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Solving a Sequence of Recurrence Relations for First-Order Differential Equations

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ABSTRACT

A whole category of engineering and economic problems can be reduced to solving a set of differential equations. Downsides of known approaches for their solutions include limited accuracy numerical methods with stringent requirements for computational power. A direct analytical solution should be derived to eliminate such flaws. This research intends to derive such a solution for an *n*-dimensional set of recurrence relations for first-order differential equations, linearly dependent on the right side. The research methodology relies on successive integration of the considered set in view of the initial conditions. The overall solution was derived as a sum of products of exponential multipliers with constant coefficients that are defined through weights of a tree graph, which is a descriptor of successive integration. An analytical solution for an *n*-dimensional set of recurrent differential equations in view of the initial conditions has been derived for the first time in this research.

Keywords: analytical condition, differential equations, differentiation, initial conditions

INTRODUCTION

A whole category of engineering and economic problems (heat engineering, transport, information, technical and economic optimization problems, etc.) can be reduced to a set of recurrence relations of first-order differential equations, linearly dependent on the right side.

$$\begin{cases} y'_1 = A_1 \cdot y_1 + (\alpha_1 \cdot y_0 + \beta_1); \\ y'_2 = A_2 \cdot y_2 + (\alpha_2 \cdot y_1 + \beta_2); \\ y'_3 = A_3 \cdot y_3 + (\alpha_3 \cdot y_2 + \beta_3); \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y'_N = A_N \cdot y_N + \left(\alpha_N \cdot y_{(N-1)} + \beta_N\right); \end{cases} \tag{1}$$

where:

$A_1, \dots, A_N, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$ are constants; $y'_1, \dots, y'_N$ are first-degree derivatives, that is, $\frac{dy_1}{dx}, \dots, \frac{dy_N}{dx}$.

These problems include thermodynamic analysis models (Sheng & Tu, 2013; Bolatturk, 2006; Kalema et al., 2008; Stepanov et al., 2000; Valero, 2014; Torío et al., 2009; Romero & Linares, 2014), heat power system models (Goryachikh et al., 2010; Kicsiny, 2014; Bau et al., 2015; Kicsiny, 2017), combustion models (Messerle et al., 2013; Messerle et al., 2014; Messerle et al., 2016; ; Messerle et al., 2016; Gao et al., 2010; Karpenko et al., 2016), local heat exchange system models (e.g., solar collectors (Kaminski & Krzyzynski, 2016; Hussain et al., 2016; Bég et al., 2016; Li & Chen, 2008; Smith et al., 2012; Thianpong et al., 2012), etc.

Sets of homogeneous and non-homogeneous differential equations are currently one of the basic tools for mathematical modeling of physical processes (Kicsiny, 2014). A set of homogeneous differential equations for adsorption chiller control is discussed by Bau et al. (2015). A solution is considered as a gradient process and no direct linear dependence is given. Richard Kicsiny discusses modern mathematical models of a piped heat system

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Contribution of this paper to the literature

- In contrast to the known methods for solving systems of first-order differential equations, this article has obtained a direct analytic dependence for the n-dimensional system.
- The resulting solution is easily integrated and differentiated, since it has only exponential coefficients.
- The resulting solution can be used in more complex mathematical models of physical processes as an integral part.
- The given example of a solution for the n-dimensional system of cooling sections of a heat network showed the simplicity of applying a direct analytical solution.

in his latest studies (Kicsiny, 2014; Kicsiny, 2017) based on differential equations of the Newton's law for pipe cooling. A rather common problem is building dynamic models of municipal heat networks to improve their efficiency. These usually rely on a set of differential equations and their solution is reduced to numerical solutions (Kicsiny, 2016; Stevanovic et al., 2009). Alternatively, a significant simplification of the model is considered via one or two equations for mean parameters, which are then solved using known dependences (Polyanin & Zaitsev, 2003).

Sets of homogeneous and non-homogeneous differential equations underlie the theory to describe combustion processes. For example, sets of differential equations with hundreds of such equations are made when building reliable mathematical models of gasification of solid fuel particles (Karpenko & Trusov, 1995). These sets are usually reduced by the finite-difference method to trivial equations, which leads to a relative loss of accuracy and no direct analytical solutions exist (Messerle et al., 2016).

The increasing use of non-conventional and recoverable sources of energy in the world shaped a separate branch of the theory of mathematical modeling of solar collector operations. Modeling of solar collectors is usually reduced to solving individual differential equations or sets of such equations. Kicsiny improved multiple linear regressions for solving differential equations that simulate a solar collector (Kicsiny, 2016). Kaminski & Krzyzynski presented a numerical and experimental study of a flat solar collector (Kaminski & Krzyzynski, 2016). At the same time, hydrodynamics and heat exchange in the collector panel are defined by transforming differential equations using the finite-difference method. Saleh et al. described a mathematical model for modeling transient processes occurring in the heated fluid of a flat collector (Saleh, 2015). This model relies on a solution of a set of linked differential equations describing heat processes of glazing, air gap, fluid, etc. The set of differential solutions is solved iteratively in the MATLAB environment. Most modeling problems are reduced to a numerical solution of a set of differential equations in this or similar system (Etter et al., 2004). Modeling projects of hybrid solar photoelectric heat systems (based on numerical solutions) can serve as an example of such studies (Khelifa et al., 2016; Gholampour & Ameri, 2015; Khelifa et al., 2015). The major drawback of such numerical solutions is that they cannot be used in more complex models.

Main downsides of known approaches to solving single differential equations or sets of such equations in applied mathematical modeling problems for physical processes are discussed in the next section. This discussion is followed by deriving a direct analytical solution of a set of recurrence relations for first-order differential equations in view of the initial conditions and by using successive integration. Then, this study considers, as a practical example, heat balance of the water in the heat system at the end of a heat network section, exposed to temperature perturbation at the beginning of such section. The last section derives conclusions and covers limitations of this study and areas of further elaboration.

