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The interplay of affect and cognition in the mathematics grounding activity: Forming an affective teaching model

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Abstract

This study aims to build a framework for affect-focused (or affective) mathematical teaching (AMT), while promoting higher-order mathematical learning (e.g., pattern finding and deep understanding). The data sources were the class mathematics grounding activity designed by Taiwan's mathematics educators, aiming to enhance students' affective performances in learning mathematics with a theoretical base on the enactivist perspective. Qualitative methodology identified features of affective mathematics teaching and formed a framework for AMT, which defines AMT as transforming natural languages to mathematical languages, highlighting student agenda of upward learning (interest, sense, utter, and present), met by teacher agenda of caring (cultivate, amuse, reflect, and explain). Finally, the enactivist embodiment activities are embedded in the pedagogical structure of 4E phases: entry, entertainment, enlightenment, and enrichment. Affect and cognition interplay in each phase.

Keywords: affect, cognition, mathematics education, qualitative methods

INTRODUCTION

Successful acquisition of mathematical knowledge is essential for academic development and all aspects of human cognition for everyday life (Menon & Chang, 2021¸ Sun et al., 2018). As a subject of "the study of patterns and relationships" (Burton, 1994, p. 12), mathematics is often set within the cognitive domain. However, there has been a call for greater emphasis on affective variables (e.g., Goldin, 2000). This is also in line with the recent emphasis on fulfilling both character (i.e., curiosity, compassion, and courage to take actions) and cognition for success in today's education (OECD, 2021).

Affect plays an important role in mathematics activities (Hannula, 2019; Zan et al., 2006). Earlier work showed that affect is especially vital in solving non-routine creative mathematical problems (Chiu, 2009). Affect is particularly evident in mathematics teaching for higher-order (cognitive) learning. Students as mathematicians experience affective challenges when learning mathematics (Burton, 1994). Projects, games, and puzzles are challenging tasks for children to exercise their creativity and control over their mathematics learning.

Feeling anxiety, exerting control, and exercising creativity are affective issues in their striving for progression and new ideas. For example, in solving discrete mathematics, five representational systems are involved: verbal-syntactic (natural language), imagistic, internalized formal notational, executive control, and affective representations (Goldin, 2004). In learning geometry, students' incomprehension mainly arises from multiple representations and representation transformations (Duval, 2006). The incomprehension raises affective responses and can be addressed by natural languages. Affective responses with natural languages, therefore, may assist learners in representation registrations and transformations, which appear to be missing in literature and would provide opportunities to fill the gap by research.

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Contribution to the literature

- Build a 4E Affective (Mathematics) Teaching (4EAT) Model, with a four-phase pedagogical structure: entry, entertainment, enlightenment, and enrichment (i.e., 4Es) based on the enactivist's perspective.
- Cross boundaries between affect and cognition by defining affective mathematics teaching as transforming natural languages to mathematical languages.
- Overcome educational systematic constraints or tensions by aligning teacher agenda with student agenda.

Recognizing the need to expand beyond the cognitive domain, the current national curriculum in Taiwan, the 12-year basic education (Ministry of Education in Taiwan, 2014), began incorporating affective aspects in learning and teaching to nurture lifelong learners capable of facing the challenges of a fast-changing lifestyle and information overload. In line with these guidelines, the mathematics curriculum envisions the ideal of mathematics as a language, practical pattern science, cultural literacy, sense-making learning opportunity, and avenue to use diverse tools (e.g., computer) (Ministry of Education in Taiwan, 2018).

In accordance with this vision, the current study aims to build an affect-centered theory on mathematics learning and teaching from both bottom-up (context) and top-down (content) methodologies. For the context aspect, the in-class mathematics grounding activity (MGA) developed by Shi-Da Institute for Mathematics Education (SDiME, 2022; Yang et al., 2021) in Taiwan can serve as a source for building an in-depth mechanism framework for mathematics learning and teaching. The rationales are that the MGA aims to enhance students' affective outcomes by building fundamental mathematical knowledge, manipulating concrete representations, and engaging in gamified activities (Wang et al., 2021). Challenging, engaging, and motivating mathematical tasks in the real classroom context will provide insights into high-quality mathematics teaching that balance or harmoniously integrate both affect and cognition from a bottom-up perspective.

For the content aspect, a top-down perspective utilizing past literature on affect in mathematics learning and teaching will add to the insights from the aforementioned empirical research. As a result, a qualitative methodology will be used for theory building, applying empirical cases, literature, and researchers of this paper as the participants to co-build the theories. Concretely speaking, this study aims to answer the following research questions.

1. What are features of instructional design that support affective mathematics teaching?
2. What is an abstract framework that can address the features identified?

Answers to RQ1 are presented in the results section. RQ2 is presented in the discussion section, which synthesizes the answers to RQ1 by drawing from the current literature and authors' insights. The following literature review first draws upon empirical and theoretical research on affect in mathematics learning and teaching. The next review goes to an enactivist perspective on mathematics education, which is the theoretical basis for the empirical data used in this study, the MGA.

The Interplay of Affect and Cognition in Mathematics Learning and Teaching

McLeod's (1992, p. 578) seminal work conceptualized affect into the dimensions of emotions (affective states such as joy or anxiety), attitudes (positive or negative predisposition toward an activity), and beliefs (learned perspective toward an object), each differing in intensity, stability, and cognitive involvement. Affect can include two kinds of systems: global (trait) vs local (state) affects (Hannula, 2012) and positive (e.g., confidence and interest) vs negative (e.g., anxiety and frustration) affect (Goldin, 2000) in mathematical learning. Subsequent scholars added other dimensions such as values (DeBellis & Goldin, 2006) and motivations (Hannula, 2012). All these constructs appear to place affect in all the processes of mathematics learning.

