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The Development of Two High School Students' Discourses on Geometric Translation in Relation to the Teacher's Discourse in the Classroom

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ABSTRACT

Research exploring student development on geometric translation in relation to classroom teaching is scarce. We used a qualitative case study design and examined two students' development of thinking on translation in relation to the teacher's discourse in a Turkish high school classroom through a discursive framework. We examined the teacher's discourse through the video-taped classroom session in which he talked about translation. We examined the students' discursive development through three video-taped task-based interviews. Finally, we compared the students' discourses on translation with the teacher's discourse. The teacher's discourse on translation in the classroom was formal and based on an algebraic realization of the concept as an addition and a geometric transformation. The students adopted particular discursive elements used by the teacher right after instruction but later abandoned some of those and used their more established realizations of translation. The findings revealed the complexity of discursive development as the students continually and actively adjusted their discourses by taking into account the teacher's discourse while also making it compatible with their own realizations of translation. We conclude that socio-cultural and discursive approaches have the potential to shed additional light on issues regarding learning and teaching of geometric translation.

Keywords: discourse, geometric transformations, high school education, teaching and learning, translations

INTRODUCTION

Transformation geometry has increasingly been considered as one of the essential topics of the K-12 mathematics curriculum by researchers and various curricular standards (e.g., Flanagan, 2001; Jones, 2000; MoNE, 2010; NCTM, 2000). Transformation geometry provides students with opportunities to think about mathematical concepts such as similarity, congruence, symmetry, and functions (Hollebrands, 2003); helps students see mathematics as an interconnected discipline (Flanagan, 2001); and engages students in higher-level reasoning using multiple representations (Hollebrands, 2003). While learning about transformation geometry, students are expected to describe patterns, make generalizations, and develop spatial competencies (Clements, Battista, Sarama, Swaminathan, & McMillen, 1997; Portnoy, Grundmeimer, & Graham, 2006). Due to the significance of the topic in the mathematics curriculum and its role in mathematical reasoning, researchers have focused on student thinking about geometric transformations. While some researchers characterized translation as the easiest concept of transformations (Clements, 2003) and found that students were more successful in performing translation tasks compared to tasks involving reflections or rotations (Moyer, 1978; Schultz & Austin, 1983), Hollebrands (2004) characterized translation as the most challenging concept of transformations for students, particularly in contexts involving translating vectors. The discrepancy in research findings suggests that additional research is needed on

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Contribution of this paper to the literature

- We use a socio-cultural, discursive approach to examine two areas requiring further attention in the literature on geometric translations: examination of how students develop their thinking on translations, and whether and how teachers' discourses shape students' thinking of translations.
- Our work is a step towards utilizing socio-cultural theories to examine what insights can be gained from such communicational approaches regarding the teaching and learning of translations in everyday classrooms.
- Our results show that the discursive framework was useful in examining the characteristics, development, and compatibility of participants' discourses on translation and the extent to which teachers' discourses influence student development on translation.

student thinking about translations, especially when vectors are involved. Therefore, we focus on geometric translations in this paper.

As will be discussed in the next section, most studies on student thinking about translation are based on cognitive perspectives without focusing on how teachers teach the concept in everyday classrooms. The challenges students face when thinking about translation highlighted by the literature indicate that teachers can play critical roles in addressing their students' difficulties with the concept. Yet, research exploring the development of student thinking of geometric translation in relation to classroom teaching is scarce. There is a need to examine how instruction shapes students' thinking of translations and how their thinking changes over time (Yanik, 2011). In addition, the existing literature does not extensively identify how teachers' language use influences students' thinking of geometric translations (Yanik, 2011). It remains unclear how students progress from viewing translations as processes involving motion to viewing them as mathematical objects (Glass, 2001), making it necessary to examine whether and how teachers facilitate this transition. We examine these issues requiring further attention in the context of a Turkish high school classroom by using a particular socio-cultural framework: Sfard's (2008) discursive approach. We chose this approach because it underlines the importance of communication in student learning; emphasizes the examination of language in conjunction with actions; and provides analytical tools with which to explore students' development of mathematical concepts with respect to the teacher's discourse. Our main goal is to examine the discourses of the teacher and two students in this classroom using Sfard's (2008) framework to examine what insights can be gained from such a communicational approach regarding student development of translation in relation to teaching.

RESEARCH QUESTION

More specifically, we address the following question: In what ways do two students develop their discourses on the concept of translation in relation to the teacher's discourse in a Turkish high school classroom? To address the question, we first describe the characteristics of the teacher's discourse on translation. We then examine the discursive development of each student and compare their discourses with the teacher's discourse on translation. Our rationale for comparing the students' and teacher's discourses is to investigate whether, and how, the teacher's mathematical discourse in the classroom shapes students' thinking on translation.

THEORETICAL FRAMEWORK

Student Thinking on Translation

Existing research on student thinking on translation identifies abstracting the properties of translations as a challenge for students due to their overreliance on visual information (e.g., orientation, location) (Boulter & Kirby, 1994; Edwards, 1991, 1997; Hollebrands, 2004, 2007; Portnoy et al., 2006). Focusing only on visual information also affects how students think about vectors. For example, some students do not think about vectors based on direction and magnitude and confuse vectors with rays since the geometric representations of vectors and rays are similar (Flanagan, 2001; Hollebrands, 2003; Laborde, 2001). Students' difficulties with vectors can be related to their difficulties with transformations (Flanagan, 2001; Harper, 2002; Hollebrands, 2003, 2004, 2007; Laborde, 2001; Yanik, 2011; Yanik & Flores, 2009). For example, students may view a translating vector as a line of reflection without realizing its role in translations (Hollebrands, 2004; Yanik, 2011; Yanik & Flores, 2009). Students also struggle with using vectors to map a shape to its image and determining the placement of the translated shape using translating vectors (Harper, 2002; Hollebrands, 2004). Students' struggles with viewing vectors as geometric objects that translate all points in the plane (Hollebrands, 2003; Sünker & Zembat, 2012; Yanik & Flores, 2009) may be related to the challenges with viewing translations as mappings.

Students often think about translation as a dynamic process of moving geometric shapes rather than a mathematical object (e.g., function, congruence transformation, mapping) (Hollebrands, 2003, 2004; Sünker & Zembat, 2012; Yanik & Flores, 2009). While a process view of translation is based on the dynamic motion of geometric figures, an object view is based on static realizations of the plane as consisting of an infinite number of distinct points and geometric figures as consisting of points that are subsets of the plane (Edwards, 2003). It is important for students to think about translations using both the process and object views to be able to work with them in different settings (Portnoy et al., 2006).