BASE ANALYSIS

Obtaining a Set of Recurrence Relations

Main downsides of known approaches to solving single differential equations or sets of such equations in applied mathematical modeling problems for physical processes include numerical methods with limited accuracy and stringent requirements for computational power for multidimensional sets, as well as limited application of the derived solutions in more sophisticated models. A direct analytical solution should be derived to eliminate such flaws.

A particular solution of set (1) is usually determined by the sequence of initial distributions of the parameters $y_1, \ldots, y_N$ with $x = 0$:

$$\begin{cases} y_1 = y_1^H; \\ y_2 = y_2^H; \\ y_3 = y_3^H; \\ \dots\dots\dots \\ y_N = y_N^H. \end{cases} \qquad (2)$$

A set of recurrent integral solutions is the solution for the set of differential equations in (1).

$$\begin{cases} y_1 = e^{A_1 \cdot x} \cdot \left[C_1 + \int e^{-A_1 \cdot x} \cdot (\alpha_1 \cdot y_0 + \beta_1) \cdot dx\right]; \\ y_2 = e^{A_2 \cdot x} \cdot \left[C_2 + \int e^{-A_2 \cdot x} \cdot (\alpha_2 \cdot y_1 + \beta_2) \cdot dx\right]; \\ y_3 = e^{A_3 \cdot x} \cdot \left[C_3 + \int e^{-A_3 \cdot x} \cdot (\alpha_3 \cdot y_2 + \beta_3) \cdot dx\right]; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{(N-1)} = e^{A_{(N-1)} \cdot x} \cdot \left[C_{(N-1)} + \int e^{-A_{(N-1)} \cdot x} \cdot \left(\alpha_{(N-1)} \cdot y_{(N-2)} + \beta_{(N-1)}\right) \cdot dx\right]; \\ y_N = e^{A_N \cdot x} \cdot \left[C_N + \int e^{-A_N \cdot x} \cdot \left(\alpha_N \cdot y_{(N-1)} + \beta_N\right) \cdot dx\right]; \end{cases} \tag{3}$$

where $C_1, \dots, C_N$ are integration constants. By substituting the initial distributions of the parameters $y_1, \dots, y_N$ with $x = 0$ (2), integration constants $C_1, \dots, C_N$ are obtained:

$$\begin{cases} C_1 = y_1^H - \int e^{-A_1 \cdot x} \cdot (\alpha_1 \cdot y_0 + \beta_1) \cdot dx \Big|_{x=0}; \\ C_2 = y_2^H - \int e^{-A_2 \cdot x} \cdot (\alpha_2 \cdot y_1 + \beta_2) \cdot dx \Big|_{x=0}; \\ C_3 = y_3^H - \int e^{-A_3 \cdot x} \cdot (\alpha_3 \cdot y_2 + \beta_3) \cdot dx \Big|_{x=0}; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_{(N-1)} = y_{(N-1)}^H - \int e^{-A_{(N-1)} \cdot x} \cdot \left(\alpha_{(N-1)} \cdot y_{(N-2)} + \beta_{(N-1)}\right) \cdot dx \Big|_{x=0}; \\ C_N = y_N^H - \int e^{-A_N \cdot x} \cdot \left(\alpha_N \cdot y_{(N-1)} + \beta_N\right) \cdot dx \Big|_{x=0}. \end{cases} \tag{4}$$

By substituting expressions (4) in equations (3), a sequence of recurrence relations is obtained:

$$\begin{cases} y_1 = e^{A_1 \cdot x} \cdot \left[y_1^H - \int e^{-A_1 \cdot x} \cdot (\alpha_1 \cdot y_0 + \beta_1) \cdot dx \Big|_{x=0} + \int e^{-A_1 \cdot x} \cdot (\alpha_1 \cdot y_0 + \beta_1) \cdot dx\right]; \\ y_2 = e^{A_2 \cdot x} \cdot \left[y_2^H - \int e^{-A_2 \cdot x} \cdot (\alpha_2 \cdot y_1 + \beta_2) \cdot dx \Big|_{x=0} + \int e^{-A_2 \cdot x} \cdot (\alpha_2 \cdot y_1 + \beta_2) \cdot dx\right]; \\ y_3 = e^{A_3 \cdot x} \cdot \left[y_3^H - \int e^{-A_3 \cdot x} \cdot (\alpha_3 \cdot y_2 + \beta_3) \cdot dx \Big|_{x=0} + \int e^{-A_3 \cdot x} \cdot (\alpha_3 \cdot y_2 + \beta_3) \cdot dx\right]; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{(N-1)} = e^{A_{(N-1)} \cdot x} \cdot \left[\begin{array}{l} y_{(N-1)}^H - \int e^{-A_{(N-1)} \cdot x} \cdot \left(\alpha_{(N-1)} \cdot y_{(N-2)} + \beta_{(N-1)}\right) \cdot dx \Big|_{x=0} + \\ + \int e^{-A_{(N-1)} \cdot x} \cdot \left(\alpha_{(N-1)} \cdot y_{(N-2)} + \beta_{(N-1)}\right) \cdot dx \end{array}\right]; \\ y_N = e^{A_N \cdot x} \cdot \left[y_N^H - \int e^{-A_N \cdot x} \cdot \left(\alpha_N \cdot y_{(N-1)} + \beta_N\right) \cdot dx \Big|_{x=0} + \int e^{-A_N \cdot x} \cdot \left(\alpha_N \cdot y_{(N-1)} + \beta_N\right) \cdot dx\right]. \end{cases} \tag{5}$$