Early mathematical learning or problem-solving theories or frameworks, however, mostly use a cognitive perspective (Polya, 1945), including sociocultural concerns (Francisco, 2013). Later studies focusing on learners appear to trigger the addition of an affective perspective (Voskoglou, 2011). Despite the intention to distinguish between cognitive and affective issues in mathematics education in order to delve into them in depth, a line of research considers that affect and cognition are indispensable or interweaving in mathematics learning and teaching.

Mason et al.'s (1996) work appears to be the first to formally address the issue of the interweaving of affect and cognition in mathematics learning. The three phases of mathematical problem-solving (entry, attack, and review) formally incorporate both cognitive and affective concerns in mathematical thinking. For example, being stuck is viewed as an inevitable part of mathematical processes. The key to conquering being stuck is to reflect on prior experiences and emotional moments. This line of research continues. Gomez-Chacon's (2000) study highlights six emotional responses (calmness, confidence, cheerfulness, being great, being blocked, and frustration) in affective and cognitive contexts during mathematics learning.

Cognitively, students experience a flash of intuition, explore the correctness of understanding, and seek strategies.

Mathematical teaching also manifests the issue of the interplay of affect and cognition. At the beginning and midway of solving problems, teachers' affective support is needed for students' calmness and active participation as problem-solvers, claim-makers, and solution-reporters (Empson, 2003). A recent study by Marmur and Koichu (2021) uses students' key memorable events (KMEs) to identify essential discursive events in undergraduate lessons. Students' affect or emotions are highly related to key mathematical teaching events (e.g., lack of understanding), highlighting students' needs for heuristic-didactic discourse (meta-level learning), which requires instructors' investment, in order to increase the opportunity of student learning affordance.

In summary, although the cognitive context appears to directly fit most goals of mathematics curricula, the affective context appears to be associated with alternative, higher-order, and broader scopes of mathematical learning. This line of research provides evidence of the interplay of affect and cognition. Using affect in mathematics may also initiate a new avenue of mathematics research to advance mathematical learning and teaching.

The MGA's Theoretical Basis: An Enactivist Perspective to Mathematics Education

The MGAs, the empirical data used in this study, base their theoretical basis on the enactivist perspective of mathematics education and are designed through the process of metaphorizing, scaffolding, and gamification (Yang et al., 2021). Enactivist perspectives insist learners learn by situating and engaging themselves in the context. What and how learners learn are co-determined by themselves as human beings and their broader ecological systems or "from cells to culture" (Hannula, 2012, p. 146). Desirable learning occurs only when learners actively engage in and/or are triggered by suitable teaching activities. This dynamic exemplifies the enactivist perspectives to learning, as encapsulated by Hannula's (2012) three major propositions, as follows:

1. **Emergence and co-emergence in fuzzy boundaries**: Learners spontaneously learn cognitively or affectively; locally or globally; and individually or socially. Learning may occur within and beyond mathematics classrooms. This is especially true with the advance of information and communication technology (Chiu, 2020).
2. **Structural affordance constraints**: Learners' action is constrained by the system. Instructors' pedagogical designs offer affordances or opportunities to learn mathematics. A salient example is that young learners' approaches to mathematics are largely influenced by school mathematics, with their confidence and interest in mathematical problem-solving decreasing over time (Hannula, 2019).
3. **Embodiment**: Mathematical learning is activated by bodily experiences including gestures (body movements), thinking (computational thinking), and linguistic expressions (Kopcha et al., 2021). Affect is a natural function of everyday human activity, including mathematics learning. Affect is essential for human survival, innovation, and interaction, although affect's role is often recognized as weaker than that of cognition.

This enactivist perspective provides a fertile, flexible ground for this study to build knowledge from authentic mathematics learning experiences. In order to find the content of 'affect', it is especially important to analyze a context where affect is the aim of the mathematics teaching design, such as the MGAs.

METHOD

Data Source and Sample

The major data source comprised all 42 MGA class videos developed through the project "Just Do Math", implemented by SDiME (2022), starting in 2014. The project is a response to a special phenomenon: Taiwanese students have high achievement but low affect (e.g., interest) in mathematics, as indicated by the Program for International Student Assessment (PISA) (OECD, 2014). The aims of the project, therefore, are to develop mathematics activities to raise students' affect and ability to learn mathematics. At the time of writing this study, the "Just Do Math" project has successfully expanded through the professional development of local school mathematics teachers (Chang et al., 2021). The original activities have also been gradually adjusted to fit the context of certain mathematics teachers.

The 42 MGA videos analyzed in this study are available on YouTube (playlist on YouTube: https://www.youtube.com/channel/UCj--Hy76_ZKsyGw_cP5HgLw/playlists?view_as=subscriber). Some related sources included relevant open data shared online (e.g., teachers' Facebook posts). The datasets generated during this study are presented in the two supplementary materials (**Supplementary Material 1** & **Supplementary Material 2**). This study was part of a larger project, which obtained the approval of the institutional review board of National Chengchi University (NCCU-REC-202105-I030).