The literature on student thinking on translation is dominantly based on cognitive perspectives that view learning as students' individual processes of changing their existing mental schemes and does not extensively focus on the roles of communication and teachers' discourses in student development. While this literature informs us in identifying student strategies and difficulties in our work, we also differ from this literature in terms of our focus on teaching in a naturalistic classroom setting and the critical roles we attribute to teachers in student development in accordance with our communicational and socio-cultural approach to learning and teaching of translations, which we discuss in the next section.

Sfard's (2008) Discursive Approach

The theoretical perspective we use challenges the universal, context-independent, and individual nature of thinking and learning processes presumed by traditional cognitive perspectives. It also provides a broader perspective to discourse analysis since, traditionally, discourse analysis is used to mainly explore linguistic phenomena such as word meaning, intonation, grammar, phonetics, and discourse flow (e.g., Howe & Abedin, 2013; Schiffrin, Tannen, & Hamilton, 2003). Sfard's (2008) discursive approach builds on the socio-cultural approaches that consider thinking as a private and modified version of interpersonal communication. Viewing cognition as a form of communication situates individuals in a contextual web of social relations, intentions, and motivations (Vygotsky, 1978); and emphasizes the centrality of discourse in communicational acts of the participants as they attempt to make others act or feel according to their intentions (Sfard, 2001). It also supports the characterization of learning as increased participation in communities of practice (Lave & Wenger, 1991). Consistent with these assumptions, Sfard's (2008) framework integrates thinking and communicating by defining cognitive processes "as individualized forms of interpersonal communication" and communication as "a collectively performed rule-driven activity that mediates and coordinates" the activities of the members of a community of discourse (p. 92). By defining thinking as an individualized version of communication, which is not necessarily inner or verbal (Sfard, 2008), this approach negates the separation between individual and social aspects of learning. According to this lens, learning is characterized as becoming a skillful participant in communities of practice through changes in one's discourse, where "discourse is a special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (Sfard, 2008, p. 297). According to Sfard (2008), discourses, including mathematical discourses, are distinguishable by their word use, visual mediators, routines, and endorsed narratives.

In mathematical discourse, word use refers to the mathematical vocabulary that participants use and forms the basis of mathematical communication. Visual mediators are the visible objects developed and used for mathematical communication. Routines are the set of meta-level rules describing the repetitive discursive patterns in the activity of participants. Sfard (2008) distinguishes between how and when routines take place. While the how of a routine refers to the course of actions performed by the participants, the when of a routine involves the circumstances in which the participants are likely to use the routine and consider the performance as complete. Endorsed narratives are the set of utterances about mathematical objects and their relationships that are considered as true by the participants. Although each of these discursive elements can be considered to have different characteristics, they influence and are influenced by each other. Sfard identifies four hierarchical stages in participants' development of word use. The first stage is passive use, where the participants are not yet able to use a mathematical word in their speech but can provide some "routine reactions to other people's utterances containing the word" (Sfard, 2008, p. 181). The next stage is routine-driven use, where participants utter a mathematical word but only in a limited number of specific routines. In phrase-driven use, the mathematical word is linked with constant phrases or narratives instead of routines and "entire phrases rather than the word as such constitutes the basic building blocks" of participants' utterances (Sfard, 2008, p. 181). Finally, in the object-driven stage, participants use a mathematical word as a noun as if it has a life of its own; they also use different realizations of the word depending on the context by using a repertoire of signifiers.

In advanced stages of discursive development, participants' word use, visual mediators, routines, and endorsed narratives become broader and more flexible; they focus on the mathematical properties of concepts rather than concentrating only on the specific routines that they perform. Sfard (2008) notes that a goal of mathematics education is to help students make their discourses compatible with those of the experts (e.g., teachers, mathematicians) so that they can more fully participate in mathematical communities of practice. Consistently, this lens views the changes in students' discourses as indicators of their development and learning. In our work, we examine two students' discursive development by examining the changes in their word use, visual mediators, routines, and endorsed narratives about translation and then compare these elements with the teacher's discourse on the concept.

CONTEXT OF THE STUDY

The Turkish education system is a centralized national system governed by the Ministry of Education. The current Turkish mathematics curriculum is influenced by constructivist perspectives that have guided the curricular reforms in many countries (Zembat, 2010). However, the main mode of teaching in Turkey is still direct instruction. The mathematics curriculum in Turkey has a spiral structure; students have exposure to geometric transformations at the elementary (grades 6-8) and high school levels (grades 9-10) (MoNE, 2009, 2010). The elementary and 9th grade curricula suggests teaching translations as physical movement without using vectors (Sünker & Zembat, 2012), whereas the 10th grade curriculum suggests teaching translations as functions mapping points to other points in the plane through the use of vectors. Tenth grade constitutes a critical juncture in learning of translations since students are expected to change their dynamic realizations of translation to static realizations. Since this juncture provides rich opportunities to examine students' discourses on geometric transformations as well as whether and how teachers attend to this transition, we focused on 10th grade students in our study.

METHODOLOGY

We examined two 10th grade (16-year-old) high school students' discursive development on translation in relation to their teacher's discourse in a medium-size urban public high school in Turkey. Our work uses a qualitative and interpretive case study design with the goal of exploring student development in relation to teaching through information-rich, contrasting cases (Creswell, 2007). We selected Mr. Can (a pseudonym) as the teacher since he considered his teaching approach as interactive and student-centered; he was open to having researchers in the classroom; and allowed us to video-tape his classroom. Mr. Can had an M.A. in Education and eight years of experience working as a mathematics teacher. We wanted to recruit expressive students willing to be interviewed throughout the study who also demonstrated different ways of thinking about translation, which is a form of purposeful sampling (Creswell, 2007; Patton, 2002). Prior to data collection, we made observations in the classroom to identify talkative students. We then sought Mr. Can's feedback regarding which students we identified would demonstrate diversity in their thinking about translation. Mr. Can recommended Okan and Eda (pseudonyms) to us; he viewed Okan as a weaker student compared to Eda based on the students' previous exam scores in the semester[1]. Okan and Eda had the same teacher in 9th grade (who was a different teacher than Mr. Can) but had different teachers through grades 6–8. These students formed contrasting cases in terms of their discursive development and the teacher's discourse influenced these students' discourses in different ways. We did not generalize our findings to all of the students in the classroom since we used a contextual, socio-cultural framework. Yet, we considered this classroom as representative of classrooms where students may develop their idiosyncratic realizations of translation, especially if particular features of the teacher's discourse remain implicit to them. Our hypothesis was that Sfard's (2008) framework could provide insights regarding the areas that require further research on geometric translation, namely examining the development of students' discourses on translation and the potential influences of the teacher's discourse in such development.

