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The effect of conceptual understanding principles-based training program on enhancement of pedagogical knowledge of mathematics teachers

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Abstract

The study aimed to investigate the effect of a training program based on the principles of conceptual understanding in enhancing the pedagogical knowledge of mathematics teachers. The study sample consisted of 34 male and female in-service mathematics teachers, who teach the middle grades (5-8). To achieve the objectives of the study, a quasi-experimental approach was used, with a pre-post design for two groups. In order to collect data, a pedagogical knowledge test was used for the content of numbers and algebra contained in the middle grades curriculum. The results showed a significant effect of the training program in enhancing the pedagogical knowledge of mathematics teachers, with two dimensions (knowledge of teaching mathematics: approaches and strategies, knowledge of students' thinking). The results also showed an improvement in performance levels on the test in general among the teachers of the experimental group, as the percentage of teachers in the level of skilled performance increased from 0% on the pre-test to 53% on the post-test. In light of the results, the study recommended to conduct more research through employing more principles of conceptual understanding to enhance pedagogical knowledge among broader samples of mathematics teachers and different mathematics content.

Keywords: pedagogical knowledge, effective teaching practices, professional development, mathematics education, in-service training

INTRODUCTION

The professional development of teachers of mathematics is an area in which research has increased dramatically in recent years. In this dynamic field, mathematics teaching practices, mathematics teacher pedagogical knowledge, and professional training in mathematics teacher education have emerged as vital areas of research (Al-Hassan et al., 2022; Hea-Jin & Vanessa, 2023; Jeschke et al., 2021; Zehetmeier et al., 2020).

Hea-Jin and Vanessa (2023) underlined on the role of training programs on developing mathematics teachers' knowledge, beliefs, and practice. Golding (2017) emphasized that the development in the process of teaching mathematics goes beyond the teacher's knowledge of the mathematical content, but rather a teacher who is skilled in the pedagogy of the mathematical content. Aviyantian (2020) and Bwalya and Rutegwa (2023) also indicated that the decisions made by mathematics teachers in the classroom are shaped by their mathematical and pedagogical knowledge. There are several goals that can be achieved through the professional development of teachers, including enhancing teachers' self-understanding of mathematics, identifying the various possible ways of understanding among students, and their misunderstanding in mathematics, as well as raising the level of pedagogical effectiveness of teachers, and their involvement in solving mathematical problems in a participatory manner.

In the context of the professional development of the in-service mathematics teacher, Schwarz and Kraiser (2019) consider it as one of the most important factors that play a role in the effectiveness of school education, due to the results of studies that indicated the

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Contribution to the literature

- Teachers practice mathematics teaching in a meaningful way, considering the integration between mathematics content and pedagogical knowledge.
- Teachers practice mathematics teaching effectively, using conceptual understanding principles, where conceptual understanding principles could be used in designing lessons and mathematics curricula.
- This study shows the importance of pedagogical knowledge in improving teaching and learning practices in mathematics learning environments. Specialized training programs based on conceptual understanding could be used as effective approach for teachers' professional development.

relationship between the professional knowledge of the mathematics teacher and the performance of his students; so they put forward two interrelated research scenarios, namely, the teacher's transition from the novice to the skilled, and the evaluation and development of the teacher's professional competence (Aviyantian, 2020; Bataineh et al., 2007; Bwalya & Rutegwa, 2023; Mapulanga et al., 2022; Yang et al., 2021).

In the same context, Ministry of Education (2018) in Jordan paid attention to preparing and qualifying teachers and their professional development as the dynamic and central point in the teaching-learning process through many programs and courses aimed at developing performance, which is positively reflected in improving students' performance and the quality of their learning. Ministry of Education in Jordan indicated in National Charter for the Teaching Profession for the year 2018 that there are seven main areas of national standards for the professional development of teachers, which are: education in Jordan, personal philosophy and professional ethics, academic and pedagogical knowledge, learning and teaching, the learning environment, sustainable professional development, and learning for life (Ministry of Education, 2018).

The development in teaching and learning mathematics comes through the use of methods and strategies that consider the mathematical content and its specificity at all levels of study. National Council of Teachers of Mathematics (NCTM) (NCTM, 2000, 2020a, 2020b) referred to the content standards for school mathematics curricula, foremost among which are numbers and operations on them, and algebra. These two content standards play an important role in mathematics textbooks, especially for the middle grades. Hence, it is necessary to focus on raising the efficiency of pedagogical knowledge of in-service mathematics teachers in teaching numbers and algebra to students, considering effective teaching practices that enhance conceptual understanding in these two standards.

At the international level, many researchers (Jeschke et al., 2021; NCTM, 2020c; Zehetmeier et al., 2020) attached great importance to the professional development of pre- and in-service mathematics teachers, as NCTM (1991) was among a group of institutions concerned with the process of reforming learning and teaching mathematics. Professional standards document for teaching mathematics included six standards for professional development for mathematics teachers: Having experience in mathematics education, knowledge of mathematics and school mathematics, knowledge of students, knowledge of the pedagogy of mathematical content and approaches to develop students' mathematical ideas, professional development of the mathematics teacher and the role of the mathematics teacher in the professional development process (NCTM, 1991, 2020c).

In terms of the professional development of mathematics teachers, global trends came to emphasize the importance of the role of mathematics teachers' possession of the knowledge that enables them to activate students' learning of mathematics. In this regard, many international studies (Buser, 2018; Mohamed et al., 2023; Olson et al., 2014; Zalami, 2020) have shown that the role of mathematics teachers is no longer limited to planning and implementing lessons, but rather how students think, and how they can be challenged for good learning and the advancement of their academic performance in general. This underscores the importance of improving the quality of teaching, for the teacher to become skilled and professional in a way that stimulates learning. This reflects the importance of continuous in-service training programs that update the teacher's knowledge and various skills in teaching and learning mathematics (Ali et al., 2023; Buser, 2018; Mohamed et al., 2023).

More specifically, NCTM (2000) asserted that effective learning needs a teacher who has a good knowledge of mathematics, an understanding of how students think, and how they construct their ideas by linking new ideas with previous ones. As well as knowledge and understanding of teaching strategies.

In addition to, several studies have indicated the importance of considering the training needs of mathematics teachers as well as the significance of developing them professionally (Barham, 2020; Fraihat et al., 2022; Mueller et al., 2014; Omar, 2014).In light of the foregoing discussion, it is necessary to identify these needs, try to find solutions to them, and develop and improve the capabilities of teachers through holding a training program based on the principles of conceptual understanding, with the aim of enhancing pedagogical knowledge of mathematics teachers.

STATEMENT OF THE PROBLEM

In-service teacher training is one of the most prominent challenges facing the teaching of mathematics in various countries of the world, as it forms the basis for the development of teachers as individuals and groups (Barham, 2020; Copur-Gencturk & Tolar, 2022; Ma'rufi et al. 2018; Mok & Park, 2022; Shaqar et al., 2020). These challenges have emerged in Jordan clearly after the change in mathematics textbooks during the last three years, which requires specialized training to familiarize teachers with the standards of change that have occurred, foremost of which is pedagogy and learning with understanding. In addition, the training programs organized by Ministry of Education in Jordan for in-service teachers are limited to general issues in the field of education and are not specialized in the field of teaching and learning mathematics.