Measures

Qualitative data analysis methods are used to analyze the videos of the MGA in class (SDiME, 2022). The initial coding scheme comprised three parts: lesson structure, teacher contexts, and students' issues. The teacher contexts and student issues with both cognitive

Table 1. Analysis framework

Coding schemes				
Lesson structure	Affective (teaching) context	Affective (learning) issues	Cognitive (teaching) context	Cognitive (learning)

and affective aspects were a combination of coding schemes used by Mason et al. (1996) (including the three phases of mathematical thinking, key moments, and affective issues), Gomez-Chacon (2000) (including local affective responses, affective contexts, and cognitive contexts), and key moments (Marmur & Koichu, 2021). While the affective aspect focuses on psychosocial behaviors and teaching materials, the cognitive aspect emphasizes students' acquisition of the (declarative and procedural) knowledge of mathematics or activities. The analysis framework is presented in **Table 1** (**Supplementary Material 1** presents the detailed analysis results of four videos).

Lesson structure

A lesson encompasses multiple teaching events in a linear progression. The lesson structure identified key phases that an excellent MGA would follow. Mason et al.'s (1996) three phases of mathematical thinking (entry, attack, and review) served as a starting framework. During the process of coding, the coders aimed to answer the question: "What is the structure (phases developing over time) of the lesson?"

Teacher affective and cognitive contexts

Teacher contexts aimed to identify key moments along the development (phases) of the lesson. The coders asked in the coding process: What are the teacher's affective and cognitive contexts (e.g., instructional vocabularies, behaviors, and material uses) that may raise students' affective and cognitive issues?

Student affective and cognitive issues

The affective and cognitive issues were students' responses to the teaching context. During coding, the coders kept in mind: What are students' affective and cognitive issues in relation to the teacher's affective and cognitive contexts, respectively?

Data Analysis

The content of the videos were qualitatively analyzed using a combination of phenomenography (Marton, 1981), grounded theory (Charmaz, 2000; Strauss & Corbin, 1990, 1998), and general qualitative data analysis methods (Miles & Huberman, 1994). The qualitative data analysis procedure included an iterative process of open coding, theme finding, and theory building. The data analysis process also included techniques of constant comparison and dialogue with literature. The video-narrative methodologies were also applied, starting with transcribing verbal and non-verbal behaviors by clips, followed by identifying critical events, coding, constructing storyline, composing narrative, and presenting results in different grain sizes to support the discussion of issues (Derry et al., 2010; Powell et al., 2003; Wilkinson et al., 2018). Detailed data analysis steps for the RQs are presented, as follows.

RQ 1 data analysis steps

1. The first two authors discussed and identified the most desirable affect-focused MGA (MGA1) among all the 42 MGAs. The MGA1 had salient pedagogies linking mathematics learning content with student affects, using student affects as teaching materials, and transforming student affects into higher-order mathematical learning.
2. **Analyzed MGA1**: The key moments or critical events of teachers and students were transcribed, photocopied, and analyzed as narratives, a procedure similar to a multimodal interaction analysis (Wilmes & Siry, 2021) and video studies (Derry et al., 2010; Powell et al., 2003). An initial coding scheme was formed.
3. Used the initial coding scheme to compare the analysis results between MGA1 and the next MGAs until the features of MGA1 merged clearly and solidly, or reached saturation, using a term of qualitative methodology (Fusch & Ness, 2015). This initial analysis was conducted using the videos' original language (Chinese) and used four videos to reach saturation (**Supplementary Material 1**). For example, challenge, curiosity, and fantasy activities (Middleton, 1995) induced learners' situational interest (Hidi & Renninger, 2006; Rodríguez-Aflecht et al., 2018).
4. Narrated MGA1 to identify features, as presented in the results section (**Supplementary Material 2**).

RQ 2 data analysis steps

5. Went beyond the identified features and generated a framework (theory or model) for affective mathematics teaching for higher-order learning.
6. Made associations between the identified features and literature.

RESULTS (RQ1): THE STORY OF 'RECTANGULAR NUMBERS'

The four steps of data analysis for RQ1 successfully identified the most excellent affective teaching, 'Rectangular Numbers' (MGA1 in this study, video available on YouTube: https://www.youtube.com/watch?v=fImiotpvqBo&list=PLylUUAvq6ZNUSGm0I1

b5OfJ71Ys4GisCI&index=23) (SDiME, 2022). MGA1's key moments emerged vividly by further continual comparisons with the other MGAs. This constant comparison between MGA1 and the other MGAs further revealed distinct features of affective mathematics teaching (or reached theoretical saturation). This process engendered a picture of an affective (affect-focused) mathematics teaching: A clear lesson structure of four phases, and the distinct features in each phase.

Lesson Structure: Entry, Entertainment, Enlightenment, and Enrichment

The 'Rectangular Numbers' was a grade-5 mathematics class on point, prime, and composite numbers (Yang et al., 2021). The teacher guided students on the graphical meanings of prime numbers and composite numbers and completed an exhaustive list of factors for a given number through a game called 'Go Pieces'. The video lasted for 12.52 minutes.

The analysis started with Mason et al.'s (1996) three phases of mathematical thinking (entry, attack, and review). However, the teaching was found to follow four phases: entry, entertainment, enlightenment, and enrichment (including formative assessment after class) ("4Es"). The reason may be the added 'entertainment' phase, in which students are immersed in mathematical games.

The 4Es capture the major characteristics of constructivist approaches to teaching and learning for conceptual changes (Driver & Oldham, 1986). The teacher started his teaching by reducing students' barriers to entry into deep understanding of basic meanings of a "rectangle" and explaining the rules of the two-player game: One posed a number, and one formed a 'rectangle'. Students won if a rectangle was formed; otherwise, they lost (Phase A). Then, the students *entertained* themselves by participating in the game (Phase B). Next, the teacher *enlightened* students with correct answers, game results, and winning strategies (Phase C). Finally, the teacher *enriched* students' understanding by linking the previous hands-on, embodied experiences (including feelings toward the numbers in the game) to formal mathematics knowledge (Phase D).