The data sources for the teacher included the 45-minute video-taped classroom session during which he talked about translation and a 106-minute video-taped task-based interview that was administered 28 days after the instruction. The purpose of the interview was to gain information about Mr. Can's discourse on translation and student thinking about the concept. The interview consisted of four tasks. In the first task, we asked Mr. Can to give an example for translation. In the second task, which included geometric and algebraic visual mediators, we asked him to translate a given triangle. The third task involved translating a polygon represented only geometrically. The last task was based on a hypothetical student's solution to a translation problem and included the dialogue between the student and a hypothetical teacher. We asked Mr. Can to examine the solution and student's approach to the task. For each task, we asked Mr. Can to talk about how he would explain the tasks to his students; the difficulty level of the tasks for his students; and the common answers he would expect from the students.

The data sources for Okan and Eda included three video-taped task-based interviews per student[2].The students were interviewed individually and each interview lasted about 30 minutes. For each student, we conducted the

[1] Okan's and Eda's scores on that exam were 25 and 60 out of 100 points, respectively.

[2] We also analyzed the classroom video for these students' discourses but Okan and Eda did not speak during the lesson on translation.

Table 1. Examples of a task across three student interviews

Interview 1	Interview 2	Interview 3
Determine the translation of the triangle OAB that has $O(0,0)$, $A(1,0)$, $B(0,1)$ as vertices in the direction of $\vec{u} = (1,2)$ in the coordinate system.	Determine the translation of the triangle ABC that has $A(3,2)$, $B(2,1)$, $C(3,1)$ as vertices in the direction of $\vec{u} = (-3,2)$ in the coordinate system.	Determine the translation of the triangle ABC that has $A(0,1)$, $B(1,2)$, $C(-1,0)$ as vertices in the direction of $\vec{u} = (1,2)$ in the coordinate system.

first interview before instruction; the second interview right after instruction; and the third interview 25 days after the instruction. Each interview consisted of four tasks on translation, which were mathematically equivalent to those used in Mr. Can's interview, but with different follow-up questions. For example, the second task, which involved translating a triangle was present in each of the student interviews through three different but equivalent tasks as demonstrated in **Table 1**. To ensure the equivalence and validity of the tasks (Patton, 2001), five experts (one professor of mathematics, one professor of mathematics education, two Ph.D. candidates in mathematics education, and one high school mathematics teacher) reviewed the tasks in terms of their parallelism and content. The tasks were finalized based on the feedback we received from these experts. The students could work on the first task (giving an example for translation) using what they learned in 9th grade but they had to use what they learned in 10th grade for the remaining tasks involving vectors and vector notation.

We conducted the interviews in the participants' native language and translated them from Turkish into English. The transcripts of the interviews and classroom observation included participants' utterances as well as their visual mediators and actions. The data were analyzed in terms of participants' word use, visual mediators, routines, and narratives (Sfard, 2008). Since there was only one classroom session in which Mr. Can talked about translation, our analysis of his discourse was descriptive whereas our analysis of the students' discourses was developmental with a particular focus on how, and whether, the students' discourses changed over time.

When analyzing participants' word use, we concentrated on the stages of the development of word use. If the participants did not use the word translation in their discourses but were able to translate objects when asked to do so, we considered their word use as passive. If the participants' use of the word translation was limited to the actions they performed when translating shapes, we considered their word use as routine-driven. If they used the word translation in general phrases they associated with the concept rather than referring to it as an object, we considered their word use as phrase-driven. If the participants talked about translation as a mathematical concept using a noun, we considered their word use as object-driven. To assure consistency of coding, we coded the stages of participants' word use independently and then compared them as suggested by Creswell (2007) and Miles and Huberman (1994). Our initial coding revealed an 85% agreement. We then collaboratively examined the cases for which our coding differed and discussed them until we reached consensus.

For visual mediators, we focused on whether or not participants used a variety of mediators (e.g., geometric, algebraic) depending on the mathematical context. For routines, we examined the repetitive actions and consistent procedures participants used when working on translation tasks and explored when and how participants used those actions as they substantiated their narratives on translation. The routines were used at least twice by the participants during an interview. Lastly, the endorsed narratives were examined in terms of what the participants considered as true about translation in relation to their word use, visual mediators, and routines. While comparing the students' discourses with the teacher's discourse, we paid attention to the four elements of participants' discourses with a focus on their similarities and differences to explore to what extent the students' discourses became compatible with the teacher's discourse on translation in the course of the study.

RESULTS

The Teacher's Discourse on Translation

During our informal conversations, Mr. Can mentioned that he assumed his students to have previous experiences with translations due to the spiral nature of the Turkish mathematics curriculum. Although Mr. Can considered his teaching approach to be interactive and student-centered, his lesson on translation was based on direct instruction with some teacher-student and student-student interactions. Mr. Can introduced translation by giving a definition of the concept and then providing examples by projecting some slides on the board.

Word use. Mr. Can introduced translation through the following definition in the classroom: "*If Q is a point in the plane defined as $\vec{u} = \vec{PQ}$, then Q is called the translated image of the point P in the direction of $\vec{u}$, which is shown as $Q = T_{\vec{u}}(P) = P + \vec{u}$*". Mr. Can's discourse on translation was consistently based on phrase- and object-driven word use throughout the class observation (12 occurrences of phrase-driven and 21 occurrences of object-driven word use)

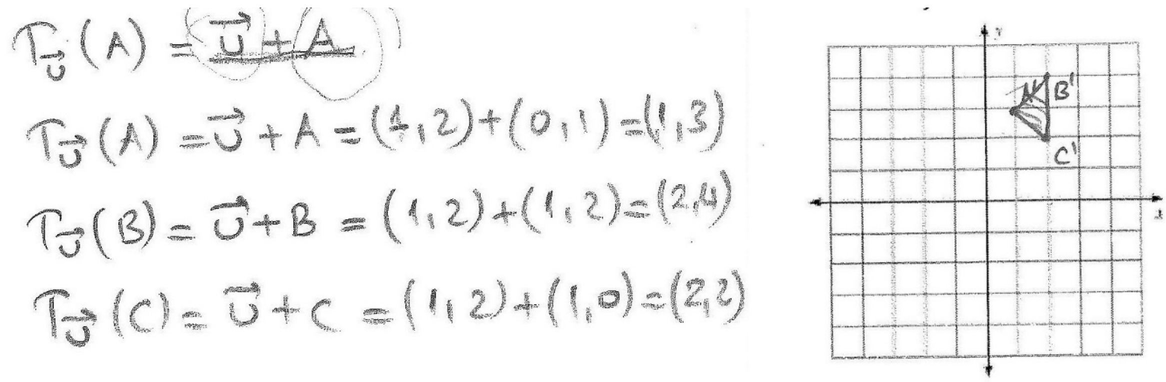

Figure 1. An example of the routine TR1 in Mr. Can's discourse

and interview (19 occurrences of phrase-driven and 29 occurrences of object-driven word use). We did not observe any passive or routine-driven word use in his discourse.