In the context of training and professional practice for mathematics teachers, Krainer (1996, 2001) refers to four dimensions in this field, which constitute an intervention strategy for their in-service training, namely, the action that generates positive trends, competition, experimentation, building a clear goal orientation, autonomy that generates responsibility, self-organized work, reflection that generates self-criticism, and a network of communication that generates collaborative work between teachers.

The issue of the study also lies in the field observations of one of the researchers who works as a mathematics educational supervisor, as these observations indicate a weakness among mathematics teachers in their teaching practices by reliance on memorization, and focusing on procedures at the expense of conceptual understanding, in addition to the lack of awareness of the concept of pedagogical knowledge in the mathematical content and its dimensions, such as: approaches and strategies for teaching mathematics, and knowledge of students' thinking. There is no doubt that the absence of this awareness causes weakness among students and a low level of their achievement.

Based on the importance of mathematics teachers' knowledge, and the significance of training them to be effective teachers in mathematics learning environments, research studies were conducted on conceptual understanding and pedagogical knowledge of mathematics teachers, as separate variables, around the world and in Jordan (Al-Barakat et al., 2022a; Al-Hassan et al., 2012; Al-Tarawneh, 2019; Bwalya & Rutegwa, 2023; Fraihat et al., 2022; Khasawneh et al., 2023; Mapulanga et al., 2022; Meier, 2021; Shaqar, 2020).

To the best of the authors' knowledge, there is a significant lack of research studies that investigated the pedagogical knowledge of in-service mathematics teachers by exposing them to training programs based on principles of conceptual understanding in mathematics. It is believed that the current study sheds light on other studies to be conducted on this issue by filling the relevant gap in the literature. In this context, the purpose of this study is to examine the effect of the training program based on the principles of conceptual understanding in enhancing the pedagogical knowledge of content (PCK) among mathematics teachers in the middle grades. To achieve this, the study tries to answer the following two questions:

1. What is the effect of the training program based on the principles of conceptual understanding in enhancing PCK among mathematics teachers in the middle grades?
2. What are the performance levels of mathematics teachers on the pedagogical knowledge test according to the training program?

THEORETICAL BACKGROUND

Teacher's Pedagogical Knowledge

The current educational trends underline on the teacher's possession of the ability, sufficient specialized knowledge, and pedagogical knowledge necessary for teaching content. Teacher's pedagogical knowledge is one of the most important factors that affect the effective teaching of mathematics at all academic levels. Shulman (1986) discussed what the teacher should know and practice, as well as the gap between content and the pedagogical process, as he indicated and emphasized that content and pedagogy cannot be separated and considered them one body of knowledge. Researchers (Cooney, 2001; Copur-Gencturk & Tolar, 2022; Ma'rufi et al. 2018; Mok & Park, 2022) stressed the importance of integration between the mathematics content and the pedagogy of the mathematical content, through the development of educational materials that integrate them, and are designed to display different mathematical situations, and make teachers engage in problem solving practice and raise pedagogical issues related to the mathematical content being taught. Through integration process, teachers are allowed to reflect on their understanding of mathematics and to establish positive beliefs about teaching mathematics, and this approach, in turn, enhances their professional development. Cooney (2001) and Ma'rufi et al. (2018) also noted that education reform is rooted in Dewey's vision, which is governed by processes that evoke justification and problem solving, not just the provision of knowledge.

In the same context, numerus researchers (Mapulanga et al., 2022; Meier, 2021; Ponte, 2001) raised several issues facing the mathematics teacher in planning and practice in the classroom; These include selecting and designing investigative situations, adapting and adopting curriculum vocabulary, selecting teaching and assessment tasks, formulating objectives, and defining the form of procedures that fit the characteristics of students and their previous experiences. Moreover, some studies (Alali & Al-Barakat, 2022; Bwalya & Rutegwa, 2023; Ponte, 2001) identified a set of characteristics of the learning tasks in the classroom, the most prominent of which are that they are rich in concepts, unfamiliar, require justification, and stimulate solving the issue in various life contexts. In addition, it was emphasized that professional development programs can be achieved through courses, projects, exchange of experiences, readings, reflection on the experiences of others, and others, and that they must balance theory and practice (Bwalya & Rutegwa, 2023; Mapulanga et al., 2022; Meier, 2021; Ponte, 2001).

In the context of the nature of the knowledge that the teacher must possess to be able to perform his role efficiently and effectively, Shulman (1987), indicated that it is divided into seven areas, the prominent of these are knowledge of content, knowledge of the curriculum, PCK, knowledge of the learner. Many researchers adopted that PCK is the integration of content and pedagogy, and they concentrated on knowledge of instructional approaches and knowledge of students, where in the knowledge of students' aspects, the researchers concentrated on students' conception of mathematics concepts and misconceptions and how to overcome these misconceptions (Aksu & Kul, 2016; Copur-Gencturk & Tolar, 2022; Ma' rufi et al., 2018; Mok & Park, 2022).

Khasawneh et al. (2022) emphasized that one of the sources of support for teachers' pedagogical knowledge in mathematics is the experiences that each of them possesses. In addition, Khasawneh et al. (2023) indicated that the classroom teaching practices of the mathematics teacher are a major indicator and determinant of students' behavior in the classroom, their progress and academic achievement. In this regard, Hill et al. (2008) emphasized the need to link PCK with student learning outcomes, the ability to know perceptions and prior knowledge they possess, and that professional development opportunities focus on developing teacher knowledge and skills in understanding students' work and their mathematical thinking

NCTM (2014) stresses that the ability of teachers to keep pace with the requirements of development, and to deal with them, constitutes a challenge to the reform process, as it can only be achieved if teachers have a deep understanding of mathematics and the ability to deal with it flexibly during the process of their teaching. In this context, Simon and Schifter (1991) suggest that positive reform of mathematics education requires important new initiatives in teacher development. Whereas Jackson et al. (2020) believe that mathematics teachers need training programs that suit their specialized teaching needs in order to keep pace with developments and activate experiences among them. Many studies (Al-Barakat et al., 2022b; Fraihat et al., 2022; Khan, 2012; Khasawneh et al., 2022) considered that it is necessary to build training programs designed to develop and support the growth of knowledge of mathematics teachers, as some mathematics teachers have limited knowledge of mathematics education, yet they teach it. Khan (2012) argues that teachers may fail to make mathematics an interesting subject if they organize their work with poor content knowledge and pedagogical skills or rely on their academic qualifications to teach them in isolation from their professional qualifications.

NCTM (2000) called for understanding-based learning through the principle of learning, and stresses in this context on conceptual understanding, through the possibility of using the knowledge and facts that have been learned flexibly and fluently and applying them in real life situations, as well-connected ideas built on a conceptual basis are more amenable to use in a variety of new situations than others.

Conceptual Understanding

Conceptual understanding is defined as connecting inseparable mathematical knowledge from each other. This indicates to the ability of student when to describe, clarify and apply the mathematical concepts in diverse methods and in different situations (Große-Heilmann et al., 2022; Yuliandari & Anggraini, 2021). In addition, the researchers emphasize that conceptual understanding refers to the student's ability to integrate the concepts she/he has learned. This understanding can take place when mathematical concepts are not isolated. It greatly contributes to enabling the student to perform operations and link concepts. Thus it enhances retention, encourages fluency and assists learning material (Desfitri, 2016; Große-Heilmann et al., 2022).