Features: Fully addressing each phase of the 4Es in the right order

Excellent affective mathematics teaching has a simple, clear structure of 4Es.

1. The entire teaching followed the phases of 4Es once only.
2. Enough time was allocated to dig deep into each task in each phase of the 4Es.
3. Except for the entertainment phase, where mathematical games engaged students through peer interaction, the other three phases (entry, enlightenment, and enrichment) focused on student-teacher interaction.
4. The teacher's language, material, and activity use gradually changed from natural/physical to mathematical/symbolic languages. This fits Piaget's theory from concrete to abstract representations or multimodal uses and from natural (informal) to mathematical (formal) language uses (Nunes, 1997).

This feature echoes a coherent, deep pedagogical approach (Stigler & Perry, 1990). Teaching of each phase adequately prepares students for the teaching of the next phase based on cognitive development principles.

Phase A. Entry

(Video time from 0:24 to 4:10; three affective and three cognitive key moments)

The lesson starts with the teacher (Mr. Chu) reviewing prior knowledge using many scaffolding questions (teacher cognitive context). Students experienced a review scaffolding from easy to hard (student cognitive issues), bridging past and new learning (coded AC1, where A=Phase A, C=cognitive aspect, and 1=the first code of 'AC').

> Teacher (T): What did I draw on the blackboard?
>
> Students (Ss): One dot.
>
> T: What can two dots link together to form?
>
> Ss: A line.

Then Mr. Chu gradually added dots (until 4 dots).

> T: Aha! Can I use no matter how many dots to form a line?
>
> Ss: Yes.
>
> T: Now I move the places of the dots. (Move four dots to form a square.) What's this called?
>
> Ss: Square.
>
> T: Can eight dots form a square?
>
> Ss: Yes.

Mr. Chu invited Jeff to demonstrate how he formed his graph on the blackboard (1:05) (**Figure 1**).

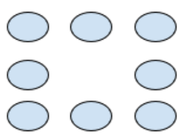

Figure 1. Jeff's method (Source: Authors' image based on the MGA1 video)

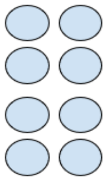

Figure 2. Forming a solid square using eight dots (Source: Authors' image based on the MGA1 video)

After viewing Jeff's method, Mr. Chu elicited other students' opinions rather than indicating Jeff's mistakes (teacher affective context). Jeff and/or all students would feel less egocentric and feel a sense of a learning community because they were not criticized directly but reached out to the community. Engaging in the community might reduce their negative emotions towards learning mathematics (student affective issue) (coded AA1, where first A=Phase A, second A=affective aspect, and 1=first code of 'AA').

> S: There is a space in the center.
>
> T: Today we must form a full spaced square … or a rectangle (1:24).

Mr. Chu invited another student to demonstrate the formula in forming a solid square using eight dots. Students underwent natural ('in a row' and 'dots' visual aids) to mathematical languages ('mathematical calculation expressions' (AC2) (**Figure 2**).

> T: How many dots in a row? … This can be recorded as 4×2.

The teacher asked a student (David) to play the game with him. Students felt involved and on equal footing with the teacher and each other (AA2).

> T: David gave me a number, "13", wanting me to make a square or rectangle. If I can form, I can obtain one point.

The teacher intentionally made mistakes, invited students to judge, and asked for reasons. Students clarified game rules by learning from mistakes of the teacher (AC3). The teacher formed the shape shown in **Figure 3**.

> T: Is it a square or a rectangle?
>
> Ss: Neither.
>
> T: Why not? (Mr. Chu invited Mary to reply)
> Mary: Because there is a hole.

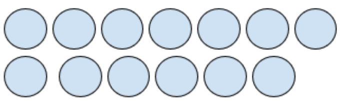

Figure 3. The teacher's formation (Source: Authors' image based on the MGA1 video)

> T: There is a hole, so it is not … (Mr. Chu waited for Mary to reply).
>
> Mary: A rectangle.
>
> T: Very good! Thank you! Please sit down (2:05).

The teacher gave compliments for correct responses and contributions to the learning community. Students would thereafter feel social recognition or confidence (AA3).

Feature: Invite to enter by reminding prior knowledge and introducing interesting games

Phase A introduced the game and reviewed prior knowledge through the teacher's interactions with students on equal footings. Students' situational interest in the game appeared to open the avenue for students' entry into learning mathematics.

The affective key moments in this phase were teachers inviting alternative opinions to supplement incomplete or incorrect student answers (rather than directly indicating students mistakes), playing with students to demonstrate how to play games, and providing positive feedback for correct responses. Through this context, students experienced the learning issues of feeling diversity, involvement with equal footing with the teacher, and social recognition in the learning community.

The three cognitive key moments include teachers' reminding the basics for the new learning content through Q & As (rather than direct teaching), connecting daily languages with mathematical languages that students had already learned, and making intentional mistakes to clarify game rules. It is inferred that students thereby could review prior knowledge, check their learning, and obtain further competency to play the games.

Phase B. Entertainment

(Video time from 4:11 to 5:57; 2 affective and 1 cognitive key moments)

Students began playing the game in pairs (4:11). Mr. Chu was not present in the video; it can be inferred that he enacted a competitive game through Phase A activities and students' behaviors. A scoring system was activated and might trigger students' desire to win (BA1). Given the playful essence of the activity, students looked happy and focused during the game (BA2).

Visual aids (times tables for 11 to 19×1-10) served as hints to support students in playing the game (5:36) (BC1).