When Mr. Can's word use was phrase-driven (e.g., "when you translate a vector in the direction of another vector, you need to add the two vectors"), he did not talk about translation as a mathematical object using a noun; instead he used the word in phrases about translation. When his word use was object-driven (e.g., "translation is an addition of vectors"), he used translation as a noun and distinct mathematical object; he also focused on its mathematical properties rather than the routines or procedures with which he translated a given shape. In the context of the study, Mr. Can did not refer to translation as physical movement; instead he talked about it as a formal concept with specific properties, which was also reflected in the initial definition he provided to his students at the beginning of the lesson on translation.

Visual mediators. The primary visual mediator in Mr. Can's discourse was algebraic notation but he also used vectors and drawn geometric shapes. He used these mediators 34, six, and four times in the classroom, respectively. When using vectors, he realized them algebraically rather than geometrically. For example, when Mr. Can asked students to translate a point $A(3,-1)$along the vector $\vec{u}=(-2,1)$ in the classroom, he emphasized the use of algebraic notation as demonstrated below:

> *You should write it like this (writes $T_{\vec{u}}(A)$) and between the parentheses, you should always write the point that you will translate. Under the letter T, you should write the vector whose direction you want to translate. Here, the translated point is A, the translating vector is $\vec{u}$. We know what this means [referring to the aforementioned definition he gave in the class]: $T_{\vec{u}}(A) = A + \vec{u}$. The sum of two [A and $\vec{u}$] when we place these in the formula would be $A(3,-1)$ plus $(-2,1)$, which is $(1,0)$.*

Mr. Can used geometric shapes as visual mediators in his discourse only when drawing a geometric shape to translate, and when drawing the translated images of those shapes.

Routines. We observed one routine in Mr. Can's mathematical discourse, which we identified as TR1 (Teacher Routine #1): *Translating a geometric shape by algebraically adding the coordinates of the vertices of the original shape with the corresponding coordinates of the translating vector and then drawing the translated shape*. For example, during the interview, we asked Mr. Can to find the translation of a triangle ABC that has $A(0,1)$, $B(1,2)$, $C(1,0)$ as vertices in the direction of $\vec{u}=(1,2)$, and draw the translated triangle in the coordinate system that we provided to him. **Figure 1** shows his use of TR1 as a routine and the triangle he drew while solving the problem.

Figure 1 demonstrates that Mr. Can initially translated each vertex of the triangle ABC by algebraically adding the coordinates of the vector $\vec{u}$ to the corresponding coordinates of the vertices of the triangle. He then marked the translated vertices he found [(1,3), (2,4), and (2,2), respectively] in the coordinate system and drew the translated triangle by connecting the vertices. During the classroom observation and interview, regardless of the geometric shape translated, TR1 was the only routine Mr. Can used in his discourse on translation. He used this routine three times during the classroom observation and four times during the interview.

Endorsed narratives. Besides the definition of translation he gave in the classroom, the most frequent narrative Mr. Can endorsed was "*translation is an addition.*" He endorsed this narrative five times in the classroom and two times in the interview. This narrative was consistent with his object-driven word use, visual mediators, and his use of TR1 as a routine, where he regularly used algebraic addition to find translation of various geometric shapes. The other two narratives he endorsed on translations were "*translation is a geometric transformation*" (endorsed twice in the classroom and interview), and "*translation is a congruence transformation that preserves the size of the original shape*" (endorsed twice in the classroom and interview). Rather than explicating the latter narratives through specific routines, Mr. Can treated them as implicit assumptions in his discourse on translation.

Development of Okan's Discourse on Translation

Word use. During the first interview, which was conducted before Mr. Can talked about translation in the classroom, Okan's word use on translation was routine-driven (10 occurrences). For example, when we asked him "What is the translation of the triangle *CAB*, where $C(0,0)$, $A(1,0)$, $B(0,1)$, along the vector $\vec{u}=(1,2)$?" he said: "*To translate, I moved it [the triangle]* 2 *units up and it came here [shows the slid triangle]. Then I moved it [shows the slid triangle]* 1 *unit to the right and it came here [shows the final image of the translated triangle]*". Okan's word use in this interview was routine-driven because, rather than talking about translation as a mathematical object, he talked about the actions and processes with which he translated geometric shapes.

In the second interview, which was conducted right after Mr. Can talked about translation in the classroom, Okan's word use was predominantly phrase-driven (11 occurrences). For example, when we asked him what translation is, he said, "*When I hear translation, what comes to my mind is the movement of geometric shapes in various directions within the coordinate system…to translate an object means to change its position without changing the original object.*" Such word use is phrase-driven because Okan's utterances were not about the specific actions he performed on the objects; instead, he used the word translation in general phrases about the concept. During this interview, we also identified four instances in which Okan's word use was object-driven. For example, he said "*translation is a transformation that enables the placement of objects within the coordinate system*". Here, Okan considered translation as a mathematical object (a particular type of transformation) and used it as a noun.

In the third interview, which was conducted 25 days after instruction, Okan's word use was mainly object-driven (10 occurrences). When talking about translation, he frequently mentioned that "translation is a transformation" and referred to the translated geometric shapes as the end states of the translation process. He did not utter words based on the actions he performed on the shapes (routine-driven word use) but talked about the "translated objects" at the end of each translation task. There were also four instances in which Okan's word use was phrase-driven during this interview. For example, while he was translating a geometric shape, he said "*when an object is translated, area, circumference, and the distance between the points [of the original shape] do not change*". In this interview, Okan talked about translation as a mathematical concept by also focusing on its mathematical properties.

Visual mediators. In the first interview, the visual mediators in Okan's discourse on translation were drawn geometric shapes, which he used four times when giving an example for translation and translating a given geometric shape in the coordinate system. He did not use any algebraic notation or vectors. Although Okan also worked with other visual mediators we provided to him (e.g., vectors, algebraic notation), he did not use these mediators in his own discourse autonomously.

In the second interview, Okan used geometric shapes, vectors, and algebraic notation as visual mediators in his discourse, which were used five, six, and four times, respectively. During this interview, Okan used vectors in all of the tasks, which contrasted with his first interview. While using vectors, he realized them geometrically as signifying dynamic movement. However, when we asked him to calculate the coordinates of a translated geometric shape, he also used vectors statically through algebraic notation. Since the analysis of a visual mediator cannot be complete without the discussion of its use, which involves the analysis of using the visual mediator as a routine, further details of Okan's use of vectors will be discussed in the next section.

