expressions and they aimed to support their verbal expressions using these mathematical representations. Actual mathematical representations related to the proof are discussed under the other code (C5).

As shown in Table 4, when we look at the PSMTs' statements, we found that some PSMTs used verbal expressions completely (PT-5, 10, 11), while some made verbal

Table 4. C4. Making verbal statements mainly when proving

PT NO	STATEMENTS
PT-1	"When I say A_m, p is a prime number. p is a prime number then it needs to be p divides m. Therefore, [*wrote down* A_m*= {p prime: p/m}*] I mean what I say m, it is a number like p times k [*wrote down m=p.k*]. When I say A_n, p is a prime number and p divides n. It was already written down. Considering n here, let's say p times t. When I say the union of those two, it'll be $A_d = A_m \cup A_n$ and p is a prime number, then it needs to be p divides n or p divides m. Since it's a union, it is enough that any of two crosschecks or it can crosscheck both [*Student wrote down* $A_m \cup A_n$*= {p prime: p/n* V *p/m}*].
PT-5	Since p divides n in the statement A_n, I think that p divides d in the statement A_d as well. Or this prime number is the union of divisors, I mean if multiples of d are m and n, when I apply m and n to a set, their union will give the division of that d. How could I say this? I think that here [*showed p/n*], p divides m and here p divides n. When I unite these two sets, it'll be equal to p divides d. When I'm constructing a proof, I can't put it into abstract concepts directly and thus I want to give examples [*then PT-5 proceeded with giving examples*].
PT-7	For example A_d, if d is equal to m times n, then A_d will be equal to $A_{m.n}$. So... Here, how could I show this? Wait a minute, let me think again... It'll be like we already chose this [*showed* A_n], but I'll say m times n, n is a prime number.... One minute... I got stuck at the end. [*then PT-7 proceeded with giving examples*].
PT-9	The researcher asked: [*You saw the statement, what did you think about it? What do you understand when you read it?*] "There is a generalization here. It shows that. I mean a number is given [showed *p/n*],and I think it asks how this number is obtained or prime numbers. I need to solve it to understand what it says. [*After some comments about understanding of the statements given in the proposition, the student tells the proposition verbally*]. For instance, I'd construct the set A_m first. And here p is a prime number. m should be a number that can divide p. m [*the number*] will be a number that can be divided by p. $A_m = \{p \text{ prime } \frac{m}{p}\}$ [*Deleted this set after writing it on the blackboard*]. Now let's say A_{m_1}. Is this number a multiple of p? Let's express the number A_{m_1} as p.k. [*then PT-9 continued his comments by using mathematical representations].*
PT-10	Now, constructing the set A_1. Actually it is not a set, I'd say it is a family of sets. I'd say A_1 is equal to p prime number. p divides one's. And ummm... p divides, yes. Prime numbers divide one, then I'd be an empty set probably. The prime number that divides one is empty, because prime numbers must be integers. They do not have rational numbers. Offf… I'm confused, it can't be rational. I mean there is no prime number that will divide one..." "Now, for example, the set we'll construct is A, it is m times n. [*the pre-service teacher meant* $A_{m.n}$]. And what does it include? Again p includes prime numbers, so... This time it consists of p prime numbers that will divide m times n... [*wrote down* $A_{m.n}$*={p prime: p/m.n} on the blackboard*]. Yes, p divides m times n. But it says d consists of m times n as you can see here [*showed the statement in the proposition*]. Then what happens? This set indicates that A_n union A_m ... Now if a number is divided by multiplication of these numbers, I mean if it divides these multiplications, if this prime number divides these multiplications, then it is possible to divide these separately. Yes, we have a prime number that divides a number. Likewise, it also divides another number. That prime number divides the multiplication of those numbers. When we get their union, there are multiplications and it is taken twice times. I mean when we sum up the number of elements of the set, those two are taken. I think that their intersection is obtained from there. I mean it'll be taken twice and it is that intersection here. So,we have the number of elements of the set A_m and elements of the set A_n. Because they are divided commonly after they are summed up. [*then PT-10 tried to give examples*].
PT-11	Thinking with a logic based on the numbers in the set A_d, if it can be divided by a prime number again, then it is a multiplier of prime number. Then, if both of two are a statement of this..." [*Here, the pre-service teacher chose a random prime number and tried to construct an example thinking that multiplications of two natural numbers which are multiples of that prime number can be divided with the same number*]. "a statement like n is equal to p times k, and m is equal to p times *l*. Then if the number d is like p square times k times *l*, then this number is directly divided by p. Actually, when I look from the logic of sets, I can't take an element twice. Therefore, I'd take it only once in the union, if we examine more... there's nothing else.
PT-13	Now we have two sets... Here [*showed p/n*] n is any number... We have two sets and when we're given the number d, then the number d is also equal to multiplication of any two numbers. And the set that is formed by those numbers is equal to the union of two sets like the set A_d. [*PT-13 wrote down mathematical expressions on the blackboard but then deleted them immediately*]. Now we can prove by induction... And considering the set A_n, it consists of numbers. If we write down these numbers, and then the numbers in the set A_m, and took the union of two... I think it can be this way...
PT-14	For example, for p is equal to 3... what can n be? [*wrote down n=3k on the blackboard*]. n is equal to 3.k, it'd be something like that. n will be like 3k. and m is any number, and I think I'll assign something to m. Let's say m is equal to 2.*l* [*wrote down m=*2.*l on the blackboard*]. Then, d is equal to 3k times 2*l*. So we'll have a set like 6k*l* [*wrote down d=3k.2l=6kl on the blackboard*]. We'll try to get this based on that. The set 6k*l* which consists of a's and d's, and A_m union A_n. Do we need to think $A_m$like that? I mean p divides m? [*after waiting for a while, the student said that he can't continue anymore*].

PT-15	[*after PT-15 stated that he wanted to give an example*] For example, I take m as 5 and n as 7 [*wrote down m=5, n=7 and d=35 on the blackboard*]. I want to see it like that... Then A_m is 5... It consists of p's, p will be a prime number, p can divide 5 [*wote down A_5={p prime: p/5} then A_5=5 on the blackboard*]. Let's go on with A_7. It consists of p's as well and let's take it as a prime number. And it'll be like that [wrote down *A_7={p prime: p/7 then A_7=7 on the blackboard*]. Now let's write down A_{35}." [*writes down A_{35}={p prime: p/35} then A_{35}={5,7} on the blackboard and puts in set brackets the sets he wrote down before*]. [PT-15 is considered under this category (C4) as PT-15 tried to present his opinion on an example only and attempted to express it based on mathematical signs and notation.]

expressions by also using mathematical representations (PT-1, 9, 13, 14, 15). Some of the PSMTs (PT-1, 9, 13) did not need to think on the example or give an example, and some of them (PT-5, 7, 10, 11, 14, 15) needed to give example at a specific stage or throughout their proving process. For example, PT-14 and PT-15 preferred to make their all statements and verify them based on an example. All pre-service teachers in this group tried to understand what kind of elements the given set includes and focused on to express the sets A_m and A_n. Some PSMTs tried to produce ideas on the set A_d itself and/or the union set based on the number d after that stage. Considering their statements carefully, it is observed that majority of PSMTs had difficulties in expressing their opinions.