Defining a Common Pattern for the Solutions

To find a common pattern, the first three solutions are written down:

$$y_1 = e^{A_1 \cdot x} \cdot \left[y_1^H - \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1)\right] + \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1); \tag{6}$$

$$
\begin{aligned}
y_2 = e^{A_2 \cdot x} \cdot & \left[y_2^H - \left\{ \begin{array}{l} \dfrac{1}{A_1 - A_2} \cdot \left(\alpha_2 \cdot y_1^H - \alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \cdot\, \beta_1) \right) + \\ + \dfrac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) \end{array} \right\} + \right] \\
+ e^{A_1 \cdot x} \cdot & \left[\frac{1}{A_1 - A_2} \cdot \left(\alpha_2 \cdot y_1^H - \alpha_2 \cdot \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) \right) \right] + \\
+ e^{0 \cdot x} \cdot & \left[\frac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) \right];
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
y_3 = e^{A_3 \cdot x} \cdot & \left[y_3^H - \left\{ \begin{array}{l} \dfrac{1}{A_2 - A_3} \cdot \alpha_3 \cdot \left(y_2^H - \left(\begin{array}{l} \dfrac{1}{A_1 - A_2} \cdot \alpha_2 \cdot \left(y_1^H - \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \cdot\, \beta_1) \right) + \\ + \dfrac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) \end{array} \right) \right) + \\ + \dfrac{1}{A_1 - A_3} \cdot \alpha_3 \cdot \dfrac{1}{A_1 - A_2} \cdot \left(\alpha_2 \cdot y_1^H - \alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) \right) + \\ + \dfrac{1}{0 - A_3} \cdot \left[\alpha_3 \cdot \dfrac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) + \beta_3 \right] \end{array} \right\} + \right] \\
+ e^{A_2 \cdot x} \cdot & \left[\frac{1}{A_2 - A_3} \cdot \alpha_3 \cdot \left(y_2^H - \left(\begin{array}{l} \dfrac{1}{A_1 - A_2} \cdot \alpha_2 \cdot \left(y_1^H - \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \cdot\, \beta_1) \right) + \\ + \dfrac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \dfrac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) \end{array} \right) \right) + \right] + \\
+ e^{A_1 \cdot x} \cdot & \left[\frac{1}{A_1 - A_3} \cdot \alpha_3 \cdot \frac{1}{A_1 - A_2} \cdot e^{A_1} \cdot \left(\alpha_2 \cdot y_1^H - \alpha_2 \cdot \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) \right) \right] + \\
+ e^{0 \cdot x} \cdot & \frac{1}{0 - A_3} \cdot \left[\alpha_3 \cdot \frac{1}{0 - A_2} \cdot \left(\alpha_2 \cdot \frac{1}{0 - A_1} \cdot (\alpha_1 \cdot y_0 + \beta_1) + \beta_2 \right) + \beta_3 \right].
\end{aligned}
\tag{8}
$$

Deriving a General Solution

An easy way to present a general solution is:

$$
y_N = e^{A_N \cdot x} \cdot y_N^H + \sum_{S=0}^{N-1} [(e^{A_S \cdot x} - e^{A_N \cdot x}) \cdot \gamma_S^P], \tag{9}
$$

where γ_S^P is a set of products of given constants $A_1, \dots, A_N$, $\alpha_1, \dots, \alpha_N$, $\beta_1, \dots, \beta_N$ and the initial conditions.

Values γ_S^P can be obtained by recurrent substitution and successive integration. Since successive integration of equations (3) gives successive products, such products can be interpreted in the form of a tree graph (**Figure 1**).

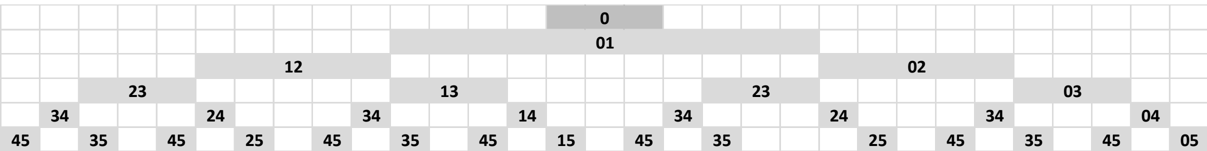

Figure 1. A standard tree graph

This stairstep presentation (**Figure 2**) gives a clear interpretation of a possible procedure of finding γ_S^P. Sum of products of the left-most multipliers ψ_S^P can be denoted in the following way:

$$
\psi_S^P = \frac{1}{A_{(P-1)} - A_P} \cdot \alpha_P \cdot \sum_{S=0}^{P-1} \gamma_S^{P-1}, \tag{10}
$$

where:

P is layer number $[P = 0, \dots, N]$; and

S is column number $[S = 0, \dots, N-1]$.

Then, γ_S^P can be found in the following way:

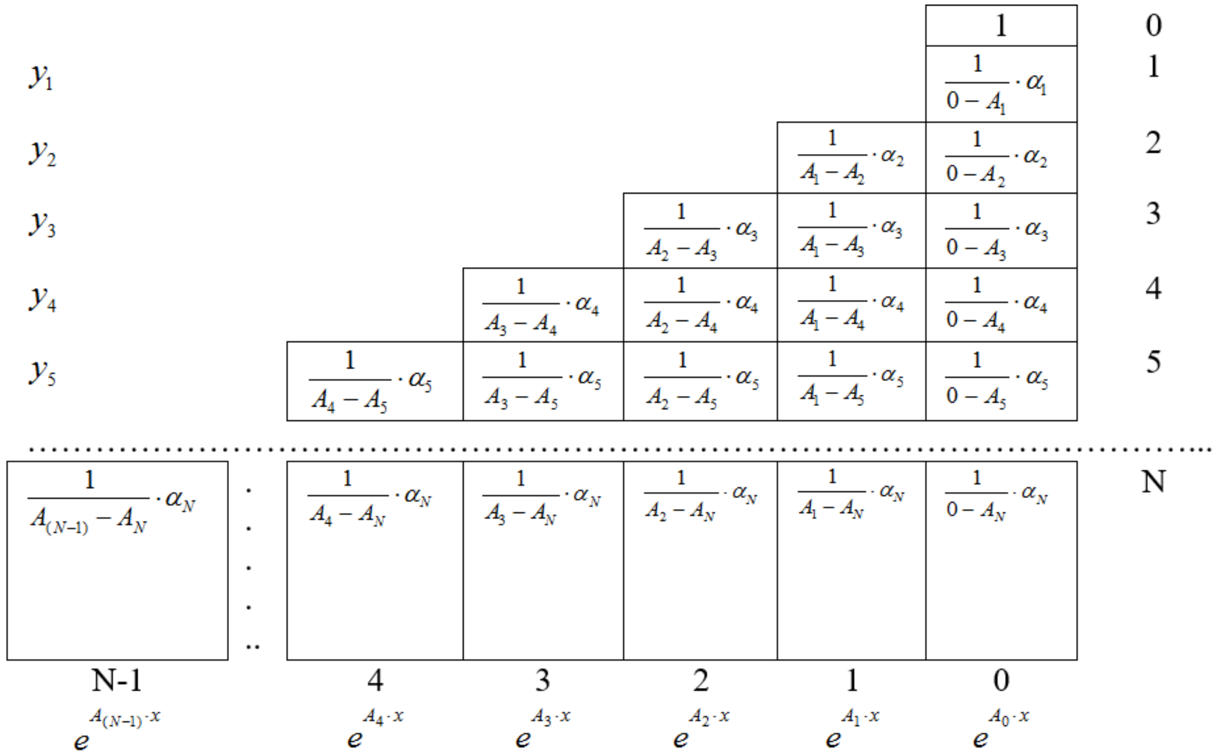

Figure 2. A stairstep tree graph

$$\gamma_S^{P=N} = \psi^{P=S+1} \cdot \prod_{R=(S+1)+1}^{N} \left[\frac{1}{A_S - A_R} \cdot \alpha_R\right]. \tag{11}$$

ψ_S^P are not known in (11). Finding of these unknown quantities is reduced to defining a sum of products of tree graph weights (**Figure 2**) in a certain layer. ψ_S^P can be written for some layers in the following way:

$$\psi_0^1 = \frac{1}{0 - A_1} \cdot \alpha_1; \tag{12}$$

$$\psi_1^2 = \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{A_1 - A_2} \cdot \alpha_2; \tag{13}$$

$$\psi_2^3 = \left[\frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{A_1 - A_2} \cdot \alpha_2 + \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{0 - A_2} \cdot \alpha_2\right] \cdot \frac{1}{A_2 - A_3} \cdot \alpha_3; \tag{14}$$

$$\psi_3^4 = \frac{1}{A_3 - A_4} \cdot \alpha_4 \cdot \begin{bmatrix} \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{A_1 - A_2} \cdot \alpha_2 \cdot \frac{1}{A_2 - A_3} \cdot \alpha_3 + \\ + \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{A_1 - A_2} \cdot \alpha_2 \cdot \frac{1}{A_1 - A_3} \cdot \alpha_3 + \\ + \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{0 - A_2} \cdot \alpha_2 \cdot \frac{1}{A_2 - A_3} \cdot \alpha_3 + \\ + \frac{1}{0 - A_1} \cdot \alpha_1 \cdot \frac{1}{0 - A_2} \cdot \alpha_2 \cdot \frac{1}{0 - A_3} \cdot \alpha_3 \end{bmatrix}. \tag{15}$$

In general, a sum of products of tree graph weights in a certain layer can be found in the following way:

$P = 1$:

$$\psi_0^1 = \prod_{i=1}^{P} \left[\frac{1}{A_{1-1} - A_i} \cdot \alpha_i\right], \tag{16}$$

where A_0=0.

$P=2$:

$$\psi_1^2 = \begin{cases} \prod\limits_{i=1}^{P}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=1}^{P-1}\left(\prod\limits_{i_1=1}^{j}\left[\frac{1}{A_{j-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{P}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)\end{cases}. \tag{17}$$

$P=3$:

$$\psi_2^3 = \begin{cases} \prod\limits_{i=1}^{P}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=1}^{P-1}\left(\prod\limits_{i_1=1}^{j}\left[\frac{1}{A_{j-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{P}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum\limits_{j_1=1}^{P-2}\sum\limits_{j_2=j_1+1}^{P-1}\left(\prod\limits_{i_1=1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod\limits_{i_3=i_2+1}^{P}\left[\frac{1}{A_{j_2-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)\end{cases}. \tag{18}$$

$P=4$:

$$\psi_3^4 = \begin{cases} \prod\limits_{i=1}^{P}\left[\frac{1}{A_{i-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=1}^{P-1}\left(\prod\limits_{i_1=1}^{j}\left[\frac{1}{A_{j-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{P}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum\limits_{j_1=1}^{P-2}\sum\limits_{j_2=j_1+1}^{P-1}\left(\prod\limits_{i_1=1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod\limits_{i_3=i_2+1}^{P}\left[\frac{1}{A_{j_2-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)+ \\ +\sum\limits_{j_1=1}^{P-3}\sum\limits_{j_2=j_1+1}^{P-2}\sum\limits_{j_3=j_2+1}^{P-1}\left(\begin{array}{l}\prod\limits_{i_1=1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdot\prod\limits_{i_3=i_2+1}^{j_3}\left[\frac{1}{A_{j_2-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\cdot\prod\limits_{i_4=i_3+1}^{P}\left[\frac{1}{A_{i_4-1}-A_{i_4}}\cdot\alpha_{i_4}\right]\end{array}\right)\end{cases}. \tag{19}$$