In the third interview, the visual mediators in Okan's discourse were drawn geometric shapes and vectors, which he used four and two times, respectively. He used vectors as signifying dynamic movement rather than using them statically through algebraic notation.

Routines. We could not identify any routine in Okan's discourse in the first interview since he did not demonstrate any consistent action or repetitive procedure when working on the translation tasks. In the second interview, there were four different routines we determined in Okan's discourse. When Okan translated a geometric shape and gave examples for translation, he used two routines. We identified the first routine as OR1 (Okan Routine #1): *Translating each vertex of a given geometric shape along the components of the translating vector using only geometric visual mediators and then drawing the rest of the shape using properties of translation*. For example, when we asked Okan to translate the polygon shown in **Figure 2**, he first identified the magnitude and direction of the components of the translating vector and labeled each vertex of the original polygon. He then labeled A' by moving the point A 8 units downwards and 14 units to the right by counting and without using any algebraic mediators. He repeated the same procedure to locate the other translated vertices of the original polygon. He then drew the translated polygon by connecting the translated vertices and using properties of translation mentioning "preservation of distance across points" and "preservation of the size and orientation of the original shape". He used this routine twice during this interview.

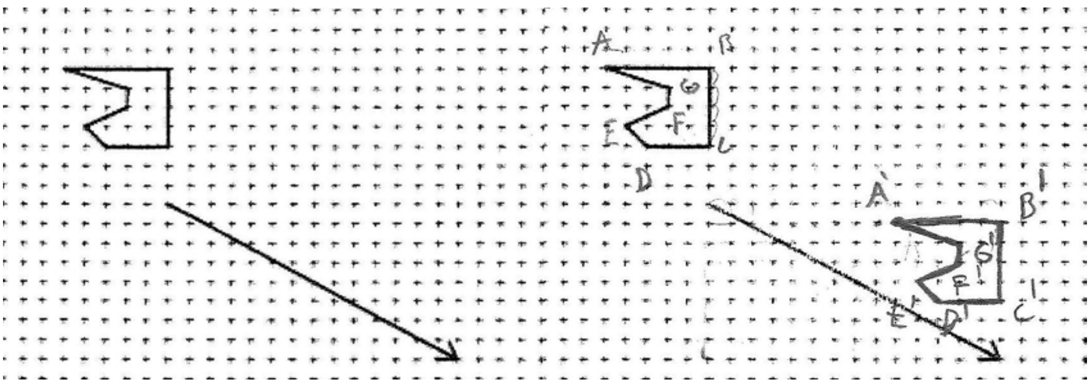

Figure 2. An example of the routine OR1 in Okan's discourse (Left hand side shows the original figure and the right hand side shows Okan's response)

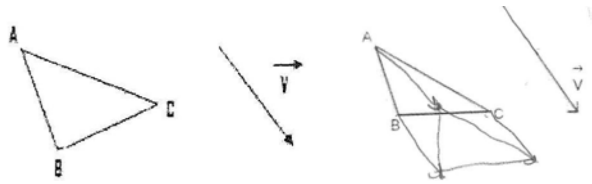

Figure 3. An example of the routine OR2 in Okan's discourse (Left hand side shows the original figure and the right hand side shows Okan's response)

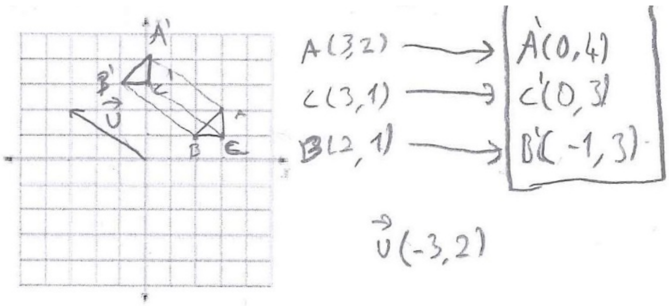

Figure 4. An example of the routine OR3 in Okan's discourse

The second routine Okan used when translating a geometric shape and giving examples for translations was OR2 (Okan Routine #2): *Translating each vertex of a given geometric shape in the direction of the translating vector by geometrically sliding the vertices to their translated locations and then drawing the rest of the shape using properties of translation*. For example, Okan used OR2 when translating the triangle shown in **Figure 3** as follows: He first slid the point *A* geometrically along the vector $\vec{v}$ and repeated the process for the vertices *B* and *C*. He then connected the translated vertices by using properties of translation (preservation of size and shape). He used this routine four times during this interview.

After Okan translated geometric shapes using OR1 and OR2, we sometimes asked him to find the coordinates of the translated vertices of the geometric shapes. Okan's responses to these questions elicited another routine, namely OR3 (Okan Routine # 3): *Finding the coordinates of the vertices of a translated shape geometrically by using the grids provided in the questions*. For example, when we asked Okan to translate a triangle *ABC*, he translated it using OR2 as a routine. After we asked him to find the coordinates of the translated triangle, he used OR3 to locate the vertices A', B',and C' geometrically by using the grids as seen in **Figure 4**. He used this routine twice during this interview.

After translating geometric shapes using OR1 and OR2, Okan regularly wanted to check his answer. In those instances, he used OR4: *Finding the coordinates of the vertices of a translated shape by algebraically adding the coordinates of the vertices of the original shape with the corresponding coordinates of the translating vector*. For example, when we asked Okan to give an example for translation, he translated a square he labeled as $ABCD$ in the direction of $\vec{u} = (2,-1)$ by using OR1. He then wanted to check his answer by adding the coordinates of the vector $(2,-1)$ to the coordinates of the vertices A, B, C, and D, respectively, to find the coordinates of the translated vertices A', B', C', and D' as seen in **Figure 5**. He used this routine three times during this interview.

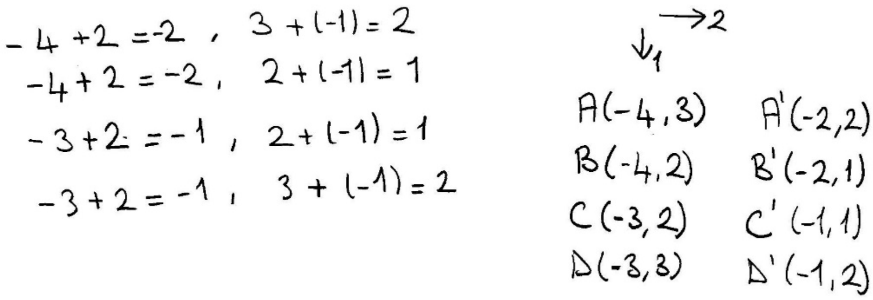

Figure 5. An example of the routine OR4 in Okan's discourse

In the last interview, Okan used only one routine, namely, OR1. He used this routine in three of the tasks on which he worked.