When constructing the proof, PT-5 said that s/he had problems with abstract concepts and it would be better to verify the proof by using an example. The discoursal stance in PT-7's statements was very ambiguous, and s/he failed to express his own opinions. PT-10 stated that A_1 is a family of sets, prime numbers must be whole numbers, and if it divides a multiple of a number (multiple of two numbers), it can divide the multiplied numbers separately. Then, s/he mentioned about the sum of the elements of other two sets in creating the set A_d by means of union. PT-11 said that if the numbers in the set A_d are divided to a prime number, then it will be a multiple of prime number. Then s/he stated that "*if we examine more... there's nothing else*". PT-13 tried to interpret the proof in a different way; although PT-13 said that the proof can be done by induction first, the way PT-13 suggested later is about writing down the sets A_n and A_m and creating their union set. Although PT-14 originally wrote down that m could be a number like 2.*l* for n=2, then PT-14 preferred later to question the elements of the set A_m. PT-15 took m as 5 and n as 7 and wrote down d as 35 as a result; then PT-15 expressed the sets as A_5=5 ve A_7=7 and finally said that the elements of A_{35} are 5 and 7.

When we consider general characteristics of pre-service teachers' verbal expressions, we can summarize them as follows:

- PT-1 made integrated and consistent statements throughout the proving process. PT-1 interpreted the rule given in the proposition and reasoned about it step by step during the process.
- PT-5 was not generally clear in his/her statements. Besides, PT-5's statements did not have a logical sequence and reasoning. In other words, PT-5 had difficulties in the method, concept and language regarding the proof.
- PT-7 failed to produce an idea on what to do in the beginning process and his/her statements were very unclear and ambiguous. Similarly to PT-5, PT-7 experienced difficulties in the method, concept and language regarding the proof.
- PT-9 gave an incorrect meaning to the statement. Although PT-9's statements improved throughout the process, his/her interpretation was wrong and therefore PT-9's opinions were not enough in constructing the proof.

- Although PT-10's initial statements were correct and focused on establishing the connections, PT-10 failed to put together his opinions in his subsequent statements and did not know what to do. Therefore, PT-10 repeated himself and made unclear statements.
- PT-11 mainly made verbal statements and most of these statements were consistent with each other, but there were some logical gaps among them.
- PT-13 failed to structure his opinions properly. In general, PT-13's opinions were unclear and logically inconsistent. PT-13's preliminary statements about the proof and the way s/he described related to the construction of proof were inconsistent.
- PT-14's statements were correct in the logic he constructed but his opinions were not connected with each other. PT-14's way of choosing examples affected his way of thinking. S/he failed to combine his/her knowledge, and stuck at some point to construct and complete the proof.
- PT-15's initial idea and the examples s/he gave were not consistent with each other. PT-15's reasoning was entirely informal and PT-15 failed to establish the right connections.

When we examine PSMTs' statements, we found that that majority of them had difficulties in making a consistent, logical and integrated reasoning and reflecting this to their statements. Besides, it is determined that PSMTs had difficulties in method, concept and language related to the proof. Only PT-1 presented opinions clearly with a consistent explanation and used both verbal and mathematical expressions properly when doing so.

C5. Behaviors based on mathematical representations

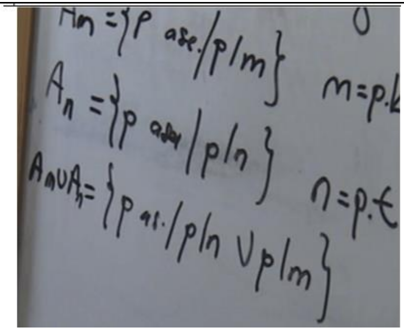
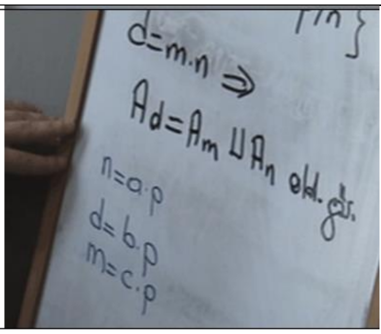
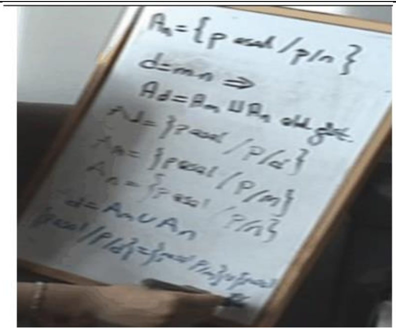
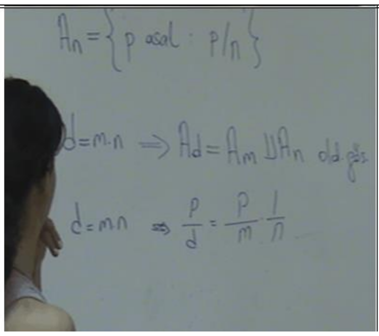
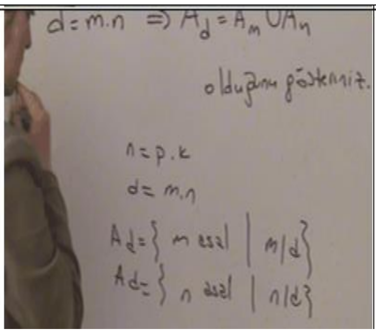
This section discusses PSMTs' statements regarding the mathematical representations that they revealed throughout the proving process. In particular, Table 5 includes the statements that were presented regarding the mathematical signs and notations to construct the proof and mathematical representations that PSMTs used throughout the process.