Feature: Entertain by playing games

In Phase B, even without formal teaching (interventions), it can be inferred that students

Table 2. Correct answer table

	Numbers and records														
Number	1	2	3	4	5	6	7	8	9	10	…	21	22	23	24	25
Record	X	X	X	2×2	X	2×3	X	2×4	3×3	2×5	…	3×7	2×11	X	2×12 3×8 4×6	5×5

continuously practiced the key lesson content during the game and intuitively gained mathematical knowledge.

Phase B had two affective key moments (BA1-2). Teachers enacted competitive games, which used scoring systems to rank players as winners or losers. Teachers could also enact playful activities (games) without any scoring system. Students would experience focus and excitement, with strong, diverse emotions in playing competitive games and positive emotions in playing games.

Cognitive key moments were limited in Phase B. The reason may be that this phase was students' play time; the teachers limited their interventions to a minimum. This MGA1 teacher provided conventional mathematical calculation support. Answering students' questions was a variation of this kind of support often observed in other MGAs.

Phase C. Enlightenment

(Video time from 5:58 to 8:44; two affective and three cognitive key moments)

After the game (Phase B: Entertain), Phase C (Enlighten) began (5:58). Mr. Chu asked a series of positive scaffolding questions about the game results, repeating student responses and inviting them to say 'yes' before checking answer correctness. Students would feel positive acknowledgement before being assessed as winner or losers (CA1).

> T: Some numbers may allow more than one approach to form (a rectangle), right?
>
> Ss: Yes.
>
> T: Perhaps two approaches?
>
> Ss: And three approaches. And four approaches.
>
> T: Yes, there may be three approaches. Is there any number which allows three approaches, but you only wrote out two?
>
> Ss: Yes.

Mr. Chu presented the correct answer table on the blackboard (**Table 2**). Based on **Table 2**, Mr. Chu designed the key question, 'Is there any miss?' and enacted group discussion. Students learned collaboratively to answer the question (CC1).

> T: OK, shall we find it? Let's check your approaches. Two in a group collaborate to check whether there is any mistake (on your (guided inquiry) worksheet).

Then, Mr. Chu invited and interacted with students to demonstrate their results, rationales, and teaching. Students became teachers, active learners, and contributors in the learning community (CA2). Mr. Chu invited two girls working in a group (Alice and Betty) onto the podium with the teacher to share their answers, where the distance between the students and teacher was shortened (CA3).

> T: In which number did you omit an approach? (6:18).
>
> Alice: 24.
>
> T: 24 has a miss. Which did you overlook?
>
> Alice: 2×12.

In the process, Mr. Chu reconstructed inaccurate answers. Students gradually used more mathematical languages (CC2).

> T: 2×12 is missing. Why?
>
> Betty: Because it is not included in the multiplication table of nine …
>
> T: How did you find (it)?
>
> Alice: We should not use multiplication, but division.
>
> T: Oh! Later we find that in addition to using multiplication, division is another choice to find more answers … However, division may also miss some answers. How can we solve this issue?
>
> Betty: Divide one by one ... like 1, 2, 3, ..., 10 (7:00).
>
> T: When can 24 no longer be divided?
>
> John: Start from two.
>
> T: Oh, so 24 divided by two equals 12, and then three, four, …
>
> Adam: Divided by two is 12. Divided by three is eight. Divided by four is six. Divided by five is

none. Divided by six returns to four. [It] cannot repeat. So, divide until six.

Then, Mr. Chu asked students their winning strategies. Students described their intuitions about the game (CC3).

T: Raise hands, the winners … What're your secrets to winning? How did you win the game? (7:54).

S: Give others the number that cannot be divided …

A meta-knowledge about the mathematics of prime and composite numbers was gradually generated through students sharing winning strategies. In the process, self-monitoring arose.

Feature: Enlighten by inviting mathematical intuitions from playing the game, including answers, results, and strategies

Phase C appeared to be a stage bridging the game (hands-on experiences) in Phase B and the lesson objective of mathematics teaching in Phase D. Students' natural languages gradually transformed to mathematical languages by Mr. Chu's raising key questions (e.g., checking answer correctness and identifying winning strategies), rephrasing students' responses, and discreetly adding more formal, conventional mathematical terms and expressions.

There were two affective key moments. First, teachers repeated student utterances and generated 'yes' teacher-student dialogues before providing answers or game results that identified winners or losers. This 'yes' atmosphere would increase students' sense of acceptance before revealing the game results. Second, teachers invited and interacted with students to demonstrate their results, rationales, and findings, during which students became teachers, contributing to the learning community.

The cognitive aspect included three key moments. The teachers designed key questions for small group discussion, reconstructed students' inaccurate answers, and asked students their winning strategies during the game. By identifying key concepts, linking mathematical concepts, and employing higher-order thinking, it is inferred that students would move towards collaborative wisdom, initial mathematical patterns, and mathematical intuitions in relation to their experience of playing the games.

Phase D. Enrichment

(Video time from 8:45 to12:36; three affective and three cognitive key moments)

Based on the emergence of self-monitoring in Phase C, Phase D formally introduced conventional mathematics knowledge by using student emotions as teaching materials. Perhaps the most distinctive event of this teaching is that the teacher asked students' feelings about playing the game. Thus, students divulged their emotions during the competitive game in Phase B (DA1).

T: If I give you '23', would you try your best to form it?

Greg: Yes.

T: Did you finally form it?

Greg: No.

T: How did you feel then? (Use a dramatic emotional voice.)

Greg: It is a bit frustrating.

T: This number makes you frustrated because you cannot form it for many times. If s/he gave you a number other than 23, like 22.

Greg: Yay (a cheerful sound)! (use two hands to show two Vs--sign for victory, with laughter on his face.) (Other students also laugh.) ...