Endorsed narratives. In the first interview, Okan talked about translation as physical movement but not as a mathematical object. His narratives were about the actions he performed on geometric shapes, which were consistent with his routine-driven word use and visual mediators where he used geometric motion when working on translation tasks.

In the second interview, Okan continued endorsing narratives that supported his realization of translation as movement of geometric shapes within the coordinate system. However, he also endorsed the narratives "*to translate an object means to change its position without changing the original object*" (three times) and "*translation is a transformation that enables the placement of objects within the coordinate system*" (twice), which indicated that he also focused on the mathematical properties of translation. Okan's narratives in the second interview were consistent with his phrase- and object-driven word use and his use of OR1 and OR2 as routines.

In the third interview, consistent with his object-driven word use, visual mediators, and use of OR1 as a routine, Okan endorsed the following narratives about translation: "*translation is a movement that preserves the dimensions of original shape*" (six times), and "*translation is a geometric transformation*" (twice). These narratives indicated that, although Okan's dominant realization of translation was through physical movement, he could also consider it as an object and focus on its mathematical properties by the end of the study.

Development of Eda's Discourse on Translation

Word use. During the first interview, Eda's word use was routine-driven (12 occurrences). For example, when we asked her to define translation, she said "*for translation, I pick a point* $(-2,2)$ *in the coordinate system. For example 4 units to the right [referring to the vector she chose, which had* $(4,0)$ *as its coordinates] and then I am moving it here [shows the translated point that is 4 units to the right of the original point]*".

In the second interview, Eda's word use was predominantly routine-driven (11 occurrences) and we also observed some phrase-driven word use (two occurrences). An example of Eda's routine-driven word use in the second interview was as follows:

> *Let's have a triangle [draws a triangle]. I select any point on the triangle [selects one of the vertices of the triangle]. I am going to translate it, let's say 5 units to the right. 1, 2, 3, 4, 5 and the point is here [translates one vertex of the triangle 5 units to the right and marks the translated point]. Again, when I do 5 units, it is here [translates another vertex of the triangle and then draws the translated triangle].*

Eda demonstrated phrase-driven word use twice in this interview when we asked her to talk specifically about the mathematical properties of translation. For example, when we asked Eda "Which geometric properties of a shape are preserved by translation?" she said "*In translation, the dimension of the shape is preserved but not its location.*"

In the third interview, Eda's word use on translation was mainly routine-driven (nine occurrences) and there was one occurrence of phrase-driven word use in her discourse. Similar to the previous interview, Eda used phrase-driven word use only when responding to our prompt that asked her to talk about the mathematical properties of translation. When we asked Eda "Do you think translation preserves the location of the points of a translated shape?" she said "*So, if a point A is translated one unit, it will move to another point so translation does not preserve the location.*" Throughout the study, we did not observe object-driven word use in Eda's discourse on translation.

Visual mediators. During the first interview, Eda's visual mediators were drawn geometrical shapes (used two times) and vectors (used three times). She drew these geometric shapes when giving an example for translation and translating a geometric shape in the coordinate system. In this interview, Eda realized vectors geometrically, as lines. The details of her use of vectors are elaborated in the next section on her routines.

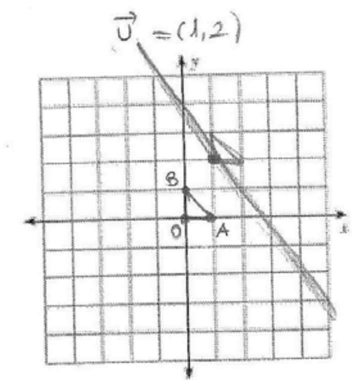

Figure 6. An example of the routine ER1 in Eda's discourse

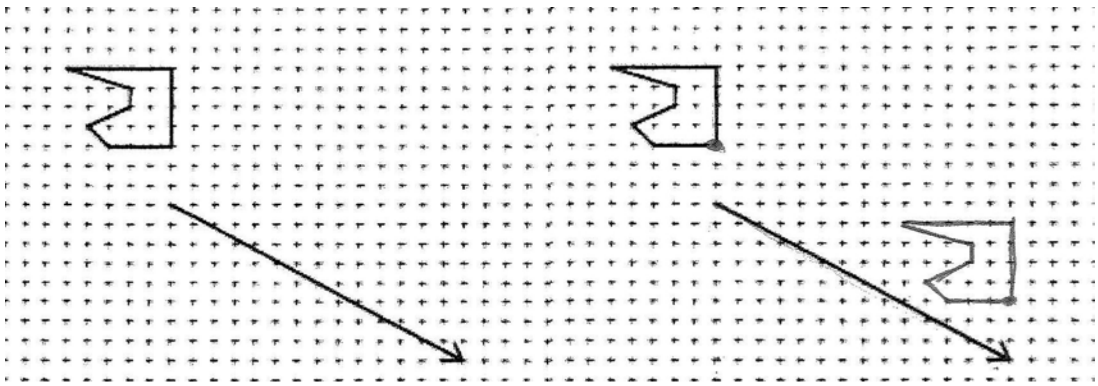

Figure 7. An example of the routine ER2 in Eda's discourse (Left hand side shows the original figure and the right hand side shows Eda's response)

In the second interview, Eda used drawn geometric shapes, vectors, and algebraic notation as visual mediators, which were used five, four, and six times, respectively. She consistently realized vectors algebraically as sets of ordered pairs. Eda also realized vectors geometrically when working on a task and, in that case, her use of vectors was mathematically correct unlike the first interview. She did not confuse vectors with lines in the second and third interviews.

In the third interview, the visual mediators in Eda's discourse were drawn geometric shapes, vectors, and algebraic notation, which were used four, three, and four times, respectively. She realized vectors statically through the use of algebraic notation, but occasionally she also realized them dynamically through physical motion.

Routines. In the first interview, there was one (mathematically incorrect) routine in Eda's discourse, which we identified as ER1 (Eda Routine #1): *Drawing a given vector as an arbitrary line that passes through the coordinates of the vector*. She used this routine twice when translating shapes and examining the work of a hypothetical student on translation. For example, when we asked Eda "What is the translation of the triangle *OAB*, where $O(0,0)$, $A(1,0)$, $B(0,1)$, along the vector $\vec{u} = (1,2)$?" she used ER1 as shown in **Figure 6**. In this task, Eda first marked the point $(1,2)$ in the coordinate system and drew $\vec{u}$ as a line passing through the point $(1,2)$. She then considered the point $(1,2)$ as one of the translated vertices of *OAB* and drew the rest of the translated triangle.