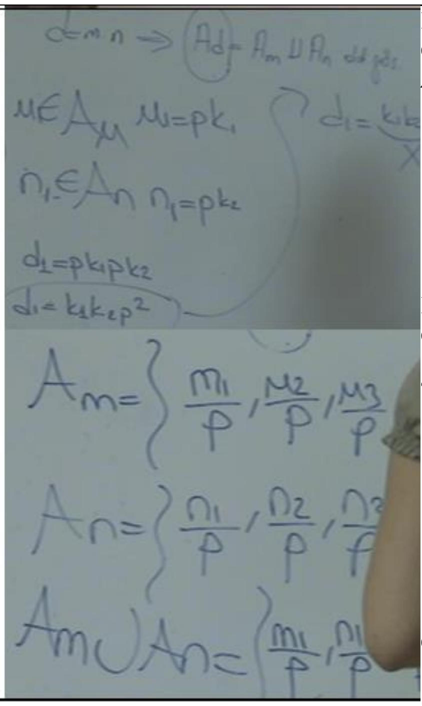
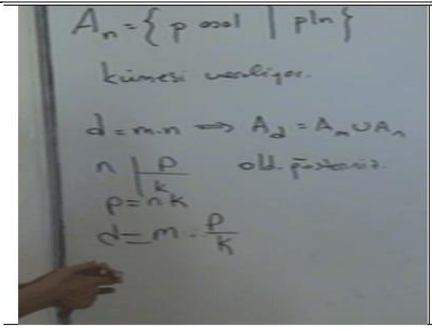
PT-1 is the most successful pre-service teacher in presenting a correct and reasonable and orderly approach from what is given to what is expected regarding the proof. Initially, PT-1 tried to understand what kind of elements the given set includes. After understanding the rule of the set correctly, PT-1 presented the set A_m mathematically and then wrote down the representations m=p.k and n=p.t. Using the reasoning on the given elements in order to express the union set, PT-1 interpreted that, "*when I say $A_d = A_m \cup A_n$ and p is a prime number, then it needs to be p divides n or p divides m.*" Based on that, PT-1 explained the union set as $A_m \cup A_n$= {p prime: p|n v p|m}. At this stage, PT-1 used the equality of the numbers m and n, wrote down the number d as p^2.k.t, considered the divisibility of this statement by p. As a result, he stated that the proof is apparent.

PT-2 initially wrote down the number n as a.p and d as b.p. Then, PT-2 wrote down m=c.p and put those in the equality d=m.n and obtained the equality b= a.c.p. But then, PT-2 tried to write down the set A_n as a list only based on n=a.p without using what he did in a manner unrelated to that equality, and formed the set as $A_n = \{\pm p, p, 2p, 3p, \ldots\}$ deciding that this set should also include negative whole numbers. Throughout the process, PT-2 had logical, conceptual and language related problems, and he got stuck at this stage without examining whether that set is correct or testing the set with an example. Afterwards, PT-2 tried to produce an example regarding the set that is given and preferred to verify the proof by using numerical values.

PT-3 explained the representation of the sets A_m and A_d by using mathematical representation of the set A_n, and then put them into their places in the equality A_d =

Table 5. C5. Behaviors based on mathematical representations

PT NO		STATEMENTS
PT-1		*A_m= {p prime: p/m}, m=p.k* *A_n= {p prime: p/n}*], n=p.t I defined m here as a number like p times k I defined A_n as p is a prime number and p divides n. It is already written $A_m \cup A_n$ = { *p prime: p/n V p/m}* *d= p.k.p.t= p^2.k.t* *A_d= {p prime: p/m.n}* *p/m.n → p/m V p/n* *İt is required to be that p divides n or p divides m. I mean, it is enough to crosscheck one of them or it crosschecks both of them.*
PT-2		[wrote down n=a.p] ‘ n is such a number, then I’d write d as b multiply by p for the set A_d [wrote down d=b.p on the blackboard, then added m=c.p] n=a.p d=b.p b.p=a.c.p^2 m=c.p b=a.c.p ‘$A_n = \{p, p, 2p, 3p, \ldots\}$ it could be an integer that means negative. I started to do it as incomplete. $A_n = \{\pm p, p, 2p, 3p, \ldots\}$ ‘ Let me write A_6, then I’d write the numbers that are up to n but smaller than n. A_6={2,3}
PT-3		[PT-3 continued to write under the proposition that was already written] *A_D= {p prime: p/d}* *A_m= {p prime: p/m}* *A_n= {p prime: p/n}* *“A_d is equal. Then, I substitute these into their places”* A_d=$A_m \cup A_n$ *{p prime: p/d}= {p prime: p/m}* ∪ *A_n= {p prime: p/n}*[wrote that statement] ‘If d equals to m times n, that is given to me’ [The pre-service teacher read the proof again, then continued by giving an example]
PT-4		“If p is a prime number, then I initially think that p divides n is a rational statement, the set A_n consists of rational numbers” [The pre-service teacher wrote the following statements on the blackboard] *A_n= {p prime: p/n}* D=m.n → Prove that A_d=$A_m \cup A_n$ *[given proposition]* d = m.n → $\frac{p}{d} = \frac{p}{m}.\frac{1}{n}$ [think about it for a while] ‘I’d like to describe the set A_d like that ($\frac{p}{d}$). It is $\frac{p}{m}$ in A_m. But, how can we describe the union of it with A_n? [After PT-4 indicated that s/he did not understand the proposition, s/he verbally explained the proposition.] [After s/he was informed that s/he could use an example] Let me take n=5. Then n=5 A_5={ *p/5…* “it consists of p divides 5” [The pre-service teachers then deleted what s/he wrote] [When his/her problem related to the interpretation of the proposition was solved, s/he wrote A_5={5}]
PT-6		“Then, I could stated n as p times k. n=p.k. Then, here, d is taken the same” d=m.n A_d=*{p prime: p/m}* and A_n= *{p prime: p/n}* “I could describe d like that” Indeed, A_d could be the union of these two. [then, PT-6 give an example]

PT-9		I initially construct the set A_m. P is a prime number. m is a number that divides p. m is a number that could be divided by p. A_m= {*p prime* $\frac{m}{p}$}[S/he then deleted this statement] $m \in A_m \quad m_1 = p.k_1$ $n_1 \in A_n \quad n_1 = p.k_2$ $d_1 = p.k_1.p.k_2$ $d_1 = k_1.k_2.p^2 = k_1.k_2.p.p$ [Then, one of the researchers guided PT-9, and s/he start to use an example. PT-9 then gave up to use an example by indicating that s/he could not construct the proof] "Let me try one more time, if I cannot do it, I cannot do it…" $A_m = \{pprime \frac{m}{p}\}$ $A_m = \{\frac{m_1}{p}, \frac{m_2}{p}, \ldots\}$ $A_n = \{\frac{n_1}{p}, \frac{n_2}{p}, \ldots\}$ $A_m \cup A_n = \{\frac{m_1}{p}, \frac{n_1}{p}, \frac{m_2}{p}, \frac{n_2}{p} \ldots\},$ [The pre-service teacher wrote all these sets, and thought for a while; then deleted all of them.]
PT-12		"d is equal to m times n, and p divides n. Prove that A_d is equal to A_m union A_n. "For instance, what can p be? [The pre-service wrote n for dividend, p for divider, and k for remainder using division sign (as shown in the picture), and then s/he wrote p=n.k] [she wrote as n is the dividend, p is the divider, and k is the remainder.] $p = n.k$ n p / k $d = m.\frac{p}{k}$ A_d= {*p prime: p*/*m.n*} [Then, the pre-service teacher tried to use and example]

A_m U A_n. Afterwards, PT-3 did not know what kind of method to use, failed to proceed from that point and thus preferred to test it using an example like PT-2.