In order to conceptual understanding takes place, mathematics teachers should provide students with clear instructions about classroom discussions and then design models for asking questions and interaction by students, as well as highlighting students' ideas and using them to enhance understanding of mathematics, in order to help students present evidence and arguments, generate examples, and develop their ability to generalize, analyze and represent mathematical ideas in multiple ways. In the same context, American National Research Council (ANRC) (ANRC, 2001) stresses the importance of conceptual understanding and its effect on student performance and considers it one of the main objectives of mathematics education, and one of the pillars of fluency in mathematics, so that students can distinguish concepts, understand the links between concepts and procedures, and helps them explain that some facts are results of other facts. In order to achieve a conceptual understanding of mathematics, there must be mathematical knowledge, pedagogical content knowledge that emphasizes the methods and approaches to teaching mathematical concepts, in addition to knowledge of students' thinking and making them the focus of the teaching-learning process. Doherty (2010) indicated that students' conceptual understanding reflects their ability to think and understand concepts, mathematical operations and relationships, which helps students solve non-routine problems.

Yuliandari and Anggraini (2021) report that conceptual understanding means having experience, and is necessary to enhance procedural fluency, as the student can interpret procedures, and if conceptual understanding is absent, there is a high probability of making mathematical errors.

Jones et al. (2013) gave conceptual understanding great importance because it facilitates the application of mathematical knowledge in new situations and facilitates the retrieval of previous knowledge. Desfitri (2016) indicates that teachers' lack in mastering concepts affects the way they present the concept, and negatively affects students' understanding of the concept. According to Copur-Gencturk (2021) teachers' understanding of the concepts they teach affects the quality of teaching and student learning. Khasawneh et al. (2022) argues that teachers' lack of understanding makes it impossible to provide a teaching-learning environment that enables students to build an understanding of the concepts they learn.

Laswadi et al. (2016) also developed three indicators of students' conceptual understanding, represented by the ability to link mathematical concepts to others, the ability to represent the mathematical problem through a set of ways, and to identify the most appropriate representation of certain situations. Yang et al. (2021) summarized a set of insights presented in the previous literature related to indicators of conceptual understanding in mathematics, most notably: representations of mathematical ideas, their multiplicity, and navigating between them, and linking previous knowledge to new knowledge in order to generate a new knowledge structure.

Hiebert and Grouws (2007) and Nahdi and Jatisunda (2020), pointed out the importance of conceptual understanding, and see that conceptual understanding is related to making the connection between mental processes and mathematical structure, developing mathematical thinking, and engaging in modeling through productive discourse.

Many researchers presented definitions of conceptual understanding, as Sands (2014) defined it as a type of thinking far from simple, although it involves the use of simple relationships, it requires coordination between different fields of knowledge. Mayer (2018) also identified that it goes beyond knowing concepts and facts to relate and organize them well into memory schemas. Jaradat (2018) defined it as the ability to perceive the definition of concepts, visualize and represent them in multiple ways, link multiple representations of a single concept, link different concepts and discover relationships between them, solve practical life issues, and interpret procedures through understanding. Moreover, National Center for Education Statistics (NCES) (NCES, 2003) considers conceptual understanding as one of the mathematical abilities to link graphic models and various representations of concepts, know and apply facts, principles and definitions.

In the context of enhancing conceptual understanding, NCTM (2000) believes that students need conceptual understanding in mathematics in order to solve the problems they face, which is one of the requirements of the twenty-first century. Since the role of the teacher is to facilitate the process of learning mathematics, and not to transfer mathematical knowledge to students, NCTM (2014) summarized eight practices of effective instructional practices that would enhance conceptual understanding, and called them the principles of conceptual understanding in mathematics: setting specific goals for learning mathematics, carrying out tasks that promote justification and solving problems, using various mathematical representations and connecting them, facilitating purposeful mathematical discourse or dialogue, asking meaningful questions, building procedural fluency through conceptual knowledge, supporting productive thinking in learning mathematics, and elicit and use evidence about student thinking. Accordingly, it can be mentioned that the effective program in training mathematics teachers is based on the following principles:

1. **Carrying out tasks that promote justification and problem solving:** NCTM (2014) emphasizes that effective teaching uses tasks as a way to stimulate student learning and help them build new mathematical knowledge through problem solving. It also emphasizes that the tasks presented to students should be based on the important and accurate mathematical content, and these tasks consist of projects, questions, problems, applications and exercises that engaged students in learning and teaching.

   Foster (2013) believes that there is an urgent need for mathematical tasks that integrate the practice of basic methods in an enriching, exploratory and investigative context, which makes the teaching of mathematics more harmonious. He also emphasizes that solving mathematical problems constitutes a more original mathematical activity, and gives students enough space to develop understanding, and opportunities to develop independent creative solutions.

Yuliandari and Anggraini (2021) also point out that understanding-based learning is possible under a variety of circumstances such as: engaging students building communication between them, sharing and explaining their answers with colleagues, which encourages students to think of arithmetic procedures as a problem-solving activity. Samuelsson (2010) supports this as the use of teaching methods in which students use their language for discussion in solving mathematical problems positively affects students' conceptual understanding. King et al. (2016) see two strategies that build deep understanding in students: problem-based learning and opportunities for students to think and justify. Al-Barakat et al. (2022b) and Fraihat et al. (2022) stress that the mathematics component is characterized by four main activities: pattern identification, guess work, providing proof and presenting an argument

2. **Using mathematical representations and linking them together:** In professional standards for mathematics teaching document, NCTM (1991) refers to the importance of representations, connectivity, and mobility among them, within the fourth standard of professional development standards for mathematics teachers, which is "pedagogical knowledge of mathematical content" by saying: that the modeling of mathematical ideas through various representations (sensory, visual, graphics, symbols, and others) is an essential move in learning mathematics. Regarding that, the mathematics teacher needs a rich and deep knowledge in the various methods of modeling different concepts, procedures and ideas in mathematics, in addition to the ability to navigate between these representations because it helps students understand mathematics and be able to build it themselves, and therefore the ability to choose, modify and build representations is at the core of mathematics pedagogy.

   In the same context, NCTM (2014) emphasizes that effective mathematics teaching engages students in diverse representations (visual, physical, symbolic, verbal, and contextual) and making connections between them to deepen understanding of mathematics concepts and procedures as problem-solving tools. Anthony and Walshaw (2009) developed ten principles of effective teaching, where the use of representations was the eighth of them. They indicated that the active teacher chooses representations of mathematical ideas that contribute to supporting students' thinking carefully and accurately and developing their understanding. In the same context, Dreher and Kuntze (2015) and Ge (2012) see that multiple mathematical representations play a dual role in teaching mathematics because they are essential to understanding mathematics, but on the other hand they can be a hindrance to learning if misused by the teacher or students.

3. **Building procedural fluency from conceptual knowledge:** NCTM (2014) clarifies that effective mathematics teaching builds procedural fluency based on conceptual understanding so that, over time, students become skilled in using procedures flexibly as they solve contextual and mathematical problems and focuses on developing both conceptual understanding and procedural fluency. It also emphasizes that procedural fluency builds on the basis of conceptual understanding, strategic justification, and problem solving, so that students who practice mathematics effectively discover that it is much more than just procedures, therefore, students must know the appropriate and most effective procedure for a particular case.