In the second interview, we observed two different routines in Eda's discourse. The first routine was ER2 (Eda Routine #2): *Translating each vertex of a given geometric shape along the components of the translating vector using only geometric visual mediators and then drawing the rest of the shape using properties of translation*[3]. She used ER2 five times when giving examples for translation, translating geometric shapes, and examining the work of a hypothetical student on translation. For example, when we asked Eda to translate a given polygon, she translated it using ER2 as shown in **Figure 7**. In this task, Eda first translated each vertex of the polygon and then drew the rest of the shape using properties of translation that it preserves size and orientation.

The second routine that Eda used in this interview when translating geometric shapes was ER3 (Eda Routine #3): *Translating a geometric shape by algebraically adding the coordinates of the vertices of the original shape with the*

[3] ER2 and OR1 are the same routines. However, when using the routine, Okan explicitly mentioned some properties of translation, whereas Eda used the properties implicitly.

$$(3,2)+(-3,2)=(0,4)$$
$$(2,1)+(-3,2)=(-1,3)$$
$$(3,1)+(-3,2)=(0,3)$$

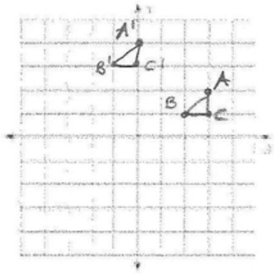

Figure 8. An example of the routine ER3 in Eda's discourse

corresponding coordinates of the translating vector and then drawing the translated shape[4]. For example, when we asked Eda "Can you translate the triangle ABC with vertices $A(3,2)$, $B(2,1)$, and $C(3,1)$ in the direction of $\vec{u}=(-3,2)$?" she used ER3 as shown in **Figure 8**. While working on the task, she added the coordinates of A, B, and C with the corresponding coordinates of the vector $\vec{u}$ algebraically and said "*as we learned in the classroom, we are adding these*". Afterwards, she located the translated vertices of the triangle in the coordinate system and connected them. While using this routine, Eda explicitly mentioned the teacher's discourse and considered ER3 as compatible with the routine Mr. Can used in the classroom[5]. She used this routine twice during this interview.

In the third interview, Eda only used ER2 as a routine (three times).

Endorsed narratives. In the first interview, Eda's narratives were only about the actions she performed when translating geometric shapes. Her word use, visual mediators and routine indicated that she realized translation as the process of repositioning a given shape.

In the second interview, Eda's action-oriented approach to translation continued through her routine-driven word use, visual mediators, and routines, indicating that she primarily realized translation as dynamic movement of geometric shapes within the coordinate system. However, during her phrase-driven word use, Eda also endorsed the following narrative about translation: "*In translation, the dimension of the shape is preserved but not its location*."

In the third interview, similar to the previous interviews, Eda's narratives signified translation as dynamic movement, which were consistent with her routine-driven word use, visual mediators, and routine. During this interview, the only narrative Eda endorsed about a mathematical property of translation was "…so *translation does not preserve the location*." Despite endorsing some narratives about mathematical properties of the concept, Eda realized translation as a process rather than an object throughout the study.

Comparison of the Students' Discourses with the Teacher's Discourse

Our comparison of Okan's discourse on translation with the teacher's discourse revealed occasions in which his discourse seemed to be influenced by the teacher's discourse as well as occasions in which Okan's and the teacher's discourses diverged. After instruction, some of Okan's discursive elements became compatible with those in the teacher's discourse. Okan's initial routine-driven word use gave way to phrase- and object-driven word use, which was compatible with Mr. Can's word use on translation. His use of algebraic notations as visual mediators was also compatible with Mr. Can's discourse. Consistently, Okan's routine OR4 was similar to Mr. Can's routine TR1[6]. Finally, some of Okan's endorsed narratives about translation after instruction were consistent with the narratives the teacher endorsed.

On the other hand, Okan did not use some of the visual mediators and routines that were compatible with the teacher's discourse during the third interview. In this interview, he abandoned using algebraic notations as visual mediators and went back to using the visual mediators that were more common in his discourse throughout the study, which were not found in the teacher's discourse. Okan also abandoned using OR4 and used OR1, which was more consistent with his overall discourse on translation. Although Okan's word use and endorsed narratives became more compatible with Mr. Can's discourse during the second and third interviews, Mr. Can did not endorse translation as movement in the classroom, whereas the realization of translation as movement was commonly found in Okan's discourse throughout the interviews.

These findings indicate that Okan adopted particular aspects of Mr. Can's discourse that were compatible with his own (previously existing and developing) realizations of translations while also abandoning other elements in

[4] ER3 and TR1 are the same routines.
[5] Eda referred to Mr. Can's discourse twice in this interview and once in the last interview.
[6] Despite this similarity between OR4 and TR1, there were also differences between these routines in terms of when and how they were used.

Mr. Can's discourse that did not match with his own realizations of translations. This active, socially influenced, and also idiosyncratic discursive adjustment that Okan demonstrated did not hinder his objectification of translation as a mathematical transformation by the end of the study. However, it resulted in the teacher and Okan using similar discursive features in different ways, and dissimilar discursive features to endorse similar narratives about translations.

Our comparison of Eda's discourse with Mr. Can's discourse revealed some occasions in which her discourse was influenced by the teacher's discourse. We observed differences in Eda's discourse before and after instruction. Similar to the teacher's discourse, Eda realized vectors statically during the second and third interviews. After instruction, Eda abandoned her (mathematically incorrect) routine ER1 and used ER3, which was identical to Mr. Can's routine TR1. Further, when using ER3, Eda explicitly mentioned that she learned about this procedure in the classroom, endorsing the role of the teacher in her discourse.

Despite these similarities, Eda's word use and endorsed narratives differed significantly from those in Mr. Can's discourse. During the study, Eda's word use on translation was not object-driven, which contrasted with Mr. Can's discourse. Similarly, Eda's endorsed narratives signifying translation as the process of repositioning geometric shapes contrasted with the teacher's realization of translation as a congruence transformation.

Although Eda mentioned the compatibility of her discourse with Mr. Can's discourse, these discourses differed in significant ways, especially in terms of the lack of objectification in Eda's discourse. The teacher and Eda used dissimilar and similar discursive features to endorse different narratives about translations. Similar to Okan's case, the existence of idiosyncratic discursive features in Eda's discourse suggested a discursive adjustment. Unlike Okan's case, such adjustment did not facilitate Eda's objectification of translation as a mathematical object.