PT-4 had problems throughout the process since s/he was not initially able to use and make sense of mathematical notations correctly. PT-4 stated that the set elements consist of rational numbers by thinking the division sign as "divided by". As part of this thought, the participant formed the equality d = m.n → $\frac{p}{d} = \frac{p}{m}.\frac{1}{n}$. PT-4 wrote down the elements of the set A_m with a similar representation and then tried to question how to link them with the set A_n. But after failing to proceed on this method, PT-4 used numerical values just like some other pre-service teachers, took n as 5 and preferred to give example.

Like some other PSMTs, PT-6 wrote down the number n as p.k, used the given equality d=m.n and made an incorrect mathematical representation like A_d={m prime: m|d} and A_d={n prime: n|d}. After this representation, he made a comment like "*Actually A_d could a union of these two*", indicating that he is not sure about what he wrote. In other words, PT-6 failed to interpret mathematical notations correctly and had problems throughout the process. Just like other pre-service teachers, PT-6 used numerical values after getting stuck at a certain stage and preferred to give examples.

PT-9 wrote down the set A_m as $\{p\ prime\ \frac{m}{p}\}$ and tried to explain the set elements in the beginning. To do that, PT-9 wrote the elements m_1, n_1, d_1 as p. k_i. The pre-service teacher put the numbers m_1 ve n_1 into their places in the equality d=m.n and obtained the statement $d_1 = k_1.k_2.p^2 = k_1.k_2.p.p$. After getting stuck at this stage, PT-9 focused on thinking about an example based on the researchers' recommendation. The pre-service teacher spent a little time while he was trying to verify the proof using examples, s/he re-attempted to prove and explained the union set mathematically as $A_m \cup A_n = \{\frac{m_1}{p}, \frac{n_1}{p}, \frac{m_2}{p}, \frac{n_2}{p} \dots\}$. However, after that stage PT-9 failed to proceed further and complete the proof.

PT-12 examined the rule of given set A_n and wrote down the number p as n.k by using the concepts of dividend, divisor and division. Afterwards, the pre-service teacher put this representation into its place in d=m.n and produced the statement $d = m.\frac{p}{k}$. At this point, PT-12 failed to produce ideas anymore and preferred to give examples just like other pre-service teachers (See Table 5).

When we examine the PSMTs' approaches regarding mathematical representations, we determined that they try to create the representations that come to their mind first as regards to the set A_d or the union set, rather than acting within a general strategy in the process (generalizing the proof). In general, they used representations that were formed like n=p.k. Also, the pre-service teachers had the tendency to try to explain the set elements without properly comprehending the elements of a given set (e.g. PT-2, 4, 6). At this stage, PSMTs had difficulties because of misunderstanding mathematical notations in the proposition. We also found that most of the PSMTs were tend to give examples by using numerical values after getting stuck at a certain stage following their efforts to construct the representations, and to verify the proof using these examples.

In the light of behaviors provided under the Theme 2, it is identified that majority of PSMTs mainly used verbal expressions rather than mathematical representations throughout the process.

T3. Using examples throughout the proving process

The Theme 3 includes the examples that were given by PSMTs throughout the proving process. Except for PT-1 and PT-13, all PSMTs preferred to verify the proof by using examples. The PSMTs attempted to produce examples and interpret these examples by themselves or as a result of the guidance of researchers. This theme is examined under two codes.

C6. Examples that are produced without guidance of researchers

Only four of the PSMTs managed to construct their own examples to construct the proof (PT-5, 10, 14, 15). These pre-service teachers preferred to construct examples by themselves during the process (In Table 6 and Table 7, R refers to researcher and PT refers to pre-service teacher).

PT-5 started constructing the example by assigning a value to p and took p as 2 and n as 4. Accordingly, PT-5 obtained the set A_n and wrote it as {2}. Afterwards, PT-5 took m as 6 and wrote the set A_m incorrectly as {3}. After the researchers encouraged PT-5 to re-consider the example, s/he checked the result again and corrected the set as {2, 3}. Again, the researcher asked what the number d would be, and the pre-service teacher obtained 24. Then, PT-5 made verbal statements and said "*Since I took A_d as 2 and 3 here, I'd say A_d is equal to union of A_m and A_m*", implying that the union set would be equal to the set A_d. It can be seen that PT-5 did not make a clear statement, and was not able to write down mathematically what s/he implied.

PT-10 needed to think about example soon after starting to construct the proof; s/he assigned a value to n and tried to construct the set itself. PT-10 correctly reasoned about it, and stated that the elements of the set A_6 should be 2 and 3. Accordingly, PT-10 stated that A_d will be the union of A_m and A_n. But then, PT-10

Table 6. C6. Examples that are produced without guidance of researchers

PT NO	STATEMENTS
PT-5	- (PT-5): When I construct the proof, I can't put it into abstract concepts directly and thus I want to use examples. For example, if p is a prime number and p divides n, let's take p as 2. Considering p divides n, let's take n as 4. [*Students wrote down p=2, n=4, A_n={2} on the blackboard*]. - (PT-5): In the set A_n, p is a prime number, p divides n. Would I show it directly as a number, or? - (R): Whatever you want. - (PT-5): I found its result as 2. Similarly, let me take m as 6 [*wrote down m=6,* A_m= *{3}*]. - (R): Are these the prime numbers that divide 6? - (PT-5): What divides 6... if p is a prime number, when I divide 6 to 2. Is it opposite? - (R): Now, what does the set A_n show here? - (PT-5): The set A_n, p is a prime. - (R): So, it consists of prime numbers. - (PT-5): Such that p divides n. All prime numbers divide this [*showed 6*]. - (R): Yes. - (PT-5): Then I'll write it like this. Prime numbers that divide this [*shows 6*]. In this case, p is 2 and 3. So prime numbers are 2 and 3. I find it as 2 and 3 [*the student wrote down the set as* A_m= *{2,3}*]. This is the results of that section. Should I write prime numbers here? - (R): Yes, ok then what is d? - (PT-5): Normally, d is m times n. so 6 times 4 is equal to 24. If its' value is p, then 2, 3... Its' divisors are 2 and 3. Since I took A_d as 2 and 3 here, I'd say A_d is equal to the union of A_m and A_n, because the union of 2 and 3 is again 2 and 3.
PT-10	- (PT-10): Umm... How could it be? For example, if we say A_6, then 6 consists of prime numbers. Then what will it be? We'll have 2, and 3. This will be the set A_6. It says that if a number d is n times m... Ok... Then the set A_d will be the union of A_m and A_n. Can I say something about its proof? - (R): Yes. [*Then the student started to make comments about how the given statement could be proved.*]
PT-14	- (PT-14): Initially, I'd try to...umm... make it concrete. I'd try to find the sets d by giving examples of p's. - (R): Ok then let's do it. - (PT): Let's say d is equal to 3... For example, for p is equal to 3... what can n be? [*Student started to make representations about how the proof could be constructed based on the examples*.]
PT-15	- (PT-15): I'd want to identify A_m and A_n. - (R): Okay, you can think about an example first. - (PT-15): I already thought about an example. - (PT-15): For example, I took m as 5 and n as 7... It's 35, I want to see it like that [*Student wrote down m=5, n=7, d=35, A_m=5 on the blackboard.*] - (R): We assigned the values, now if we try to construct the sets... - (PT-15): A_m is 5. - (R): For what did you say 5? Is m? - (PT-15): A_m. - (R): How about 5? You said 5 for what you wrote? Number m. - (PT-15): Number m. - (R): Then will you try to find the set A_5? [*Student wrote A_5=5, then A_7=7 and then the set A_{35} as A_{35}={p prime: p|35}... Then put a set sign for A_5 and A_7 and wrote A_{35}= {5,7}.*]

gave up using an example and preferred to prove it formally. It is inferred that the pre-service teacher focused on the example only for a short time, and it is perceived as a behavior that the pre-service teacher presented to test whether he understood the statement correctly. PT-10 decided to give up using the example after noticing that the example is correct.