   Bautista (2013) believes that students' procedural fluency is affected by their mathematical knowledge and abilities. In the opinion of the ANRC (2001) and NCTM (2020a), procedural fluency and conceptual understanding are often seen as competing in school mathematics, but mastery of skill versus understanding generates a false separation between them, they are interrelated, as understanding makes learning an easy and less prone to making mistakes and forgetting.

Previous Studies

In the research field related to the topic of the current study, it was found that there are few studies that dealt with the principles of conceptual understanding combined and their impact on enhancement of pedagogical knowledge. On the other hand, there are studies that dealt with pre-service teacher preparation programs and in-service professional development programs, and others that investigated the principles of effective teaching that enhance conceptual understanding, and studies that investigated the pedagogical knowledge of mathematics teachers using the descriptive approach.

The training programs varied, some of which were based on mathematical modeling and its effect on modeling skills, mathematical thinking, decision-making skills and problem solving skills of mathematics teachers (Koc and Elci, 2022), which revealed the effectiveness of the programs in teachers' performance on tests that dealt with the different variables. In the same context of training programs, Kim et al. (2018) concluded that there are fundamental effects of a training program on mathematics knowledge, teaching strategies of mathematics and use of representations. Moreover, Bozkus and Ayvas (2018), Desfitri (2016), and Ding et al. (2014) showed that teachers do not have comprehensive and sufficient knowledge about mathematical justification, most of the mathematics teachers were at the relational level after analyzing their understanding of mathematical justification and mathematics teachers who have the ability of representing mathematics concepts in different ways, are more flexible in presenting mathematics concepts.

In the same context, Al-Tarawneh (2019) concluded that mathematics teachers' use of mathematical representations (pictures, shapes, models, symbols, and life situations) was at low rates, while the teachers' use of written symbols and verbal representations was high. Based on professional development course, Santos et al. (2022) indicated that the course had an effect on enabling teachers to deepen their understanding of the processes of generalization and justification, and their ability to enhance these processes with their students. In addition, Goos et al. (2021) recognized a change in the knowledge of mathematics teachers and their ability to teach more advanced mathematics, and they recorded teaching practices consistent with the developed curriculum that adopted problem solving, after they were exposed to an in-service training program (professional diploma) in mathematics. Whereas Khasawneh et al. (2023) revealed the possibility of benefiting from professional development programs in improving the mathematical thinking strategies of primary school teachers, and transformations in understanding mathematical thinking by engaging in the presentation of lessons, preparing tasks and including them within the lesson.

It is noted by reviewing the theoretical and research background that it is necessary to pay attention to designing and building special training programs and materials, which mainly aim to enhance teachers' knowledge related to mathematics, including pedagogical knowledge.

Moreover, the professional development in mathematics learning environments requires the presence of teachers who have knowledge, confidence, competence and enthusiasm to teach mathematics at all levels of school education (Amendum & Liebfreund, 2019; Hudson et al., 2015), as the mathematics teacher plays a pivotal role in influencing the progress of students' mathematical knowledge and skills. Accordingly, mathematics teachers should be well qualified in mathematical knowledge and pedagogical knowledge and should continue their professional development (Fraihat et al., 2022; Hea-Jin & Vanessa, 2023; Machaba, 2018; Malatjie, 2012).

Based on the above discussion, the current study gains importance as it offers a training program based on the principles of conceptual understanding, in order to enhance pedagogical knowledge in mathematics. It is one of the first specialized programs to train mathematics teachers in Jordan, which is in line with the recent amendment of mathematics textbooks. At the field level, the practical importance of the study is shown through the possibility of using it by teachers to improve their specialized pedagogical knowledge, and to employ that knowledge in their teaching practices (Abu Naba'h et al., 2009; Blömeke & Delaney, 2012; Voss et al., 2011).

It is also expected that improving teachers' pedagogical knowledge will improve students' academic achievement and benefit from mathematics in daily life. Based on the foregoing, this study presents a perception for decision-makers in Ministry of Education in Jordan about a training program based on conceptual understanding, and its impact on teachers' knowledge of mathematics pedagogy, and may help reconsider the training programs provided to in-service teachers and make them specialized training programs.

METHODOLOGY

Research Design

To find out the effect of conceptual understanding principles-based training program on enhancement of pedagogical knowledge of mathematics teachers, the quasi-experimental approach was used with a pre-post design for two equal groups (Cohen et al., 2017), one of them was an experimental group that underwent the training program, and the other was a control group that did not undergo any training program during the implementation of the program.

Participants

The study sample consisted of 34 participants (19 female teachers, and 15 male teachers). All the subjects of the study were randomly selected from Jordanian public schools in north of Jordan for the first semester (2021/2022). All respondents teach mathematics for middle grades (5-8). They were randomly distributed into two equal groups, an experimental group (seven male teachers and 10 female teachers) that underwent the training program, and a control group (eight male teachers and nine female teachers) that was not subjected to training.

It should be noted that the experimental and control groups did not attend any previous training programs.

Moreover, the two groups were randomly selected from 25 schools located in one area in the north of Jordan with similar conditions. This means that all the subjects of the study are similar in academic qualifications, professional development, teaching experiences, economic and social conditions.

Training Program

The training program for in-service mathematics teachers has been prepared based on the conceptual understanding principles issued by one of NCTM (2014) documents, that deal with the effective instructional practices that promote conceptual understanding in mathematics, in addition to the experiences of the authors in this field. Three conceptual understanding principles were adopted in this program: carrying out tasks that promote reasoning and problem solving, using mathematical representations and linking them together and building procedural fluency from conceptual knowledge. The program dealt with its general and specific objectives, its mathematical content, training methods, and the timetable for its implementation, in addition to its validation. The program aims to enhance the performance of in-service mathematics teachers by relying on the principles of conceptual understanding and employing them during teaching within the three above principles.

A set of special objectives of the program were extracted, summarized in clarifying the concept of conceptual understanding, designing a learning environment that supports conceptual understanding, designing instructional tasks that support student learning and challenge their abilities through the three principles of conceptual understanding, designing activities that support students' acquisition of reasoning and problem solving, procedural fluency, mathematical representations and navigating between them and planning lessons according to the principles of conceptual understanding. All objectives were achieved in the content of numbers (fractions, regular and decimal, rational numbers, ratio and proportion, real numbers, prime number, number sense, and operations on numbers), and algebra (equality, variable, patterns and relationships, functions, equation, solving linear and quadratic equations, notation, and algebraic sense), in middle grades mathematics curriculum.

The program dealt with several themes: The sequence of the mathematical concepts included in the textbooks of grades (5-8) on the topics of numbers and algebra, and how they are related, an overview of conceptual understanding and its link to conceptual knowledge and procedural knowledge, an overview of instructional practices that support conceptual understanding, with a focus on the three principles previously mentioned and practical applications in the fields of numbers and algebra according to the three principles of conceptual understanding. The training program took 30 hours of face-to-face training, divided into 10 hours for the theoretical aspect of the program, and 20 hours for the application aspect.

As for the training methods and media, the objectives of the program, the teachers' backgrounds, their academic and educational level, their experiences in teaching mathematics, and the training courses they were previously exposed to, were taken into consideration, knowing that they were not exposed to specialized courses in teaching and learning mathematics. Training methods included: direct training; that is, seminars, lectures, group discussion, role plays (simulation), educational videos, and workshops to confront a problem and work to solve it. The program also included indirect training methods by assigning the trainees with home self-tasks.