Our findings about these students contrasted with how Mr. Can viewed them. Mr. Can viewed Eda as a successful student based on her exam scores and considered Okan as a weaker student. However, in the context of our study, Okan was a more skillful participant in the discourse on translation compared to Eda. Okan demonstrated a richer inventory of visual mediators, routines, and endorsed narratives in his discourse and was also able to objectify translation as a mathematical object, whereas Eda's discourse remained restricted to the specific procedures that she performed on geometrical shapes.

CONCLUSIONS AND DISCUSSION

We explored how two students developed their discourses on the concept of translation in relation to the teacher's discourse in a Turkish high school classroom. While doing so, we examined the teacher's discourse on translation in the classroom and the discursive development of two students in the classroom. We compared the students' discourses with the teacher's discourse to explore whether, and how, the teacher's mathematical discourse in the classroom shaped the students' thinking on translation. The results indicated that the teacher's discourse on translation in the classroom was formal and based on an algebraic realization of the concept as an addition and a geometric transformation. The students adopted particular discursive elements used by the teacher, especially right after instruction. However, they abandoned some of those elements in the third interview and used their more established realizations of translation. These findings revealed the complexity of discursive development as the students continually and actively adjusted their discourses in a way that took into account the teacher's discourse while also making it compatible with their own realizations of translation. The observed differences between Okan's and Mr. Can's discourses did not hinder Okan's objectification of translation, whereas the differences between Eda's and Mr. Can's discourses were significant enough that Eda did not objectify translation. Despite the incompatibilities, Eda viewed particular elements of her discourse as aligned with the teacher's discourse. Although Okan did not explicitly refer to Mr. Can's discourse during the study, his endorsed narratives aligned more with Mr. Can's endorsed narratives. These findings suggest that the students were not necessarily aware how their discourses were influenced by or differed from the teacher's discourse, indicating a miscommunication between the teacher and students, especially when communication is characterized as "an attempt to make other people act or feel according to one's intentions" (Sfard, 2001, p.38).

The differences between the students' and Mr. Can's discourses may be related to the curricular approach to translation in the Turkish curriculum. It is likely that the students' discourses were influenced by their prior experiences with translation since a realization of translation as dynamic movement was present in their discourses but not in Mr. Can's discourse in the classroom. It is possible that the short lesson during which Mr. Can talked about translation did not provide the students with sufficient time to fully adjust their discourses with the teacher's discourse. The students' discourses could be influenced by their peers' discourses on translation during the semester and their interactions outside the classroom, since not all learning took place in classrooms. The students' discursive development may also be connected to their interests, motivations, attitudes, and beliefs about mathematics and geometric transformations.

In our work, Sfard's (2008) framework provided insights regarding teaching and learning of geometric translation. The framework helped provide in-depth analyses of the students' discursive development by juxtaposing their utterances and narratives with their visual mediators and routines. It also helped identify the instances where the students' discourses were influenced by and differed from the teacher's discourse. The finding that the students' and teacher's discourses on translation differed may not be unexpected; a particular affordance of this framework was to provide the analytical tools with which we could specify *when* and *how* the participants' discourses diverged from or converged to each other. By doing so, the discursive approach helped us highlight the contexts of possible mathematical miscommunication in the classroom. Our discursive analysis using Sfard's (2008) framework revealed a different picture about these students compared to how Mr. Can viewed them, showing that many subtle features of mathematical discourse may not be captured by a mere focus on students' written performances. A discursive approach underlines the importance of reflecting on what being a successful mathematics student means and how to assess learning. Further, it highlights the communicational nature of learning as indicated by the adjustments the students made to their initial discourses in relation to the teacher's discourse on translation.

During instruction, Mr. Can did not consider the transitions students needed to go through in 10th grade regarding their realizations of translation. Mr. Can's lack of focus on those transitions and his assumption that the students were already familiar with the concept may be why his discourse was formal and why he and the students did not realize the discrepancies between each other's discourses. Our findings confirm that teachers' lack of attention to their own and their students' discourses may contribute to communicational failures in the classroom (Güçler, 2013; 2014). Discursive differences that remain implicit in the classroom are particularly difficult to elicit during direct instruction. An implication of Sfard's (2008) framework for teaching is the importance of giving students more opportunities to talk and act in the classroom to get information about their thinking and development. If Mr. Can gave such opportunities to his students and focused beyond the procedural aspects of translating shapes in his lesson, he might have realized that Okan's discourse on translation was more mathematically sophisticated than Eda's. Another implication of Sfard's (2008) approach to teaching is the need for teachers to explicate the discursive elements underlying different realizations of translation to enhance communication in their classrooms. For translation, this may require talking explicitly about the dynamic and static realizations of vectors and the accompanying visual mediators to support each distinct realization; and addressing how mathematical language, visual mediators, routines, and endorsed narratives differ for the objectified realization of translation as a congruence transformation compared to its realization as a process of repositioning shapes within the coordinate system. Teachers' explication of mathematical discourse and making it an explicit topic of discussion is a form of meta-level learning that has the potential to help students become aware of their teachers' and own discourses, and make communication more transparent in the classroom (Bar-Tikva, 2009; Güçler, 2016; Kjeldsen & Blomhøj, 2012). Explicating elements of mathematical discourse can also be useful for teachers in developing strategies to address student difficulties in the classroom (Güçler et al., 2015).

In this study, we could not provide an analysis of Okan's and Eda's discourses in the classroom since they did not speak during the lesson on translation. Due to direct instruction, we could not analyze these students' discourses in a more interactive setting. However, our work is representative of the characteristics of mathematical communication in classrooms where teachers' stated teaching approaches can be incompatible with their enacted approaches. In such classrooms where the elements of mathematical discourse can remain implicit for the students, it is possible for them to develop their idiosyncratic realizations of translation as exemplified by Okan's and Eda's discourses. We acknowledge that the particular lesson during which we observed Mr. Can may not be representative of his overall teaching approach. Yet, it was the only lesson in which the students learned about translation and the teacher's discursive patterns in this lesson impacted how the students thought about translation during the course of the study. Although our analysis demonstrated that Okan and Eda formed contrasting cases regarding their discursive development, the teacher's diagnostic competence when initially recommending these students for our study may also have influenced the findings of the study.

Our work can be considered as an initial step in utilizing a communicational framework based on the need to have further research that examines the development of student learning in conjunction with classroom teaching of geometric translation. More research is needed to learn about how student thinking about geometric transformations change over time, especially during the critical junctures in their development of thinking about these concepts. Future research also needs to focus more on the influence of teachers in student thinking of translations (Yanik, 2011) and how instruction can motivate students to form useful process- and object-based realizations of translations (Glass, 2001). Sociocultural frameworks in general, and Sfard's (2008) framework in particular, are potentially useful lenses that future studies can use to explore these critical issues in classrooms so that teaching and learning are examined together within their context.

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