Initially, PT-14 said "*Let's say d is equal to 3 ... For example, for p is equal to 3 ... what can n be?*" and wrote down n=3.k after thinking what n can be if p is 3 in order to concretize the given statement, which was neither a completely mathematical method or a complete example. Based on a similar reasoning, PT-14 expressed m as 2.*l* and indicated d as the multiplication of these numbers. Although PT-14 assigned a value to d and then to p, s/he did not construct any example related to other elements of the statement.

PT-15 initially focused on assigning a value to the unknowns in d=m.n and took m as 5 and n as 7, and found d as 35. Afterwards, PT-15 tried to construct the sets A_m and A_n using those values. PT-15 explained his or her ideas regarding the sets by writing $A_5=5$ and $A_7=7$. PT-15 managed to interpret the given statement correctly and made correct interpretations on what the elements of given sets are/could be. But s/he failed to correctly present the sets. After the researcher warned him/her, the pre-service teacher used the correct way of representation.

It is important to note that PSMTs, who preferred to use examples without any guidance, used examples mainly for interpreting, explaining and concretizing what is given in the proposition. Also, we could not clearly indicate that PSMTs were entirely successful in constructing their own examples.

C7. Examples that are constructed with guidance of researchers

During the interviews, some of the PSMTs constructed examples with the guidance of researchers and tried to verify the proof (See Table 7). For example; PT-2 had a problem to construct the proof, and the researchers encouraged and guided him/her to construct an example. Although PT-2 emphasized that constructing an example was helpful to think easier and the statement more concrete, s/he failed to construct a complete example. Firstly, PT-2 took n as 6 for the set A_n and wrote that 2 and 3 are the elements of the set A_6. The pre-service teacher did not proceed to verify the proof after that. Although PT-2 constructed the example correctly, the pre-service teacher did not prefer to construct an example for other unknowns.

PT-3 initially decided to assign a value to the unknowns in the equation d=m.n and took n as 2 and m as 3 and equalized d to 6. Afterwards, the pre-service teacher had an ambiguity when trying to interpret the prime number in the given statement. While s/he was having difficulty in deciding on whether n or p s/he should think, the researcher guided him or her. Then, s/he continued to construct the example with the values s/he originally used. When constructing the proof, PT-3 tried to write down numerical values for the elements of the sets A_n, A_m and A_d. First of all, the pre-service teacher wrote down the sets A_2 and A_3, obtained the set A_6 from there, defined the union set and then obtained the statement $A_6 = A_2 \cup A_3$.

When PT-4 was guided to construct an example, s/he indicated that s/he is not sure about whether constructing an example will light the way for constructing the proof. In this process, it was observed that the pre-service teacher was not very eager to construct an example. First of all, the pre-service teacher took n as 1 and tried to construct an example quite simply and took n as 5 after the question of researcher (See Table 7). When the pre-service teacher was asked what the elements of the set A_5 will be, PT-4 took the set as p/5. This showed that the pre-service teacher was not able to interpret the given statement. However, when giving an example on the statement s/he wrote, s/he realized that s/he wrote incorrect statement and corrected that mistake. Then, s/he wrote the set as $A_5=\{5\}$ saying that prime numbers will divide 5.

PT-6 started to construct an example by taking p as 5. After s/he realized that she misinterpreted the given statement, the pre-service teacher stated that n will be multiples of p and have an increasing pattern. Regarding this thought, s/he wrote the set as $A_n= \{5,10,15, \ldots\}$. The pre-service teacher could not construct the interrelationship between n and p and therefore s/he constructed the set

Table 7. C7. Examples that are constructed with guidance of researchers

PT NO	**STATEMENTS**
Some of pre-service teachers started to give example with d, n or m, while others started with p.	
PT-2	- (R): Would you feel more comfortable if you give an example? Just to define your way. - (PT-2): It'd make thinking easier as it concretizes the statement. [*Before constructing an example, the pre-service teacher preferred to interpret the statement again.*] - (R): I mean you could think directly with a numerical example. I mean you'd remember easier if you write down any special set of A. - (PT-2): What if I write A_6? [*at first, the student wrote 1 as an element in the set A_6 and then deleted it*]. Then I'd already write numbers smaller than n, up to n". [wrote $A_6 = \{2,3\}$ *and then ended constructing the example.*]