T: '13' (9:22).

John: Very worried and stressful. I want to beat "13" then (with a sad sigh and humorous voice, while gesturing as if beating something) ...

T: May I ask if s/he gives you '22'?

John: So happy (with a cheerful smile on his face, both hands forming V (victory), and dancing) …

Mr. Chu reflected the process like telling a story, from playing the game, expressing feelings, and associating their feelings with the number. Students revisited the whole process through personal stories and stayed focused (DA2).

T: So happy, so happy. Look at this (correct answer) table now. There are many numbers in this table. These numbers are just numbers, but because you have played the game, … you will have a feeling for these numbers. It may be joy, it may be anger, it may be sadness, and it may be despair. Those are your moods. Let's first divide it into two categories, according to your mood, or the pattern you see, and give it a name.

After that, Mr. Chu asked students to name the mathematical phenomenon they perceived. They wrote on their worksheets and created their own terms (DC1).

T: What do you call those numbers which cannot form (a rectangular or square)?

Table 3. Students' answers

Invalid numbers	Valid numbers
'Bad eggs'	'Good eggs'
Line and dot numbers (because the numbers can only form line or dot)	General numbers
Weak	Potent
Surprise numbers (because we both want to give it to each other in the game)	Square-rectangle number
Explosion numbers	Very good

S: Bad eggs.

T: Bad eggs. How about the numbers that can?

S: Good eggs.

Mr. Chu asked several more students and wrote their answers on the blackboard (**Table 3**).

Mr. Chu then summarized the discussion by associating students' creations and mathematics as an aid for pattern recognition, which was then used to introduce formal mathematical terms. Students were exposed to both their own natural languages and formal mathematical languages, thereby building mathematical knowledge in cognitive aspects (DC2).

T: Mathematicians divide these numbers into two categories. One is that they can be arranged in rectangles or squares. Mathematicians think that this category, no matter which number, can be decomposed into two other numbers and multiplied together. In other words, it is synthesized from two numbers ... Therefore, mathematicians call these numbers of composite numbers ... Line point numbers only have lines and points.

T: Which number is special? (Mr. Chu ask questions for deeper understanding, which strengthen student learning [DC3]).

S1: '1'.

T: Why?

S2: Because it cannot form a line.

T: Although you feel the invalid numbers are surprising or weak, mathematicians do not. They think they are numbers that cannot be decomposed any further. What are they called? (Mr. Chu showed "prime number" on the .ppt).

Ss: Prime numbers (12:06) (Only until now was conventional mathematics introduced, gradually from students' natural languages to mathematical languages).

To deepen student understanding, Mr. Chu pretended to be oblivious and asked them to teach him. Students rephrased what they had learned gradually from unclear to clearer mathematical concepts by becoming a tutor (DA3).

T: I'm now a student who does not understand mathematics. Please tell me what is a prime number? How will you explain it?

S1: A number with only one and itself.

S2: No multiplication (on the correct answer table) (12:20).

More students stated their experienced mathematical phenomenon. Interaction between Mr. Chu and students continued until the meaning of prime numbers could be clearly addressed by the students using their own words. Here, students experienced a sense of presence in a mathematics learning community.

Feature: Enrich by linking natural (emotional) terms to abstract mathematical terms

Phase D addresses formal, conventional knowledge in mathematicians' world. A unique pedagogy is the process of arousing students' emotions by revisiting the game, reporting, and classifying their emotions into two categories based on experiences of playing the games in Phase B, and connecting students' and mathematicians' lexicons with mathematical rationales, which were also experienced by the students during the game.

The final phase features presence, where students feel gifted through higher-order mathematical knowledge and skills. The affective key moments manifested by directly asking students' feelings about the numbers used in the game, using story-telling throughout the whole process to link students' emotions to the numbers, and students' teaching. Thus, students divulged their strong emotions about the game (results or objects [numbers]), riveted in personal stories, and became helpers (teachers) in the learning community.

The three cognitive key moments were the teacher's initiating activities for students to create terms for a mathematical phenomenon they experienced while playing the game, linking student creations to conventional mathematical knowledge, and asking questions (e.g., "why"). It is inferred that these activities would deepen the understanding of the newly acquired mathematical knowledge.

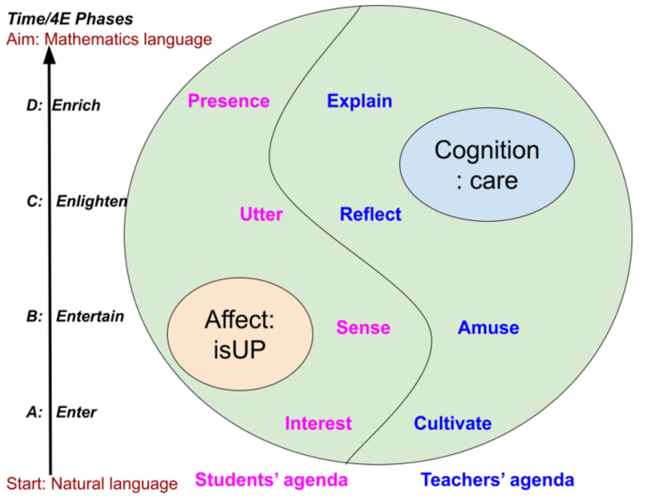

Figure 4. 4E affective (mathematics) teaching (4EAT) model (Source: Authors' own elaboration)

Student Outcomes and Teacher Reflections

Mr. Chu shared a student's work after the lesson on an open Facebook post (Chu, 2020):

> I named the numbers that could not form a rectangular (or square) as 'dedication numbers'. Reason: They must be 'prime numbers' because only 1 and itself can become them (e.g., 1×19=19), not others. It's like our fathers love our mothers … exclusively forever …
>
> (On the other hand,) the numbers that can form a rectangle (or square) are the 'companion numbers'. Reason: Even numbers must be (the 'companion number'). There are many partners multiplied together, and they will not feel lonely (e.g., 8=1×8, or 2×4) ...