Then, generalization for $P=N$ gives:

$$\psi_{N-1}^N = \left.\begin{cases} \prod\limits_{i=1}^{N}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=1}^{N-1}\left(\prod\limits_{i_1=1}^{j}\left[\frac{1}{A_{j-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{N}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum\limits_{j_1=1}^{N-2}\sum\limits_{j_2=j_1+1}^{N-1}\left(\prod\limits_{i_1=1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod\limits_{i_3=i_2+1}^{N}\left[\frac{1}{A_{j_2-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\underbrace{\begin{array}{l}\sum\limits_{j_1=1}^{N-(N-1)}\sum\limits_{j_2=j_1+1}^{N-(N-2)}\cdots\cdots\cdots\cdots \\ \sum\limits_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\sum\limits_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\sum\limits_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}\end{array}}_{N-1}\left(\underbrace{\begin{array}{l}\prod\limits_{i_1=1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot\prod\limits_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}\left[\frac{1}{A_{i_{(N-2)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod\limits_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_N-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N}\right)\end{cases}\right\} N. \tag{20}$$

Deriving a Solution in View of Constituents of Initial and Steady-states

In order to define a general solution for (9) for layer N, γ_S^N can be divided into three constituents:

$$\gamma_S^N = \gamma_S^N(y_0) + \gamma_S^N(y^H) + \gamma_S^N(\beta). \tag{21}$$

Since extra β_i is introduced for $\gamma_S^N(\beta)$ for each layer of a tree graph, a notation for a sum of products of tree graph weights in a certain layer must be introduced, while staring from the given layer and not from the first layer:

$\boldsymbol{\psi}_{P=1}^{P=R}$ is a sum of products of tree graph weights in a certain layer $P = R$, starting from the first layer $P = 1$ (general definition (20));

$\boldsymbol{\psi}_{P=1}^{P=R_2}$ is a sum of products of tree graph weights in a certain layer $P = R_2$, starting from the given layer $P = R_1$.

The sum of products of tree graph weights in a certain layer $P = N$, starting from the given layer $P = R_1$, can be defined in the following way:

$$\underset{P=R_1}{\overset{P=N}{\psi_{п}}} = \left\{ \begin{array}{l} \prod_{i=R_1}^{N}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum_{j=R_1}^{N-1}\left(\prod_{i_1=R_1}^{j}\left[\frac{1}{A_{j-1}-A_{i_1}}\cdot\alpha_j\right]\cdot\prod_{i_2=i_1+1}^{N}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum_{j_1=R_1}^{N-2}\sum_{j_2=j_1+1}^{N-1}\left(\prod_{i_1=R_1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod_{i_3=i_2+1}^{N}\left[\frac{1}{A_{j_2-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\underbrace{\sum_{j_1=R_1}^{N-(N-1)}\sum_{j_2=j_1+1}^{N-(N-2)}\cdots\cdots\sum_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\sum_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\sum_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}}_{(N-1)-R_1}\left(\underbrace{\begin{array}{l}\prod_{i_1=R_1}^{j_1}\left[\frac{1}{A_{j_1-1}-A_{i_1}}\cdot\alpha_{i1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdots\cdots\cdots\cdots\cdots \\ \cdot\prod_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}\left[\frac{1}{A_{i_{(N-2)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_N-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N-R_1}\right) \end{array}\right\} N \quad (22)$$

$$-R_1.$$

In order to define $\gamma_S^N(y^H)$, a definition of the sum of products of tree graph weights in a certain layer on the right side or on the left side must be introduced. $\boldsymbol{\psi_n}_{P=R_1}^{P=N}$ The right side is defined according to (20) and differs from the left side by the presentation of the first products. The concept of right side is defined in a stairstep graph (**Figure 2**) for $\prod_{i=R_1}^{N}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right] = \prod_{i=R_1}^{N}\left[\frac{1}{0-A_i}\cdot\alpha_i\right]$ located on the right. To define a sum of products of tree graph weighs in a certain layer for the left side, the first products can be substituted by:

$$\underset{P=1}{\overset{P=N}{\psi_{л}}} = \left\{ \begin{array}{l} \prod_{i=1}^{N}\left[\frac{1}{A_{i-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum_{j=1}^{N-1}\left(\prod_{i_1=1}^{j}\left[\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{N}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum_{j_1=1}^{N-2}\sum_{j_2=j_1+1}^{N-1}\left(\prod_{i_1=1}^{j_1}\left[\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod_{i_3=i_2+1}^{N}\left[\frac{1}{A_{j_3-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\underbrace{\sum_{j_1=1}^{N-(N-1)}\sum_{j_2=j_1+1}^{N-(N-2)}\cdots\cdots\sum_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\sum_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\sum_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}}_{N-1}\left(\underbrace{\begin{array}{l}\prod_{i_1=1}^{j_1}\left[\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdots\cdots\cdots\cdots\cdots \\ \cdot\prod_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}\left[\frac{1}{A_{j_{(N-1)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_{(N-1)}-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N}\right) \end{array}\right\} N. \quad (23)$$

Particular solutions for equation (1) given in (3) are described by a partial sign alternation. A tree graph of successive products with y_0 in view of sign alternation is given in **Figure 3**.

															00															
							-01																01							
			-12								12								-02								02			
	-23				23				-13				13				-23				23				-03				03	
-34		34		-24		24		-34		34		-14		14		-34		34		-24		24		-34		34		-04		04

Figure 3. A standard tree graph with signs distributed

The following notation can be introduced for further presentation:

$$\prod_{i=n}^{N}{}^{[f(i)]}[F(i)] = \begin{cases} f(i), & at \quad N = n; \\ f(i) \cdot \prod\limits_{i=n+1}^{N}[F(i)], & at \quad N > n. \end{cases} \tag{24}$$