PT-3	- (R): Could you give an example? Before starting to construct the proof... Does it make things easier if you give a numerical example? - (PT-3): For instance, if I took this [*showed n*] as 2, this [*showed m*] as 3, this [*showed d*] as 6, [*wrote down the numbers in the equation d=m.n*], for instance I'd take this [*wrote down under p in the set A_n={p prime: p|n}*] as 12. - (R): to p? - (PT-3): Sorry... [*student deleted what he wrote down*] Ummm... If it is a prime number, it must divide itself or 1 to be divided by n. - (R): n is not a prime, p is. - (PT-3): Umm... I don't know, I can't think anything right now. - (R): You just said 6, 2, 3. Consider this. [*student wrote down 6, 3, 2 under the equation d=m.n*]. - (R): Now, how about the elements of the set above? How about he elements of the set A_n? Here, how will you construct $A_2$for instance? - (PT-3): For example, A_n should be equal to a prime number. Prime... - (R): For example, could you do it for 2, for A_2? - (PT-3): Umm... A_2... So... Then... - (R): Then? - (PT-3): I need to find a prime number that it divides 2. But there is no such prime number? - (R): Are you sure? - (PT-3): We have 2. OK… OK… Then it is 2 divides 2. [*at first student wrote $A_2 = \{2/2\}$, but then corrected it as $A_2 = \{2\}$ after being guided*.] - (R): How about A_3? The set A_m. You took m as 3. - (PT-3): It's 3. [*wrote down A_3= {3}*]. - (R): How about A_6? - (PT-3): A_6…? It is 2 and 3... - (R): So we saw that the statement is true. - (PT-3): Yes. [*Student wrote the set A_6= $A_2 \cup A_3$ on the blackboard*]. [*Afterwards, student said that the proof can be constructed by means of generalization from there, but was not able to construct the proof.*]
PT-4	- (R): How about you do it with an example, do you have any idea about what to do this time? What if you try to construct an example? - (PT-4): Sos? - (R): I mean what kind of set could the set A_n be? Which elements can it have? If we write it first, does it make things clear? - (PT-4): Maybe. - (R): Then let's choose a set A_n. I mean try to write A_n for a number n. For instance, choose a number for n. - (PT-4): Let's take n as 1 [*wrote n=1*]. - (R): Choose a greater one. - (PT-4): Then let's take 5. - (R): A_5 set? Ok, 5. Then how will the set A_5be? Which elements will it have? The set A_5? - (PT-4): It'll be p divided by 5. [*wrote n=5, A_5= {p/5*] - (R): What does that mean? p divided by 5? [the pre-service teacher deleted the last *p/5*]. - (PT-4): 1 divided by 5, 2 divided by 5. Sorry, no 1 divided by 5. [*Researcher asked some questions to correct the pre-service teacher's misinterpretation.*] - (PT-4): I just showed that set [*The pre-service teacher indicated the set he wrote first.*] - (R): Ok, you understood correctly now. - (PT-4): Prime numbers that divide 5. - (R): Ok, then what will it be? -(PT-4): It'll be only 5. [*wrote n=5 A_5={5}*]. [*Then, the interview proceeded regarding how the proof can be constructed.*]
PT-6	- (R): For instance, can you give an example, a numerical one to do that. - (PT-6): For example, p is a prime number. Let's take 5. n, 5, 10, 15... and so on. [*wrote down 5, 10, 25 on the blackboard and also* A_n= *{5,10,15,...} under it*]. And this d creates the set A_n. d is m times n. I don't know whether m and n are prime numbers. - (R): Think about the set A_n, again. Take n as a number. - (PT-6): If I take n as a number. [deleted the set A_n= *{5,10,15,...}*]... If I take n as 1, then A_1becomes 5. - (R): Let's crosscheck it? - (PT-6): It does, if I take as 1... Sorry, I can't take 1, it's a prime number... [*The researcher guided the student to interpret the statement correctly.*] - (PT-6): How about the elements of A_n? Is it n or p? - (R): Of course it will have p's. - (PT-6): Let's take 6. If it is a prime number, then it is an empty set because there is no prime number that is divided by 6. - (R): How? Can't you divide 6 by 2? - (PT-6): Sorry… [*Student did not continue to construct the example.*]

PT-7

- (R): Could you give an example for the set A_n? What kind of set is that?
- (PT-7): p is a prime number... Then it'll be like... p can divide n, so if I take 2, n will be multiples of 2, so it'll go on like 2, 4, 6, 8... It'll divide p and d is equal to m times n... [*the pre-service teacher mumbled*]. Anyway, let's take 2 and 3 or 3 and 5 [*talked about m and n, respectively*].
- (R): For example, assign a number to n for the set A. What is A_5 if n is 5?
- (PT-7): [*wrote* $A_5 = \{5,10,15,\dots\}$ *and after that the student focused on making sense of what is given in the statement.*]
- (PT-7): Then it'll be like... It'll be a prime number and divide itself, and so it'll be equal to itself.
- (R): What happens if n is 6? What would the set A_6 be?
- (PT-7): 2 and 3 will be the prime numbers that divide 6. [*after that stage, the pre-service teacher presented his opinions on how the proof can be constructed.*]

PT-8

- (R): Would it be easier if you give an example?

  …
- (R): What is the set A_6?
- (PT-8): I couldn't do it, I couldn't find it.
- (R): What would the set A_6 be if you take n as 6?
- (PT-8): Here the set A_6, for example, is 3, a prime number. 2 and 3 divided by 6. [*wrote down* $A_6 = \{2, 3\ prime\ numbers\}$ *on the blackboard*.]
- (R): Yes, 2 and 3, good.
- (R): What would it be if let's say A_{10}?
- (PT-8): 5 and 2.
- (R): 5 and 2, ok. In this case?
- (PT-8): If d is equal to m times n, then prove that A_d is equal to the union of A_m and A_n.
- (R): When you think based on that example, is it correct?
- (PT-8): …
- (R): For instance, n= 6.
- (PT-8): I couldn't construct a relationship with 2 and 3.
- (R): Now look, we took m as 10 and n as 6, right?
- (PT-8): Yes.
- (R): Then d is 60, right?
- (PT-8): Yes [*The student made no comment after that stage.*]

PT-9

- (R): And assign a numerical value to m and n. What would d be? [*PT-9 wrote down 2 under m and 3 under n in the equation d=m.n on the blackboard.*]
- (PT-9): It'll be 6.
- (R): Ok, now construct those sets. A_m, A_n, A_d. Try to construct it for those numbers.
- (PT-9): Let me deleye those. [*The pre-service teacher deleted what was written on the blackboard*.]
  For instance, let me take p as 2. Let me say 4 and 6. [*wrote down 4 under m and 6 under n in the equation d=m.n*]
- (R): What would d be?
- (PT-9): It'll be 24.
- (R): In this case, let's write down the sets A_m, A_n, A_d.
- (PT-9): A_4.
- (R): What would the set A_4 be?
- (PT-9): The prime numbers which divide 4. What would it be? 2. A_6… The prime numbers which divide 6 would be 2 and 3. And it's A_{24}. And the prime numbers which divide 24 would be 2 and 3. [*The pre-service teacher stated that he understood the given statement, and then focused on constructing the proof again*.]

PT-11

- (R): Do it there. Give an example if you will.
- (PT-11): For example, if I take a prime number like 7, I think about two numbers that can be divided by 7. I think the union of these numbers can also be divided by their multiplication. And its mathematical representation... [*Afterwards, the pre-service teacher focused on constructing the proof.*]

PT-12

- (R): If you want, give an example for the set A_n?
- (PT-12): A_n'e ne örnek verebiliriz? What would I give for A_n?
- (R): For instance, give a numerical value...
- (PT-12): If we take 12, 3 is divided by 12.
- (R): What else?.
- (PT-12): Let me generalize it, what would A be?
- (R): Only 3 for the set A_{12}?
- (PT-12): Ok. If p is 4. It can't be 4, but 2. The prime numbers of that could be 2 and 3. [*wrote down the set* A_{12}*={2,3}*].
- (PT-12): So it consists of prime numbers.
- (R): Yes, that's true.
- (PT-12): Then would the set d consist of any m numbers and prime numbers?
- (R): If you write down n instead of m, A_m becomes a set as well.
- (PT-12): Instead of m?
- (R): A_n, for instance you took n as 12. And you can take m as another number. For example, take 14.
- (PT-12): Does the same rule apply to that?
- (R): Of course, sure.