Mr. Chu also shared his own reflection.

> (Students) learned more than just mathematics; more importantly, thinking like a mathematician … The student said, 'prime number' and 'composite number' but couldn't explain the reasons why mathematicians used these names. 'Names given and explained by yourself' gives name essence of the humanities of mathematics.
>
> (Students provided the following names for prime numbers and composite numbers)
>
> 'Hate number' and 'like numbers'.
>
> 'Oh my God number' and 'Oh yes number'.
>
> 'Dedication number' and 'companion number'.
>
> 'Introvert number' and 'extrovert number' …
>
> (This is a) real experience, like a mathematician forming mathematical concepts.

In summary, Mr. Chu's teaching of 'rectangular numbers' was selected by the research team as excellent affective mathematics teaching. The major reasons are that students' affective or emotional states are strongly stimulated through playing games and directly transformed to connect with the targeted mathematical topics. The teacher's interactions with students created a desirable atmosphere of mathematical learning in both affective and cognitive aspects.

DISCUSSION (RQ2)

MGA1 appears to feature the merits of the best mathematics teaching practices such as engaging, listening, using questions, preparing, assimilating with rich representations, learning community, discovering, and uplifting students' roles (Maher et al., 2014; Schulman, 2013). A 4E affective (mathematics) teaching (4EAT) model (**Figure 4**) is posited on the basis of the dialogue between the answers to RQ 1 and the literature.

Beyond Boundaries (Enactivist Proposition 1) by Defining Affect as Natural Languages to Approach Mathematical Languages

This addresses the proposition of the enactivist perspective in going beyond fuzzy boundaries. Given the focus of this study, the following definitions aim to define affective and cognitive mathematics separately and transcend their boundaries.

Affective mathematics=Natural languages for (learning and teaching) mathematics

Affect as natural languages. This pedagogical design is consistent with the notion that affect represented by natural languages are more automatic and precedent, which should be resolved before cognitive, intentional issues (Duval, 2000). One salient example is that an affective mathematics teacher would invite students' features of the enactivist MGA and contribution in the learning community. This includes unlocking students' hidden emotions from entertaining games. The teacher accepted students' languages by repeating them.

Affective mathematics emerges here and now, co-emerging in fuzzy boundaries with cognitive mathematics and all entities in and beyond mathematics classrooms (Chiu, 2020; Hannula, 2019). Affective mathematics materials are stories, daily languages (e.g., emotional responses of frustration and joy), and daily activities (e.g., games, home design, and financial simulations). Affective mathematics events are psychosocially-related activities, involving psycho-somatic behaviors, interaction, and perception, which prompts a learning community.

Cognitive mathematics=Professional languages in mathematics

Mathematics is a study of abstract patterns and relationships (Burton, 1994), which moves mathematics toward a pure, context-free, and cognitive domain of knowledge. The national mathematics curriculum was designed by mathematics experts and educators who have been learners in the system and endorses the mission to transmit the cultural heritage to the next generations. This forms instructional or structural affordance constraints.

Affective mathematics teaching=Transforming natural languages to mathematical languages

Affect can serve the cognition essence of mathematics. Affect or emotions naturally emerge from students' daily lives and playing the games (in Phase B), while cognition is the main goal of mathematics with abstractive, conventional knowledge rooted in teachers' minds and the mathematics curriculum. Perhaps one naturally generated affective response is the anxiety of self-identification and social recognition as being winners or losers if a competitive game is activated. Using affect naturally emerging from playing games as teaching materials is to use emotional arousal (e.g., anxiety and pride) and reflections to engender a deep, experiential understanding of mathematics.

Affective mathematics teaching will fully transform students' informal daily activities, events, or languages to formal mathematical activities or languages as practiced in the natural curriculum and the professional world (Nunes, 1997; Stipek et al., 1998). This will reduce the difficulty of learning mathematics with affective responses represented by natural languages as automatic, precedent issues coming before mathematical, cognitive, and intentional issues (Duval, 2000). Further, these key moments (features) delves into the essence of mathematics-thinking like a mathematician and being consistent with a discovery-oriented teaching (Askew et al., 1997).

A salient practice of affective mathematics teaching is to use students' affective responses to a mathematical game as teaching material, as in the enrichment phase of MGA1, where the teacher invited students to map the names of negative and positive emotions (mainly toward the numbers and the results in the game) to prime and composite numbers. Subsequently, students' affective responses (personal, natural languages) from playing the game (natural activities) are used as teaching materials and fully transformed to cognitive mathematician-like thinking (mathematical languages and activities).

Beyond Constraints (Enactivist Proposition 2) by Fitting Teacher Agenda to Student Agenda

According to the enactivist perspective to mathematical learning, learning opportunities are entrenched in the teaching context or system. Learning opportunities can be manifested in students' agenda and teaching context in teachers' agenda.

Student agenda: Upward learning=isUP (interest, sense, utter, and present)

The theme of students' learning agenda is the issue of upward learning or growth mindset (Yeager et al., 2019). Students are born potentially curious about the world and mathematics. However, despite striving to learn mathematics (Burton, 1994), most students gradually lose their confidence and interest throughout educational stages (Hannula, 2019).