The training program was subjected to a group of specialists in the field of preparing training programs, and a number of specialists in mathematics pedagogy in order to ascertain the ability and suitability of training activities and methods to enhance the skills and experiences of teachers, their compatibility with the principles of conceptual understanding, and the time required to implement the program. The suggestions of the referees were considered in light of the objectives of the program. In this context, the referees suggested concentrating on using home self –learning tasks such as reading pamphlets, printed study materials, and audio-visual educational materials (video tapes, recording tapes, computer programs), noting that much homework was discussed in face-to-face sessions.

The following examples, illustrate sample of activities practiced in the training program according to each principle of conceptual understanding.

Carrying out tasks that promote justification and problem solving

Teachers should be familiar with low level tasks and high level tasks, with the concentration on the later that needs non-algorithmic thinking, practicing mathematics, students' engagement, cognitive effort and different strategies for solving it. In this context, teachers should use and formulate tasks that promote justification and problem solving.

Activity1: *Discuss with your group the cognitive processes and the resources that students need in solving these two problems and decide if each is rich task or not.*

1. *Solve the equation system, $2x-y=12$, $2x-4y=24$.*
2. *If $x<1$, then $x^2>x$, x is a real number. Is this statement true? If your answer is yes justify, and if your answer is no, justify and correct the statement.*

Activity2: *Within your group, write four rich mathematical tasks that promote reasoning and problem solving in the context of algebra and numbers for grades 6-8, and exchange the tasks between the different groups in order to reflect on it, followed by collective discussion.*

Using mathematical representations and linking between them

This principle is translated into practice for the purpose of using different representations (visual,

Table 1. Examples of the tasks included in the test

Task	Definition
Task 1	*Ahmed and Khaled have a number of balls, if you know that ($\frac{2}{5}$) of them are with Ahmed, and ($\frac{3}{10}$) of them with Khaled, how can you use models (representations) to show the student who has more balls, Ahmed or Khaled?*
Task 2	*One of the students addressed to you "I do not understand the meaning of ($1\frac{2}{3}$x$\frac{4}{7}$) how can you explain the meaning of this process to the student? Suggest a method or approach to help the student.*
Task 5	*Hassan is a math teacher for eighth grade students, but his students find it difficult to accept the idea that there are systems of linear equations with two variables that have no solution in R. How can you help Hassan explain the idea to his students? Design two different activities to help Hassan.*
Task 6	*Teacher asked the following question to the fifth-grade students: "The verbal expression of ($4\div\frac{1}{4}$) is: (a) Four whole divided into four equal parts, (b) $\frac{1}{4}$ part of 4 whole, and (c) How much $\frac{1}{4}$ in four whole. One student's answer was: Four whole divided into four equal parts, what is your response to that answer? If it is inappropriate, how to address the error or the misconception made by the student?*
Task 7	*The following question was presented to eighth grade students: "Mustafa used to put his money in two savings boxes, if he put in first box an amount of 200 dinars, and put in second box an amount of 100 dinars, then he started adding 25 dinars per month in first box, and 50 dinars per month in second box, in which month are two funds equal?" (a) How do you know how well students understand problem? Suggest some questions you could ask them to make sure, (b) What concepts & prior knowledge should students possess to solve such a problem?, and (c) Describe two different ways to solve this problem.*

pictorial, physical, symbolic, real-life context) of mathematics ideas, and translating between it in order to promote conceptual understanding. Examples are:

(1) *analyzing and assessing given students' representations,*

(2) *observing videos related to different representations of functions, equality concept, and how to translate between them,*

(3) *how the different representations of equality can help to teach the concept of equivalent algebraic expressions, and*

(4) *using physical, Iconic and symbolic representations for the concepts: function, solving linear equations and fractions.*

Building procedural fluency from conceptual knowledge

Procedural knowledge is the ability of applying procedures flexibly, precisely and effectively. To find the following multiplication 3.14×4.5, through conceptual understanding, the students recognize that the result is between 12 and 20, and they estimate the result precisely that it is very close to 14, while in procedural knowledge, students use algorithm to calculate 3.14×4.5=14.13 and use conceptual understanding to evaluate if the result is reasonable.

Activity: Within each group of teachers, clarify how can you promote procedural fluency for your students in the following task: If $3(2x-1)^3=24$, find the values of : $4x^2-4x+3$, $6x-4$, and $(2x-1)/4$.

Instruments (Pedagogical Knowledge Test)

The pedagogical knowledge test was designed to answer the two questions of the study, by referring to a group of studies (Aksu & Kul, 2016; Copur-Gencturk & Tolar, 2022; Diko & Feza, 2014; Kim et al., 2018). Based on the definition of pedagogical content knowledge in the current study, which included two dimensions (knowledge of teaching approaches and strategies, and knowledge of students' thinking), 13 open-ended tasks, in the form of teaching situations and scenarios, were prepared by the researchers based on their experience in mathematics education. They dealt with the topics of numbers and algebra related to the middle grades curricula in Jordan. The test dealt with the topics: comparison of numbers, operations on numbers, patterns, functions, variables, algebraic expressions, linear equations, solving linear equations, and solving systems of linear equations.

Indicators of the dimension of knowledge of teaching approaches and strategies have been identified with the teacher's ability to: propose appropriate teaching methods and approaches to teach numerical and algebraic concepts, to provide and suggest appropriate activities that contribute to helping students identify and understand concepts and operations and to suggest multiple representations that contribute to a deeper understanding of different ideas. The indicators of the dimension of knowledge of students' thinking were also determined by the teacher's ability to: determine students' prior knowledge of a concept, to identify students' errors and misconceptions and explain it, to suggest ways to address these errors that students may encounter in solving a given problem. Examples of the tasks included in the test are presented in **Table 1**.

In order to score the test, qualitative performance indicators were described for each task of the test, and a three scaled rubric was prepared, based on what was mentioned by Shaqar et al. (2020), to rate the in-service mathematics teachers' responses to each task. Three scores are given for high performance indicators, two scores for medium performance indicators, and one score for low performance indicators. Thus, the maximum score for test is 39, and minimum score is 13.

Table 2. Means & standard deviations (SDs) of teachers' performance in pre- & post-pedagogical knowledge test as a whole according to group

Group	Number	Pre		Post		Adjusted post-means	Standard error
		Means*	SD	Means*	SD		
Experimental	17	23.32	3.87	29.85	4.22	29.70	0.92
Control	17	22.18	3.24	23.79	3.35	23.95	0.92

Note. *Maximum score=39

The maximum score 18 and 21 was distributed to the two dimensions of pedagogical knowledge, namely: knowledge of teaching approaches and strategies, and knowledge of students' thinking, respectively. For example, to score task 1 included in **Table 1**, three qualitative performance indicators were used: suggest one or more correct representation (areas, countable objects, others) and explain how to use it (high level, score 3), suggest one or more correct representation without explaining how to use it (moderate level, score 2), suggest incorrect representation, or using procedures, such as common denominators (low level, score one).

Reliability and Validity of Instruments

To ensure the validity of the content of the pedagogical knowledge test, it was presented to a group of specialists, and they were asked to verify the affiliation of the test tasks to each dimension of pedagogical knowledge, its accuracy, linguistic integrity and clarity. In light of their comments, none of the test tasks was omitted, but suggested modifications were made. These amendments relate to making linguistic corrections. The test was also exposed to an exploratory sample that included 10 male and female teachers from outside the sample, in order to ascertain the time required for the test, the validity of the test's internal consistency, and its reliability.