- (PT-12): I thought the rule applies to n only.
- (PT-12): Then if we took 14 for 2, it'd be 7.
- (R): Where's A_d?
- (PT-12): A_d is the union of both, it'd be 2, 7, 3. So it means we'll find the common elements.
- (R): What's d? m times n. Then, it is 12 times 14.
- (PT-12): And the result... Umm... 168. So it's prime multiples of 168. [*Afterwards, the pre-service teachers made*

- (PT-12): I thought the rule applies to n only.
- (PT-12): Then if we took 14 for 2, it'd be 7.
- (R): Where's A_d?
- (PT-12): A_d is the union of both, it'd be 2, 7, 3. So it means we'll find the common elements.
- (R): What's d? m times n. Then, it is 12 times 14.
- (PT-12): And the result... Umm... 168. So it's prime multiples of 168. [*Afterwards, the pre-service teachers made comments on how the proof can be constructed.*]

incorrectly. Similar to this thought, the pre-service teacher then indicated that s/he does not know whether n and m are prime numbers as regards to the equality d=m.n. After the researcher asked the pre-service teacher to take n as a numerical value instead of p, the pre-service teacher stated that A_1 will be 5. In other words, he meant that the first element of the set A is 5. After the researcher asked whether the elements of the set are n or p, the pre-service teacher answered as p, and then stated that the set will be an empty set for 6. Afterwards, the pre-service teacher said that there is no prime number that can be divided to 6 and s/he did not continue to construct and explain the example. It is clear that the pre-service teacher was not able to interpret the elements of the set.

Similar to PT-6, PT-7 started the process by assigning a value to p and took p as 2, saying that n will be multiples of 2. Afterwards, the pre-service teacher preferred to assign a value for n and m in the equation d=m.n, and took them as 2 and 3 or 3 and 5. After the researcher asked what the set A_5 would be for 5, the pre-service teacher wrote down the set as A_n= {5,10,15, …}. To encourage the pre-service teacher to realize the mistake s/he made in interpreting the proposition, the researchers asked various guiding questions (See Table 7). After these questions, PT-7 correctly answered the question what the set A_6 will have if n is taken as 6. Then, she did not complete the example.

After the researchers guided PT-8 to use an example, s/he preferred to interpret what is given in the statement first, but after testing it for a while, the pre-service teacher said that s/he can't do it and find anything. During this period, the researcher asked to what the set A_6 could be equal to help the pre-service teacher use a special example (See Table 7). Then, in order to enable the pre-service teacher construct a similar example, the researcher asked what the elements of the set A_{10} could be if n is taken as 10. In the following step, the pre-service teacher was asked about what could be done, and s/he did not give any answer regarding this question; so, the researcher decided to guide. Meanwhile the pre-service teacher again read the given statement aloud and stated that s/he can't associate 2 and 3 after the researcher said, "*for example n=6...*". Then, the interview with the pre-service teacher was ended.

PT-9 started to construct an example based on d=m.n and took n as 2 and m as 3 regarding the guidance of the researchers. After that, the pre-service teacher decided not to use these numbers and took 4 and 6, respectively. Based on those numbers, when the pre-service teacher was asked about what the elements of the sets A_m, A_n, A_d could be, s/he was able to find the elements of all three sets and revealed that s/he understood the statement.

Similar to other PSMTs, PT-11 tried to assign a value to p first, thinking that n and m are common. The pre-service teacher took p as 7, and thus considered 21 and 70, the multiples of 7, and stated that multiplication of those two can be divided to 7. The pre-service teacher decided to re-consider the proof and focused on mathematical representations without constructing the sets based on those numbers, and s/he examined if there is any relationship between them as in the proposition.

PT-12 started to construct the example by assigning a value to n in order to find the set A_n. The pre-service teacher took n as 12 and stated that the element of the set A_n consist of only 3 (See Table 7). After the researchers guided PT-12, s/he said that 2 and 3 are the elements of the set A_{12} and wrote this set on the blackboard. When constructing the example, s/he had difficulties in understanding what kind of a set the set A_m is and its relationship with d. Afterwards, by the help of researchers, the pre-service teacher took m as 14 and stated that the elements of the set A_m will be 2 and 7. Then the pre-service teacher said that the elements of the set A_d will be 2, 3 and 7 based on d and these numbers are prime multiples of 168, which is the multiplication of 12 and 14.

When PSMTs' examples were examined, it was found that some of them assigned a value for n (or m) first (PT-2, 3, 5, 9, 12), while others did that for p (PT-6, 7, 11). PT-3 initially assigned values for n, m and d, but s/he failed to make the right decision for the connection of n and p when constructing the set. Two of three pre-service teachers who preferred to assign a value to p when constructing the example thought that the set consists of the multiples of the prime number p. It is also observed that some pre-service teachers initially had various problems when they constructed the sets based on numerical values (e.g. PT-5 in presenting the set, PT-6 in defining A_1). In the light of findings, we can interpret that PSMTs were not successful enough to construct proper example(s), but perceived the connections between the given statements after the guidance of the researchers.

DISCUSSION AND CONCLUSION

The following interpretations and results obtained based on PSMTs' general performance, behaviors and verbal and mathematical expressions that they revealed throughout the process. In this study, it is found that PSMTs generally had problems during the process from getting started on the proof to ending it. PT-1 was the only pre-service teacher who revealed most acceptable approaches for reaching the proof among 15 pre-service secondary mathematics teachers. Different from other pre-service teachers, PT-1 presented correct statements and behaviors as regards to the proof. In this process, PT-1 explained both mathematical and verbal statements as well as s/he did not prefer to verify the proof using numerical values and examples different from other pre-service teachers.

It is also found that other PSMTs had some difficulties to construct the proof. They presented three different behaviors as soon as they saw the proposition. Some of the PSMTs read the proposition as it was written, while others expressed the proposition with their own sentences. As a third behavior, some PSMTs skipped the stage of reading or explaining the proposition, and attempted directly to make comments and produced ideas considering what is given and asked in the proposition. It is also determined that PSMTs had problems to interpret the statement since they were not able correctly perceive and understand the mathematical notation (especially the division sign) as regards to what kind of elements the given set includes.

Considering the PSMTs' opinions regarding the construction of proof, it is found that PSMTs produced right ideas with no mistakes. Some of the PSMTs reasoned on the set A_n or the number d only, while others produced ideas related to the mechanism of proof, as well. It is also examined that there are more verbal statements than statements related to mathematical expressions, but majority of PSMTs had difficulties in making a consistent, logical and integrated reasoning and reflecting this into their statements. Besides, PSMTs were not able to conclude the proof correctly since they had method, concept and language related problems.

In regards to mathematical representations, PSMTs constructed the first representations that came to their minds related to the set A_d or the union set rather than establishing a general strategy to construct the proof. PSMTs, who got stuck and were not able to use mathematical representations and verbal statements to construct the proof, preferred to use examples or were guided to use examples by the researchers. It is also found that the PSMTs used examples without any guidance aimed to interpret, understand and concretize what is given in the proposition. In particular, a substantial part of PSMTs preferred to verify the proof by assigning numerical values to the statement. It is also revealed that PSMTs who tried to construct examples with guidance of researchers were not successful enough to construct proper example(s), but perceived the connections between the given statements with the guidance of researchers.