The results of RQs 1 and 2 highlight four phases of student learning issues in experiencing the MGAs: interest, sense, utterance, and presence. Students should be motivated by situational interest and reminded of prior learning, experience hands-on playful activities, utter the facts of the playful experiences, and celebrate the rewards of obtaining abstract mathematical knowledge and skills.

Teacher agenda: Cultural teaching=CARE (cultivate, amuse, reflect, and explain)

The theme of teachers' agenda is acknowledging mathematics as a cultural product and implementing the mathematics curriculum by the cultural educational system. This theme, however, forms both opportunities and constraints.

To avoid the structural affordance constraints, teachers need to care about students' agenda. Teachers' agenda needs to begin by cultivating student interest and foundations, followed by enacting playful activities to enrich sense-making, inviting students to reflect on the activities, and finally explaining the newly learned conventional mathematics knowledge and skills as addressed in the curriculum by eliciting students' previous experiences in the lesson.

Tension or harmony?

Both teachers and students may be constrained by their themes and agendas. With teachers' superior status, they assume the role of building a mathematics classroom with tension or harmony. As a saying by 'Zhuangzi (莊子)', 'It's hard to tell a worm that lives only until summer about ice (夏蟲不可語冰)'. Teachers or mathematicians are the survivors in learning mathematics; they are capable enough to live until the winter and know what ice is (the cold, abstract, and cognitive knowledge of mathematics).

Tensions occur if teachers fail to care about students, who strive to learn professional (cognitive) mathematics. Students will potentially lose interest and reject learning mathematics, like a summer worm with warm, affective mathematics competencies (e.g., natural languages with playful tendencies) dying before winter's (cold, cognitive mathematics) arrival. Harmony occurs if the proposed teachers' agenda fits students' agenda.

Beyond Embodiment (Enactivist Proposition 3) by Enacting Affective Mathematics Teaching Through 4Es Phases

Affective experiences naturally arise from embodiment or daily activities, including mathematics learning, though it is typically perceived as cognitive experiences. Affective mathematics teaching can successfully link natural embodiment activities with mathematical learning through four phases: entry, entertainment, enlightenment, and enrichment ('4Es '). The 4Es capture the major characteristics of enactivist approaches (Hannula, 2012; Yang et al., 2021) and constructivist approaches to teaching and learning for conceptual changes (Driver & Oldham, 1986).

The reasons for the appropriateness of 4Es may be that they align with the basic structure of the traditional three-phase lesson design (motivating, main, and synthesis activities), but adds an 'entertaining' element which specifically tackles the issue of affect-focused and enactivist design of the MGA (SDiME, 2022). Further, the 4Es fit the four steps of typical Chinese writing: introduction, elucidation, transition, and conclusion. It also matches our natural, physical experiences of the four seasons, making the lesson structure easily acceptable and possibly automatically adopted, though with some adjustments by teachers.

4Es' affective and cognitive mathematics

While enactivist 4Es develop (or move) along the four teaching phases, affective and cognitive mathematics are interwoven along the 4Es phases. However, relative or sequential roles of affective and cognitive mathematics teaching in each phase can be derived from the answers to RQs 1 and 2. The interplay between cognitive and affective mathematics would build a positive atmosphere for learning mathematics.

Phase A: Cognitive to affective mathematics: Entry starts with reminding prior mathematical knowledge and ends in game preparation. Situational interest is the key through inviting students to play the games.

Phase B: Affective with cognitive mathematics: Entertainment involves students actively experiencing playful activities and intuitively sensing mathematics. Students' natural language use dominates this phase of game playing, while the cognitive mathematical learning is implicit or embedded.

Phase C: Cognitive with affective mathematics: Enlightenment of mathematical minds confronts students with facts (game results), mistakes, and patterns, while building a safe affective atmosphere. When performance is the concern, potential affective issues (e.g., frustration, pride, and confidence) arising from social comparison and recognition may deserve notice by educators.

Phase D: Affective to cognitive mathematics: Enrichment builds upon students' creation of terms (starting with affective/emotional languages/representations) for the mathematical phenomenon, experiencing the lesson like a story, and linking formal mathematics knowledge, skills, and terms. The underlying mechanism may be that emotional languages may not completely satisfy the experienced (mathematical) phenomenon. Pattern recognition gradually emerges and leads to the creation of concise, abstract mathematical languages.

CONCLUSION

Contribution

This study used a qualitative methodology to analyze mathematics teaching lessons focusing on promoting student positive affective responses to learning mathematics using an enactivist perspective. Dialogues between the lesson analysis results and literature identify features of affective mathematics teaching. A framework for affective mathematics teaching (4EAT model) is further built to add theoretical interests and pedagogical insights to enactivist's perspectives in mathematics education.

1. Cross boundaries between affect and cognition by defining affective mathematics teaching as transforming natural languages to mathematical languages.
2. Overcome educational systematic constraints or tensions by aligning teacher agenda (with care through the pedagogical phases of cultivating, amusing, reflecting, and explaining) with student agenda (with upward learning tendency through the phases of interest, sense, utterance, and presence).
3. Extend embodiment activities to a four-phase pedagogical structure: entry, entertainment, enlightenment, and enrichment with relative emphasis and sequence between affective and cognitive mathematics teaching in each phase.

Limitations of This Study and Suggestions for Future Research

This study conducted in-depth case studies, which are widely used by studies on student local affect during mathematical teaching (e.g., Marmur & Koichu, 2021). While qualitative methodologies rely on contextual cases to infer theories, this inductive nature suggests using quantitative methodologies to validate the findings. Experimental studies can also be conducted to validate the theory and models. Large-scale surveys can validate the model further.

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