It was found through the exploratory sample that the time required for the test was estimated at two hours, and that the correlation coefficients of each task with its dimension, and with the test as a whole were, respectively, within the two intervals 0.38-0.86, 0.47-0.88 and with statistical significance ($p<0.05$), which are acceptable values for the purposes of the current study (Odeh, 2010). To assess the reliability of the internal consistency of the pedagogical knowledge test, Cronbach's alpha coefficients were respectively 0.75, 0.78, and 0.88, the knowledge of teaching approaches and strategies, knowledge of students' thinking, and the test as a whole. Moreover , intra-rater reliability and inter-rater reliability of scoring the test were calculated using Cooper's equation (Krippendorff, 2004) and it were 0.89 and 0.87, respectively.

Data Analysis

To analyze the data of the study, statistical package in the social sciences (SPSS) was used to achieve the following:

1. Means and standard deviations of the teachers' performance in the pre- and post-pedagogical knowledge test as a whole according to the group.
2. One-way analysis of covariance (ANCOVA) for mathematics teachers' performance in pedagogical knowledge test according to the training program.
3. Means and standard deviations of pre- and post-measurements of dimensions of pedagogical knowledge test according to the group.
4. One-way multivariate analysis of covariance (MANCOVA) on dimensions of pedagogical knowledge test combined.
5. One-way ANCOVA for each dimension of pedagogical knowledge.
6. Frequencies and percentages were used to answer the second research question.

RESULTS

The results of the study were presented based on its two parts, as follows.

Part One: Results of the Research

This part aims to reveal the effect of the training program on enhancing pedagogical content knowledge. To achieve this aim, means and standard deviations of the teachers' performance in the pre- and post-pedagogical knowledge test as a whole were calculated, according to the group, as shown in **Table 2**.

It is clear from **Table 2** that there are apparent differences between the means of mathematics teachers' performance in the post pedagogical knowledge test as a whole according to the group. In order to find out whether the apparent differences on the post-test as a whole are statistically significant, one-way ANCOVA was used, as shown in **Table 3**.

It is clear from **Table 3** that a one way between groups analysis of covariance was used to compare the effect of a training program designed to enhance pedagogical knowledge. The independent variable was the training program (based on conceptual understanding principles, without training), and the dependent variable consisted of the scores on the pedagogical knowledge test administered after the treatment was completed. Teachers' scores on the pre-test of the pedagogical knowledge were used as the covariate in this analysis.

Table 3. One-way ANCOVA for mathematics teachers' performance in pedagogical knowledge test according to training program

Variance source	Squares sum	Degrees of freedom	Squares average	F-value	Level of significance	ETA square
Pre	28.39	1	28.39	2.02	0.17	
Program	274.03	1	274.03	19.51	0.00	0.386
Error	435.52	31	14.05			
Total	775.94	33				

Table 4. Means (Ms) & standard deviations (SDs) of teachers' performance in pre- & post-dimensions of pedagogical knowledge test according to group

Dimensions	Group	Number	Pre-M	Pre-SD	Post-M	Post-SD	A-post-M	SE
Knowledge of teaching strategies*	Experimental	17	10.32	2.11	13.62	2.29	13.60	0.61
	Control	17	10.44	1.97	11.24	2.50	11.26	0.61
Knowledge of students' thinking**	Experimental	17	13.00	2.42	16.24	2.48	16.17	0.58
	Control	17	11.74	1.99	12.56	2.11	12.62	0.58

Note. SE: Standard error; A: Adjusted; *Maximum score=18; & **Maximum score=21

Table 5. One-way MANCOVA on dimensions of pedagogical knowledge test combined

Effect	Multivariate test	Multivariate test value	F-value	Freedom degree of hypothesis	Freedom degree of error	Probability of error	Size effect
Training program	Hotelling's trace	0.67	9.75	2	29	0.00	0.40

After adjusting for pre-treatment scores, there was a statistically significant difference between the two groups on the post- pedagogical knowledge test, $F(1, 31)=19.51$, $p<0.05$, partial eta square=0.386. The statistical difference was in favor of the experimental group, as shown in **Table 2** through the adjusted means.

In addition, the effect of the training program on the dimensions of the pedagogical knowledge test was tested, where means and standard deviations for the teachers' performance on pre- and post-pedagogical knowledge test were calculated according to the group (experimental, control), as shown in **Table 4**.

Table 4 shows that there are apparent differences between the means in the post measurement of the two study groups for the dimensions of the pedagogical knowledge test. In order to verify the statistical significance of these differences, one-way MANCOVA was conducted on the test dimensions combined. This is as shown in **Table 5**.

Table 5 shows that a one-way between groups multivariate analysis of covariance was conducted. Two dependent variables were used: knowledge of teaching strategies and knowledge of students' thinking. The independent variable was the training program (based on conceptual understanding principles, without training). There was a statistically significant difference between the two groups on the combined dependent variables, $F(2, 29)=9.75$, $p<0.05$, partial eta square=0.40.

In order to determine on which dependent variable the impact of the training program was significant, ANCOVA was conducted for each dimension according to the program, as shown in **Table 6**.

Table 6 shows the results for the dependent variables separately, by conducting the one-way analysis of covariance on each of the dependent variables. There were statistically significant differences between the two groups on each of the dependent variables after controlling for the covariate. For knowledge of teaching strategies, $F(1, 30)=7.1$, $p<0.05$, partial eta square=0.191, and for knowledge of students' thinking, $F(1, 30)=17.81$, $p<0.05$, partial eta square=0.372, where the significant differences were in favor of the experimental group that was subjected to the training program, in light of the post adjusted means shown in **Table 4**.

Part Two: Results of the Research

This part tried to find out the performance levels of mathematics teachers on enhancement of the pedagogical test. The responses of the in-service mathematics teachers on the pre- and post-pedagogical knowledge test were analyzed based on an analytical rubric with performance indicators for each task of the test, as three scores are given for high performance indicators, two for medium performance, and one for poor performance. A total score was calculated for each teacher on the pedagogical content knowledge test, as the maximum score was 39 while the minimum was 13. Using the equation of the range (Odeh, 2010), the performance levels of the teachers were classified according to the extent of their performance on the total test into: a novice teacher, whose score ranged from 13 to less than 21.67, a trainee teacher, whose score ranged from 21.67 to less than 30.34, and a skilled teacher whose score ranged between 30.34 to 39.00.