One of the main findings of the study is that almost all PSMTs (except for PT-1) were not successful throughout the process. Although their incorrect reasoning and approaches were related to various individual reasons, it is possible to suggest some common reasons. These are failing to know where to getting started on a proof, prejudice towards construction of proof, feeling uncomfortable when constructing a proof, lack of knowledge related to mathematical language and notation, method, concept and communication related problems in the proving process, and lack of content and strategy knowledge regarding the proof. It is observed that some PSMTs' statements reflect these common reasons while they were explaining their reasoning about the proof. For example, PT-4 emphasized that he does not know where to start the proof and said, "*I think I don't know where to start. But even if you tell me where to start, I couldn't do it again, because I don't know how to proceed*". Moreover, the same pre-service teacher revealed his lack of methodological, conceptual and strategic knowledge about the proof by indicating that he did not know how to proceed with the proof.

It is also inferred that PSMTs preferred to use the thoughts that immediately came to their mind to reach the end of the proof rather than using a certain strategy in the proving process. Furthermore, it can be argued that PSMTs had difficulties to explain their opinions, and their thoughts as well as opinions regarding the proof is not based on logical, multi-perspective and essential background knowledge. Although the proposition shows that the elements of given set are p prime numbers [like A_n={p prime : …] and pre-service teachers stated it verbally, it is observed that they did not consider this basic knowledge enough in what they did throughout the process. Also, it can be argued that pre-service teachers did not properly use their existing knowledge which could be used to construct the proof. All these difficulties are directly related to the fact that PSMTs were did not adequately used the examples to construct the proof.

Considering the existing literature, it is found that that the PSMTs presented similar behaviors with participants in other studies. For example, we found similar findings to Moore (1990, 1994) and Sarı Uzun and Bülbül (2013), undergraduate students (freshmen and even seniors) in their studies did not have an adequate understanding in getting started on a mathematical proof, comprehending the concepts of proof and constructing the proof as the pre-service teachers who participated in our study. Consistent with Moore's (1990) doctoral dissertation, we found that PSMTs in our study had difficulties to reach the result of proof as they did not know how to get started on a proof, failed to use language and notation correctly and construct their own examples. The PSMTs also revealed similar behaviors to the participants in Baker and Campbell (2004), Remillard (2010), Segal (1999), and Sarı Uzun and Bülbül's (2013) studies.

In this study, similar to the participants in Knapp (2005), Remillard (2010), and Weber (2006), it is found that PSMTs had problems to apply mathematical language and notations related to the proof to reach the result in the proving process, and they had method-related problems since they did not have sufficient knowledge about the content and strategy. In other words, these PSMTs had difficulties in reaching the result since they failed to examine with a logical reasoning which way they will pursue during the process. For example, as in the studies of Köğce (2013) and Demiray (2013), pre-service teachers preferred to verify by assigning numerical values to a statement in order to construct a proof. However, it is found that these PSMTs were not successful enough in constructing (as the participants in Güler, Kar, Öçal and Çiltaş's (2011) study) and making use of example(s), but perceived the connections between the given statements with the guidance of researchers.

Even though PSMTs had knowledge to construct the proof, they were not able to construct and complete the proof since they did not have complete and adequate understanding of the proof and proving process. In his study, Weber (2001) examined college students' ability to construct and complete a proof related to group homomorphisms and isomorphisms in an abstract algebra course, and reported that the students failed to construct the proof and had difficulties in the process despite the fact that they had necessary and enough knowledge required to construct proofs as the PSMTs in this study. In another study, Doruk and Kaplan (2013) identified that participants were not able to evaluate and construct the proof, although they were successful in the proof-related course and knew the theorems. As in the studies of Weber (2001) and Doruk and Kaplan (2013), the PSMTs did not present any lack of important knowledge about prime numbers, sets, unions in sets, different concepts and representations of sets, but it is found that none of the pre-service teacher managed to construct the proof completely. The reasons of this situation can be listed as misapplication of notations (Baker & Campbell, 2004; Knapp, 2005; Moore, 1990, 1994; Segal, 1999; Remillard, 2010; Sarı Uzun & Bülbül, 2013), misunderstanding of proof (Knuth & Elliot, 1997) the state of not knowing how to get started on a proof (Segal, 1999; Moore, 1994), using examples insufficiently or deficiently (Köğce, 2013), pursuing an inadequate or deficient method to construct the proof (Knapp, 2005; Remillard, 2010; Weber, 2006) or failing to define logical structure of the statements in the theorems (Selden & Selden, 1995). To sum up, it is examined that the findings of this study are similar to the existing studies' findings, and main difficulties that the pre-service teachers had in the proving process are based on similar reasons that existing studies indicated.

IMPLICATIONS AND FUTURE DIRECTIONS

Based on the findings of this study, we could suggest that pre-service teachers should be given and taught more proof throughout their undergraduate studies. In this process, an education that is offered through activities with active participation of pre-service teachers rather than faculty members will be more helpful. Moreover, it is necessary to ensure that future teachers should not see the proof as a ritual that is offered at certain stages of their study or a problem that they are asked during the exams, and they should realize that proof is one of the most important elements of mathematics and mathematics education. Thus, they could be encouraged to motivate themselves to make more effort in reading, understanding and constructing a proof. One of the things that can be done in that regard is that pre-service teachers should be asked to thoroughly examine some basic books/reports about the teaching of proof, and given the opportunity to work on them. Accordingly, it will be helpful to share and discuss with the pre-service teachers the reports and research results published by some leading education institutions [e.g. NCTM (National Council of Teachers of Mathematics), NCETM (National Centre for Excellence in the Teaching of Mathematics), ERME (European Society for Research in Mathematics Education), AMS (American Mathematical Society) PME (The International Group for the Psychology of Mathematics Education)].

Although the foreign literature has many studies on this subject, our country has a limited number of studies related to the proof and proving in this field. Therefore, more studies should be conducted related to pre-service teachers' ability of understanding, learning, using, constructing, and completing the proof. Findings of these studies should be evaluated and reported comparatively. In our study, direct observation of PSMTs and their verbal statements regarding what they did on the blackboard enabled that the findings are more detailed and the analysis is performed in categories. Thus, our recommendation to the researchers who aim to conduct a study in this field is that they should make a plan in a way to observe student behaviors directly and interactively in the data collection process (even including the teacher-student and student-student interaction). Although only one proof problem is used in this study, we managed to thoroughly analyze the pre-service teachers' behaviors in the proving process (patterns, categorical structure of their behaviors etc.). In future studies, researchers can follow a similar data collection process. They could use only one proof problem but is repeated regularly at certain intervals (e.g. practice 4-6), and thus participants' general ideas and changing patterns of the behaviors related to the ability of understanding, learning, using, constructing, and completing the proof can be examined.

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