Then, a sum of products of tree graph weights (**Figure 3**) in a certain layer with view to sign alternation and left-side bypass is:

$$\underset{P=0}{\overset{P=N}{\psi_{\text{л}}}} = \begin{cases} \prod\limits_{i=0}^{N}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=0}^{N-1}\left(\prod\limits_{i=0}^{j}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{N}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum\limits_{j_1=0}^{N-2}\sum\limits_{j_2=j_1+1}^{N-1}\left(\begin{array}{l}\prod\limits_{i_1=0}^{j_1}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdot\prod\limits_{i_3=i_2+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_1}-A_{i_3}}\cdot\alpha_{i_3}\right]}\left[(-1)\cdot\frac{1}{A_{i_3-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\end{array}\right)+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\underbrace{\begin{array}{l}\sum\limits_{j_1=0}^{N-(N-1)}\sum\limits_{j_2=j_1+1}^{N-(N-2)}\cdots\cdots\cdots\cdots \\ \sum\limits_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\;\sum\limits_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\;\sum\limits_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}\end{array}}_{N-1}\left(\underbrace{\begin{array}{l}\prod\limits_{i_1=0}^{j_1}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot\prod\limits_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}{}^{\left[(-1)\cdot\frac{1}{A_{i_{(N-1-2)}}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]}\left[(-1)\cdot\frac{1}{A_{j_{(N-1)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod\limits_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_{(N-1)}-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N}\right)\end{cases} \tag{25}$$

Considering (25), a sum of products of tree graph weights (**Figure 3**) in a certain layer with view to sign alternation and right-side bypass is:

$$\underset{P=0}{\overset{P=N}{\psi_{\text{п}}}} = \begin{cases} \prod\limits_{i=0}^{N}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]+ \\ +\sum\limits_{j=0}^{N-1}\left(\prod\limits_{i=0}^{j}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]\cdot\prod\limits_{i_2=i_1+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right)+ \\ +\sum\limits_{j_1=0}^{N-2}\sum\limits_{j_2=j_1+1}^{N-1}\left(\prod\limits_{i=0}^{j_1}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]\cdot\prod\limits_{i_2=i_1+1}^{j_2}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot\prod\limits_{i_2=i_1+1}^{N}\left[\frac{1}{A_{i_2}-A_{i_3}}\cdot\alpha_{i_3}\right]\right)+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\underbrace{\begin{array}{l}\sum\limits_{j_1=0}^{N-(N-1)}\sum\limits_{j_2=j_1+1}^{N-(N-2)}\cdots\cdots\cdots\cdots \\ \sum\limits_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\;\sum\limits_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\;\sum\limits_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}\end{array}}_{N-1}\left\{\underbrace{\begin{array}{l}\prod\limits_{i=0}^{j_1}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]\cdot \\ \cdot\prod\limits_{i_2=i_1+1}^{j_2}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot\prod\limits_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}{}^{\left[(-1)\cdot\frac{1}{A_{i_{(N-1-2)}}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]}\left[(-1)\cdot\frac{1}{A_{j_{(N-1)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod\limits_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_{(N-1)}-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N}\right\}\end{cases} . \tag{26}$$

The sum of products of tree graph weights in a certain layer $P = N$, starting from given layer $P = R_1$, in view of signs and right-side bypass is defined in a way similar to (22):

$$\underset{P=R_1}{\overset{P=N}{\psi_{\text{П}}}} = \left\{ \begin{array}{l} \prod_{i=R_1-1}^{N}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right] + \\ + \sum_{j=R_1-1}^{N-1}\left(\prod_{i=R_1-1}^{j}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]\cdot\prod_{i_2=i_1+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\right) + \\ + \sum_{j_1=R_1-1}^{N-2}\sum_{j_2=j_1+1}^{N-1}\left(\begin{array}{l}\prod_{i_1=R_1-1}^{j_1}{}^{[1]}\left[\frac{1}{A_{1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdot\prod_{i_3=i_2+1}^{N}\left[\frac{1}{A_{i_2}-A_{i_3}}\cdot\alpha_{i_3}\right]\end{array}\right) + \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \underbrace{\sum_{j_1=R_1-1}^{N-(N-1)}\sum_{j_2=j_1+1}^{N-(N-2)}\ldots\ldots\sum_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\sum_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\sum_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}}_{(N-1)-R_1}\left(\underbrace{\begin{array}{l}\prod_{i=R_1-1}^{j_1}{}^{[1]}\left[\frac{1}{A_{1-1}-A_i}\cdot\alpha_i\right]\cdot \\ \cdot\prod_{i_2=i_1+1}^{j_2}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]}\left[(-1)\cdot\frac{1}{A_{i_2-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \cdot\prod_{i_{(N-1)}=i_{(N-2)}+1}^{j_{(N-1)}}{}^{\left[(-1)\cdot\frac{1}{A_{i_{(N-1-2)}}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]}\left[(-1)\cdot\frac{1}{A_{j_{(N-1)}-1}-A_{i_{(N-1)}}}\cdot\alpha_{i_{(N-1)}}\right]\cdot \\ \cdot\prod_{i_N=i_{(N-1)}+1}^{N}\left[\frac{1}{A_{i_{(N-1)}-1}-A_{i_N}}\cdot\alpha_{i_N}\right]\end{array}}_{N-R_1}\right)\end{array}\right. . \quad (27)$$

The condition of sign alternation (**Figure 3**) directs that the equation for $\gamma_S^N(y_0)$ (9) comprises $\boldsymbol{\psi_n}_{P=0}^{S+1}$ as the sum of products of the tree graph weights in a certain layer solely with negative last multipliers, instead of all products. Extra notations must be introduced to account for this condition:

$\boldsymbol{\psi_n}_{P=R_1}^{P=N_{(-)}}$ is products of tree graph weights in a certain layer $P = N$, starting from given layer $P = R_1$, in view of signs and left-side bypass with the products having negative (–) last multipliers:

$$\underset{P=R_1}{\overset{P=N}{\psi_{\text{Л}}^{(-)}}} = \left\{ \begin{array}{l} \prod_{i_1=R_1-1}^{N}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right] + \\ + \sum_{j_1=R_1-1}^{N-2}\sum_{j_2=j_1+1}^{N-1}\left(\begin{array}{l}\prod_{i_1=R_1-1}^{j_1}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \cdot\prod_{i_3=i_2+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_1}-A_{i_3}}\cdot\alpha_{i_3}\right]}\left[(-1)\cdot\frac{1}{A_{i_3-1}-A_{i_3}}\cdot\alpha_{i_3}\right]\end{array}\right) + \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \underbrace{\sum_{j_1=R_1-1}^{N-(N-1)}\sum_{j_2=j_1+1}^{N-(N-2)}\ldots\ldots\sum_{j_{(N-3)}=j_{(N-3)}+1}^{N-3}\sum_{j_{(N-2)}=j_{(N-3)}+1}^{N-2}\sum_{j_{(N-1)}=j_{(N-2)}+1}^{N-1}}_{(N-1)-R_1}\left(\underbrace{\begin{array}{l}\prod_{i_1=R_1-1}^{j_1}{}^{[1]}\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_1}}\cdot\alpha_{i_1}\right]\cdot\prod_{i_2=i_1+1}^{j_2}\left[\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]\cdot \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \cdot\prod_{i_{(N)}=i_{(N-1)}+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_{(N-2)}}-A_{i_{(N)}}}\cdot\alpha_{i_{(N)}}\right]}\left[(-1)\cdot\frac{1}{A_{j_{(N)}-1}-A_{i_{(N)}}}\cdot\alpha_{i_{(N)}}\right]\cdot\end{array}}_{N-R_1}\right)\end{array}\right. . \quad (28)$$

$\boldsymbol{\psi_n}_{P=R_1}^{P=N_{(-)}}$ is products of tree graph weights in a certain layer $P = N$, starting from given layer $P = R_1$, in view of signs and right-side bypass with the products having negative (–) last multipliers:

$$\underset{P=R_1}{\overset{P=N}{\psi_{\text{п}}^{(-)}}} = \left\{ \begin{array}{l} + \sum_{j=R_1-1}^{N-1} \left(\prod_{i=R_1-1}^{j}{}^{[1]} \left[\frac{1}{A_{1-1} - A_i} \cdot \alpha_i \right] \cdot \prod_{i_2=i_1+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]} \left[(-1) \cdot \frac{1}{A_{i_2-1} - A_{i_2}} \cdot \alpha_{i_2} \right] \right) + \\ + \sum_{j_1=R_1-1}^{N-3} \sum_{j_2=j_1+1}^{N-2} \sum_{j_3=j_{21}+1}^{N-1} \left(\begin{array}{l} \prod_{i_1=R_1-1}^{j_1}{}^{[1]} \left[\frac{1}{A_{1-1} - A_{i_1}} \cdot \alpha_i \right] \cdot \prod_{i_2=i_1+1}^{j_2}{}^{\left[(-1)\cdot\frac{1}{A_{i_1-1}-A_{i_2}}\cdot\alpha_{i_2}\right]} \left[(-1) \cdot \frac{1}{A_{i_2-1} - A_{i_2}} \cdot \alpha_{i_2} \right] \cdot \\ \cdot \prod_{i_3=i_3+1}^{J_3} \left[\frac{1}{A_{i_2} - A_{i_3}} \cdot \alpha_{i_3} \right] \cdot \prod_{i_4=i_3+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_3-1}-A_{i_4}}\cdot\alpha_{i_4}\right]} \left[(-1) \cdot \frac{1}{A_{i_4-1} - A_{i_4}} \cdot \alpha_{i_4} \right] \end{array} \right) + \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \underbrace{\begin{array}{l} \sum_{j_1=R_1-1}^{N-(N-1)} \sum_{j_2=j_1+1}^{N-(N-2)} \cdots\cdots\cdots\cdots \\ \sum_{j_{(N-3)}=j_{(N-3)}+1}^{N-3} \sum_{j_{(N-2)}=j_{(N-3)}+1}^{N-2} \sum_{j_{(N-1)}=j_{(N-2)}+1}^{N-1} \end{array}}_{(N-1)-R_1} \left(\underbrace{\begin{array}{l} \prod_{i_1=R_1-1}^{j_1}{}^{[1]} \left[(-1) \cdot \frac{1}{A_{i_1-1} - A_{i_1}} \cdot \alpha_{i_1} \right] \cdot \prod_{i_2=i_1+1}^{j_2} \left[\frac{1}{A_{i_1-1} - A_{i_2}} \cdot \alpha_{i_2} \right] \cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot \prod_{i_{(N)}=i_{(N-1)}+1}^{N}{}^{\left[(-1)\cdot\frac{1}{A_{i_{(N-2)}}-A_{i_{(N)}}}\cdot\alpha_{i_{(N)}}\right]} \left[(-1) \cdot \frac{1}{A_{j_{(N)}-1} - A_{i_{(N)}}} \cdot \alpha_{i_{(N)}} \right] \cdot \end{array}}_{N-R_1} \right) \end{array} \right. . \quad (29)$$

Then, considering (9) and (28-29), constituents γ_S^N can be defined in the following way:

$$\gamma_S^N(y_0) = y_0 \cdot \left(\underset{P=0}{\overset{S}{\psi_{\text{л}}^{(-)}}} \cdot \prod_{R=S+1}^{N} \left[\frac{1}{A_S - A_R} \cdot \alpha_R \right] \right); \quad (30)$$

$$\gamma_S^N(\beta) = \sum_{I=1}^{S} \left[\frac{\beta_I}{\alpha_I} \cdot \underset{P=I-1}{\overset{S}{\psi_{\text{п}}^{(-)}}} \cdot \prod_{R=S+1}^{N} \left(\frac{1}{A_S - A_R} \cdot \alpha_R \right) \right]; \quad (31)$$

$$\gamma_S^N(y^H) = \sum_{I=1}^{S} \left[y_I^{\text{H}} \cdot \underset{P=I}{\overset{S}{\psi_{\text{л}}^{(-)}}} \cdot \prod_{R=S+1}^{N} \left(\frac{1}{A_S - A_R} \cdot \alpha_R \right) \right]. \quad (32)$$