Teachers' levels were also categorized according to the performance scale for each dimension of pedagogical knowledge. In the dimension of knowledge of teaching strategies, teacher was classified into a novice teacher if his/her score ranged from 6 to less than 10, a trainee

Table 6. One-way ANCOVA for each dimension of pedagogical knowledge

Variance source		Squares sum	Degree of freedom	Squares mean	F-value	Probability of error	Size effect η2
Pre (covariate)	Knowledge of teaching strategies	4.67	1	4.67	0.79	0.380	
Pre (covariate)	Knowledge of students' thinking	1.81	1	1.81	0.33	0.567	
Training program	Knowledge of teaching strategies (post)	41.79	1	41.79	7.10	0.012	0.191
	Knowledge of students' thinking (post)	96.14	1	96.14	17.81	0.000	0.372
Error	Knowledge of teaching strategies (post)	176.52	30	5.88			
	Knowledge of students' thinking (post)	161.97	30	5.40			
Adjusted total	Knowledge of teaching strategies (post)	232.07	33				
	Knowledge of students' thinking (post)	284.89	33				

Table 7. Frequencies & percentages of in-service teachers' performance levels on pre- & post-total pedagogical knowledge test & its dimensions by group

Dimension of pedagogical knowledge	Control group				Experimental group			
		Novice	Trainee	Skilled		Novice	Trainee	Skilled
Pedagogical knowledge pre-test								
Knowledge of teaching strategies	Number	4	12	1	Number	7	10	0
	Percentage	23.5%	70.5%	6.0%	Percentage	41.0%	59.0%	0.0%
Knowledge of students' thinking	Number	7	10	0	Number	6	10	1
	Percentage	41.0%	59.0%	0.0%	Percentage	35.0%	59.0%	6.0%
Total	Number	7	10	0	Number	6	11	0
	Percentage	41.0%	59.0%	0.0%	Percentage	35.0%	65.0%	0.0%
Pedagogical knowledge post-test								
Knowledge of teaching strategies	Number	5	8	4	Number	3	4	10
	Percentage	29.5%	47.0%	0.0%	Percentage	17.5%	23.5%	59.0%
Knowledge of students' thinking	Number	5	12	0	Number	1	7	9
	Percentage	29.0%	70.5%	0.0%	Percentage	6.0%	41.0%	53.0%
Total	Number	5	12	0	Number	2	6	9
	Percentage	29.5%	70.5%	0.0%	Percentage	12.0%	35.0%	53.0%

teacher if his/her score ranged from 10 to less than 14, and a skilled teacher if his/her score ranged from 14 to 18, noting that the minimum score for the dimension of teaching strategies is six, and the maximum score is 18.

As for the knowledge of students' thinking, it was classified into a novice teacher if his score ranged from seven to less than 11.67, a trainee teacher if his score ranged from 11.67 to less than 16.34, and a skilled teacher if his score ranged from 16.34 to 21.00, noting that the minimum score for the dimension of knowledge of students' thinking is seven, and the maximum score is 21. The distribution of teachers' frequencies and their percentages on these categories was based on the results of the pre- and post-tests for each of the experimental and control groups, as shown in **Table 7**.

Table 7 shows a convergence between the distribution of in-service mathematics teachers in the novice and trainee levels, through their performance on the results and dimensions of the total pre-test, the percentage of the novice level teachers was 41.0% in the control group compared to 35.0% in the experimental group, and at the trainee level the percentage of teachers was 59.0% in the control group and 65.0% in the experimental group. While none of the teachers of the two groups was in the skilled level based on their results in the whole pre-test. This gives an indication of the in-service teachers' lack of the pedagogical knowledge in the mathematical content of numbers and algebra before the experiment, especially in the dimensions of strategies and students' thinking.

As can be seen from **Table 7**, the number of mathematics teachers at the skilled level increased in the experimental group sample on the pedagogical knowledge post-test. This is an indication of the effect of the training program in enhancing the pedagogical knowledge of mathematics teachers during the service, as the percentage of teachers at the skilled level increased from 0% in the pre-test to 53.0% in the post-test, while none of the teachers of the control group scored at the skilled level on the whole post-test. It also appears from **Table 7** that the percentage of the novice level teachers in the results of the post-test of pedagogical knowledge decreased and reached 12.0% for the experimental group, and the performance level of the control group focused on the post-test at the trainee level and the novice level, while in return it focused on the trainee level and the skilled level for the experimental group. Moreover, the convergence in the percentages of teachers appears in the skilled level of the experimental group on the post-test and for each of the two dimensions of knowledge of teaching strategies and knowledge of students' thinking. This confirms the importance and effectiveness of the training program in improving teachers' performance to a higher level.

In order to support the quantitative results of the study, a reflective qualitative analysis of the responses of the participants on a sample of the tasks related to the pedagogical knowledge test was conducted, considering knowledge of teaching strategies and knowledge of students thinking, in addition to the different levels of teachers' performance on the post-test (novice, trainee, skilled). The responses of the experimental group were more flexible, precise and elaborative in their suggested teaching strategies, activities that contribute conceptual understanding, and using multiple representations, especially those ranked in the skilled level of performance.

For example, in the first task in **Table 1**, which request comparing between $\frac{2}{5}$ and $\frac{3}{10}$, the teachers at the novice level in both groups used the usual common denominator procedure, without using models. The area model was used commonly by the trainee and the skilled teachers, of both groups, while multiple representations by the skilled teachers were used, such as, number line, area model and countable objects. Moreover, few participants used benchmark strategy to compare between the two fractions, such as $\frac{1}{2}$, but they could not decide, which of the two fractions is closer to $\frac{1}{2}$. For the second task in **Table 1**, which request explanation of the meaning of ($1\frac{2}{3} \times \frac{4}{7}$), teachers in the novice level in both groups used multiplication procedure of two fractions and many teachers in the control group faced difficulty to give a reasonable method to explain that. In addition, few teachers in the trainee level succeeded to explain it by using area model, while many of skilled level in the experimental group successfully used area model and real life situation.

In light of convincing students that some linear equations systems do not have solutions in real numbers, one teacher at the trainee level gave a system of equations: y=3x+4 and y=3x-5, which leads to a contradiction that (9=0), so this system has no solution in the set of real numbers (R). Moreover, skilled teachers in the experimental group gave more than one method, where there are three cases of lines, intersecting (there is a solution), parallel lines (no solution) and congruent lines (infinity of solutions). Regarding task 6 in **Table 1**, many respondents at the novice level in both groups addressed the misconception by dividing $4 \div \frac{1}{4}$ to give 16, but this procedure does not help for understanding, while in the trainee level they used area model to clarify and to correct the misconception, and many at the skilled level from the experimental group introduced easier problem "how many binary set in 10" corresponding to "how many fourth in 4".

In terms of task 7 in **Table 1**, the responses of the experimental group were distributed on the three levels (novice, trainee, skilled), while the responses of the control group were distributed on novice and trainee levels only. The responses of the experimental group were more precise in introducing prerequisites that help students to solve the given problem, such as: equity, unknown, variable, algebraic linear equation, pattern, also they constructed different solutions to solve the given problem, by using table and equations (25x+200=50x+100).

DISCUSSION

The findings of the study can be attributed to the nature of the training program, as teachers were given the opportunity to familiarize themselves with the principles of conceptual understanding that included effective teaching practices, in the forefront of which is the provision of tasks that support students' ability to justify and solve problems, and provide multiple representations of algebraic and numerical ideas, coupled with the navigation between those representations, in addition to linking between conceptual knowledge and procedural knowledge and how to develop procedural fluency through that link. Moreover, the practice of those principles played an effective role in enhancing the pedagogical content knowledge, through the proposed activities and collaborative workshops, with a focus on the integration between these principles, which cannot be separated because they are strongly interrelated.

The findings of the current study agree with the results of some studies that there is an effect of training programs in improving the pedagogical knowledge of mathematics teachers during service, such as (Barham, 2020; Khan, 2012; Kim et al, 2018), where the results showed that the performance of mathematics teachers after undergoing the training program was better than their performance before undergoing the training program in terms of mathematical content and pedagogical knowledge.

The training program in the current study focused on the principles of effective teaching, which developed and deepened conceptual and procedural understanding in numbers and algebra for middle grades (5-8). These results were supported by previous studies (Jeschke et al., 2021; Mapulanga et al., 2022; Shaqar et al., 2020) in the field of teaching mathematics. These studies emphasized the role of the teacher's knowledge of mathematics, which has been studied in university courses, is not sufficient to provide knowledge to students, but rather goes beyond that towards identifying the topics presented in textbooks and methods of teaching them, and this is what was presented through the activities of the program. Watching videos, using role playing between training groups, analyzing the simulation procedures for the teaching process, and discussing them collectively is a key element in better identifying and acquiring pedagogical knowledge of the content.

Moreover, the findings of the study accord with the previous studies (Aviyantian, 2020; Hill, 2007; Yang et al., 2021), that highlight the teachers' acquisition of knowledge and understanding of mathematical concepts, especially those concepts contained in textbooks, and working in an integrated manner with pedagogical knowledge makes the acquisition of pedagogical knowledge flexible and within reach, as the direct activities of the program within group dialogue sessions and their inclusion of discussion, role-playing, educational videos and others contributed to the enhancement of pedagogical knowledge in terms of focusing on many skills, such as the skill of analyzing mathematical situations, and finding solutions to the activities presented. The trainees also contributed to developing means and methods that support prior knowledge of mathematical concepts and enhance subsequent knowledge and link it together with meaningful understanding, which enhances students' conceptual understanding, through the indirect (home) activities assigned to the trainees.

In its activities and training, the program focused on the importance of procedural fluency, and its construction from both the theoretical and applied sides, as it supports procedures and algorithms with flexibility, effectiveness and accuracy, and also contributes to transferring procedures to different issues and contexts, which led to improving teachers' use of multiple strategies in solving mathematical problems, which contributed to raising and enhancing their pedagogical knowledge. In addition, the training program did not overlook the multiple representations of mathematical ideas and the movement between them, which also contributed to understanding unfamiliar mathematical situations and enabled the search for solutions to difficult problems and facilitated the understanding of students' thinking by teachers of how to interpret and treat their errors and misconceptions and how to justify their expected solutions to mathematical problems.

Returning to the integration between pedagogy and the mathematical content, it provided a context for the participants to reflect on their understanding of school mathematics topics, which was reflected in their knowledge of the pedagogy of the mathematical content, its reconstruction and organization, as emphasized by Cooney (2001). Hence, the performance levels of teachers participating in the program improved, and 53% of them moved to the level of skilled in the post pedagogical knowledge test, while this level was scored by 0.00% of the teachers in the pre-test. Dewey, referred to in Cooney (2001), sees that reflective processes are the key to the teaching-learning process because they add a new meaning to knowledge in its various forms, so the reflective activities that were presented in the program, whether individual or group, have appropriately enhanced the teachers' performance indicators. These indicators were represented in the ability to use physical, semi-physical and symbolic representations, offering teaching approaches to algebraic and numerical content that support conceptual understanding, and suggesting educational activities and tasks, in addition to the questioning strategy that shows the teacher the extent to which students understand mathematical ideas, knowing how students deal with problem-solving strategies, and discovering, interpreting and treating students' conceptual and procedural errors.

There is no doubt that the training program has formed a community of practice in one aspect, which led the novice teacher to learn from the skilled teacher, and this in turn led to an improvement in performance levels on the pedagogical knowledge test for a number of participants in the program. This community has been affected by several factors, including: teachers' aspiration for more knowledge in teaching and learning mathematics, teachers' beliefs about mathematics, and its teaching and learning, the nature of the activities presented in the program and covered three basic practices that develop conceptual understanding, the aspirations of the trainees in raising the level of mathematics teaching and learning among middle school students and deepening the knowledge of students' thinking by teachers. These factors, in turn, led to raising the level of responsibility of the teacher to achieve the aspirations of his students, and this kind of responsibility is one of the important values of the mathematics teacher as indicated by Bozkus and Ayvaz (2018).

In addition to the above, the program's balance between theory and practice on the one hand, and practice and training on the other hand, helped the participating teachers share knowledge with their peers and learn from experience as well, and this is what was supported by Matos et al. (2009). It is clear that the participants themselves played a role in enhancing their pedagogical knowledge because of their belief in the necessity of reforming education, and focusing on participation in order to change educational experiences and not only acquire knowledge, through their participation and their serious involvement in activities, and this is what was supported by NCTM (2000) that the reform of mathematics education requires the teacher to play a role in his/her professional development.

CONCLUSIONS

In light of the study results, it can be concluded that focusing on the professional development of in-service mathematics teachers, through training programs based on the principles of conceptual understanding can improve pedagogical knowledge of mathematical content, especially in the areas of teaching strategies and knowledge of students' thinking. This gives an indication that teachers training programs raise the level of classroom performance from the novice to the skilled.

In addition, it can be said that mathematics teachers have achieved a high level of understanding of the integration between the school's mathematical content and the pedagogical knowledge.

Moreover, the study concluded that the way in which mathematical tasks selected and structured can improve mathematical justification and problem solving. This reflects to support the teacher's knowledge of students' thinking. In addition to the foregoing, it can be concluded that building procedural fluency from conceptual knowledge, mathematical representations, and translation between them can contribute positively to achieving integration between teaching mathematics and indicators of understanding students' thinking by the teacher.

It should be noted that the balance between theory and practice on the one hand, and practice and training on the other hand, in professional development programs can support mathematics teachers' pedagogical knowledge, and provide opportunities for discussion, self-reflection and the sharing of ideas, which increases the chances of increasing the educational experiences of teachers. In summary, the principles on which the training program was based in developing conceptual understanding constitute a model for a training program that is consistent with modern theories of learning and teaching mathematics.

Recommendations

Based on the results of the study, it can be recommended to emphasize the importance of in-service mathematics teacher training programs on pedagogical knowledge that makes students to become the centered learning process. In addition to the inclusion of professional development programs for mathematics teachers with the principles of enhancing conceptual understanding, the study also recommends balancing the practice-based and training-based professional development programs for in-service mathematics teachers.

Furthermore, the study recommends conducting more studies that employ and adopt other principles of effective teaching that enhance understanding of concepts as a framework for professional development programs for in-service mathematics teachers, with the aim of revealing the effectiveness of such training programs in developing pedagogical knowledge and teaching practices in the classroom, and conducting studies that depend on the specialized training of mathematics teachers in other mathematical subjects such as measurement, geometry, statistics and probability and its effect on new dependent variables such as teachers' beliefs about mathematics and its teaching and learning.

Limitations and Delimitations of the Study

The findings of the study were limited by a set of limitations that open horizons for future studies in mathematics education. One of these limitations is that the data collection was restricted to an available sample of mathematics teachers in north Jordan, which prevents the generalization of the findings over a wider scope. So, future studies should consider collecting data from south and middle of Jordan, which allows better generalization of the results. Further, data collection relied on the pedagogical knowledge test is limited to knowledge of the strategies and approaches to teaching mathematics content, and knowledge of students' thinking.

To go deeper into the teachers' pedagogical knowledge, future research should rely on semi-structured interview and class observation with the purpose of observing models of mathematics learning environments. In addition, one main limitation of the current study is that the training program was limited to three principles that improve conceptual understanding as reported by the NCTM (2014) and include carrying out tasks that enhance mathematical reasoning and problem solving, using and linking mathematical representations and building procedural fluency from conceptual knowledge.

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