Considering (30-32), the solution of the initial set (1) is:

$$y_N = e^{A_N \cdot x} \cdot y_N^H + \sum_{S=0}^{N-1} [(e^{A_S \cdot x} - e^{A_N \cdot x}) \cdot (\gamma_S^N(y_0) + \gamma_S^N(y^H) + \gamma_S^N(\beta))]. \quad (33)$$

One should mention a special case of constituents γ_S^N with $S = 0$:

$$\gamma_0^N(y_0) = y_0 \cdot \prod_{I=1}^{N} \left(\frac{1}{A_0 - A_I} \cdot \alpha_I \right); \quad (34)$$

$$\gamma_0^N(\beta) = \sum_{I=1}^{S} \left[\frac{\beta_I}{\alpha_I} \cdot \prod_{R=I}^{N} \left(\frac{1}{A_0 - A_R} \cdot \alpha_R \right) \right]; \quad (35)$$

$$\gamma_0^N(y^H) = 0. \quad (36)$$

DISCUSSION

The overall solution derived for an n-dimensional system matches known results for a case of a set that comprises of two or three differential equations [30]. This presentation of the solution is convenient for immediate use and for use in more complicated mathematical models. One should mention that the solution derived is reliable, provided denominators of coefficients $\frac{1}{A_i - A_j}$ are not zero. In other words, the following extra conditions must be imposed on the derived solutions:

$$(A_i - A_j) \neq 0. \quad (37)$$

An example of practical application of set (1) can be heat balance of heat system water flow at the end of a heat network section exposed to temperature perturbation at the beginning of such section. This balance can be written down in the following way:

$$V_1 \cdot c_P \cdot \rho_B \cdot dt_1 = -v_1 \cdot c_P \cdot \rho_B \cdot (t_1 - t_1^y) \cdot d\tau, \quad (38)$$

where:

c_P is heat system water capacitance [KJ/kg °C];

ρ_B is heat system water density [kg/m³];

t_1 is water temperature at the end of a heat network section at a certain time point $d\tau$ [°C];

t_1^y is water temperature at the end of a heat network section at a certain time point $\tau \to \infty$ [°C];

V_1 is heat network section volume [m³]; and

v_1 is volume water flow at a section [m³/sec].

Heat losses at the section can be accounted for by writing down a heat balance equation in steady-state:

$$\left(t_0 - t_1^{\text{y}}\right) \cdot v_1 \cdot \text{c}_\text{P} \cdot \rho_\text{B} = K_1 \cdot \pi \cdot l_1 \cdot (1 + \mu_1) \cdot \left(\bar{t}_1^{\text{y}} - t_\text{н}\right), \tag{39}$$

where:

t_0 is temperature perturbation at the beginning of a heat network section [°C];

$\bar{t}_1{}^y$ is mean heat system water temperature at a heat network section [°C];

v_1 is volume water flow at a section [m³/sec];

μ_1 is coefficient of local heat losses of a heat network section;

$K_1 \cdot \pi$ are linear heat losses; and

l_1 is piping length of a heat network section.

The solution of equation (38) relative to t_1^y and in view of $\bar{t}_1^y = \frac{t_1^y + t_0}{2}$ is the following:

$$t_1^{\text{y}} = t_0 \cdot \frac{2 - \varepsilon_{\text{TC}1}}{2 + \varepsilon_{\text{TC}1}} + 2 \cdot t_\text{н} \cdot \frac{\varepsilon_{\text{TC}1}}{2 + \varepsilon_{\text{TC}1}}, \tag{40}$$

where $\varepsilon_{\text{TC}1} = \frac{K_1 \cdot \pi \cdot l_1 \cdot (1 + \mu_1)}{v_1 \cdot \text{c}_\text{P} \cdot \rho_\text{B}}$ is a dimensionless number that describes heat losses through the heat insulation of the heat network section relative to the flow that passes through. Equation (39) will then be transformed into a set identical to (1):

$$\begin{cases} t_1^{\text{y}} = t_0 \cdot \dfrac{2 - \varepsilon_{\text{TC}(1)}}{2 + \varepsilon_{\text{TC}(1)}} + 2 \cdot t_\text{н} \cdot \dfrac{\varepsilon_{\text{TC}(1)}}{2 + \varepsilon_{\text{TC}(1)}}; \\ t_2^{\text{y}} = t_1^{\text{y}} \cdot \dfrac{2 - \varepsilon_{\text{TC}(2)}}{2 + \varepsilon_{\text{TC}(2)}} + 2 \cdot t_\text{н} \cdot \dfrac{\varepsilon_{\text{TC}(2)}}{2 + \varepsilon_{\text{TC}(2)}}; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ t_{n_{\text{TC}}}^{\text{y}} = t_{n_{\text{TC}}-1}^{\text{y}} \cdot \dfrac{2 - \varepsilon_{\text{TC}(n_{\text{TC}})}}{2 + \varepsilon_{\text{TC}(n_{\text{TC}})}} + 2 \cdot t_\text{н} \cdot \dfrac{\varepsilon_{\text{TC}(n_{\text{TC}})}}{2 + \varepsilon_{\text{TC}(n_{\text{TC}})}}. \end{cases}$$

CONCLUSION

1. Contrary to well-known methods of addressing sets of first-order differential equations, this research produces a direct analytical dependence (34).
2. The solution produced integrates and differentiates easily, since it comprises of solely exponential factors.
3. Again, the solution can be used as a constituent of more sophisticated mathematical models of physical processes.
4. The solution for set (1) is constrained with extra condition $\left(A_i - A_j\right) \neq 0$.
5. This research gives an example of practical application of the solution to an n-dimensional set of cooling sections of a heat network. Easy implementation of the direct analytical solution determines high practical value of the results.
6. Areas for further elaboration of this research can include practical problems of modeling physical processes that can be solved using the solution and derivation of set (1), which is not constrained with limiting condition (37).

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