MODESTUM

Review Article

The State of South African Mathematics Education: Situating the Hidden Promise of Multiple-solution Tasks

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Abstract
We live in the challenging times of the 21st-century with the increased need for humans to possess specific skills that will help them to be successful in this era. This means that education should in learners, develop these skills effectively. Different global countries have begun to recognize the significance of multiple solutions tasks in the teaching and learning of mathematics in the 21st-century. However, this practice is not visible in South Africa. Hence, the current study explores and synthetize the sparsely available literature on MSTs to answer the question: What is the significance of multiple-solution tasks (MSTs) in mathematics education and why is it relevant for South African mathematics education to make the exercise of producing multiple solutions accessible to learners? The literature that is being synthetized here is viewed through the optic lens of the social constructivism theory as proposed by Vygotsky and explicated in Jean Lave and Etienne Wenger`s *Situated learning: Legitimate peripheral participation*. In the conclusion I engage in an argumentation that illuminates the significance of MSTs in mathematics education and provide reasons why it would be beneficial for the South African mathematics curriculum to incorporate MSTs.

Keywords: multiple solution tasks, problem solving, creativity, geometry

INTRODUCTION AND PROBLEM STATEMENT

"Mathematics teachers should strive to provide multiple and varied opportunities for learners to develop their mathematical reasoning skills – the capacity for logical thought, reflection, explanation, and justification" (Department of Basic Education, 2019, p. 8).

This theoretical article explores the power of mathematical problem-solving in the development of mathematical knowledge, thinking and reasoning. It puts the development of advanced mathematical thinking and mathematics-specific 21st-century skills, like creativity, at the centre of mathematical problem-solving, from routine to non-routine problem-solving. A good problem should always allow problem-solvers to compute multiple solutions. Such problems are referred to as multiple-solution tasks (MSTs). MSTs allow problem-solvers to present multiple solutions or

multiple proofs of a problem (Levav-Waynberg & Leikin, 2012b). These multiple solutions differ in their "representations of a mathematical concept, … properties (definitions or theorems) of mathematical concepts from a particular mathematical topic, … mathematical tools and theorems from different branches of mathematics, or … tools and theorems from different subjects (not necessarily mathematics)" (Leikin, 2014, pp. 68-69). Furthermore, MSTs have different solution spaces: an *individual solution space* representing all sets of solutions generated by a single person without the assistance of another person; a *collective solution space* representing solutions generated by a group of people working together; and an *expert solution space* produced by expert mathematicians (Leikin, 2014; Levav-Waynberg & Leikin, 2012b). The requirement of multiple solutions to a problem transforms it from being a routine problem to a challenging task which requires more insight and multiple-solution strategies (Leikin, 2014; Silver, 1997) because computing multiple solutions to a problem requires reflective (Dewey, 1933) and higher

> **Contribution to the literature**
> - This study dissects the literature done on multiple solutions tasks globally and positions multiple solutions as an option to teach mathematical problem solving in South Africa.
> - The study identifies a gap that South African mathematics education and mathematics education research has not considered the use of multiple solutions tasks to teach mathematics.

order[1] (Miri et al., 2007; Resnick, 1987) thinking. Encouraging learners to construct multiple solutions for a problem is considered an attribute of high-quality teaching (National Council of Teachers of Mathematics, 2000). Theoretically, the use of MSTs is well-founded in mathematics and mathematics education. Polya (1973) acknowledges that problems are better understood by using two or more proofs which allow further understanding of the problem and viewing it from different perspectives. Furthermore, mathematical thinking and reasoning can be fostered by showing different paths to solving mathematical problems (Schoenfeld, 1985). Dhombres (1993) argued that one proof is not always enough because of the universality of mathematics as a domain – various approaches are required. Recent educational transformations in different countries have recommended that mathematics learners be exposed to multiple methods of solving mathematical problems (Department of Basic Education, 2019; Große, 2014; National Council of Teachers of Mathematics, 2000; Rittle-Johnson, Star, & Durkin, 2012). However, South African teachers still do not value learners' multiple solutions, have difficulty in grading them (Bingolnali, 2011; Mhlolo, 2017) and might not be aware of the benefits of nurturing multiple-solution strategies in learners.

Active learning, critical and creative thinking, high knowledge and skills are some of the basic principles of the Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education, 2011). Furthermore, the Department of Basic Education (DoBE) sees mathematics as a human activity in which both teachers and learners actively engage in solving societal problems (Department of Basic Education, 2011, 2019). A recent document from the DoBE has highlighted the significance of multiple solutions in mathematics through the framework for teaching mathematics for understanding (Department of Basic Education, 2019). However, South Africa has a longstanding trend of poor performance in both national and international tests. The country has been ranked second last in the recent Trends in Mathematics and Science Study (TIMMS) (Reddy et al., 2016). While results from the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) show improvement in Rasch scores of South African learners' performance in mathematics

(Department of Basic Education, 2017), the World Economic Forum (2015) has ranked South Africa the lowest in higher education mathematics and science education. Different factors influencing poor mathematics performance have been investigated in local studies, from language and learners' background information (Howie, 2003; Phakeng & Moschkovich, 2013), home and school resources (Visser, Juan, & Feza, 2015) to teacher classroom practice (Arends, Winnaar, & Mosimege, 2017) and other perspectives. As useful as they are, the suggestions from these studies are limited as regards their classroom applications. In this article, I argue for an unchartered territory in South African mathematics education as a possible approach to enhance learners' problem-solving skills, critical thinking and higher-order thinking – that is, nurturing the production of multiple solutions to mathematical problems. While the emphasis of this article is on problem-solving as a 21st-century skill that has been on the mathematical agenda for several years, my approach is new and has not been empirically tested in both teachers and learners within the context of South Africa. In this article, I address the following question: What is the significance of multiple-solution tasks (MSTs) in mathematics education and why is it relevant for South African mathematics education to make the exercise of producing multiple solutions accessible to learners?

MULTIPLE SOLUTION TASKS (MSTs)

In George Polya's formulation of mathematical problem-solving heuristic, he mentions the *looking back* stage where learners are encouraged to reflect on their solutions and search for alternative solutions (Polya, 1973). This highlights the process of reflecting as critical during mathematical problem-solving. Reflection kindles metacognition – another integral part of problem-solving (Flavell, 1976, 1979; Schoenfeld, 1985, 1992) – that is promising in nurturing multiple solutions. Engaging in the activity of producing multiple solutions through MSTs allows learners to practice nonroutine problem solving, showcase their creative and critical thinking and most importantly construct their own knowledge, all of which are activities learners can be engaged in to develop higher order mathematics thinking skills (Apino & Retnawati, 2017). The use of MSTs in mathematics education has been shown to have

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[1] "Higher order thinking can be conceptualised as a non-algorithmic, complex mode of thinking that often generates multiple solutions" (Miri, David, & Uri, 2007, p. 355). In his characterization of higher order thinking skills, Resnick (1987) mentions multiple criteria and solutions, self-regulation and effortful, all of which are characteristics that are closely linked to self-directed learning, a global imperative for the 21st-century.

a positive influence on learners' development of different mathematical competencies and may be used to assess some mathematical abilities. Hence, in the following sections, I discuss the benefits of incorporating MSTs in mathematics education as they appear from the sparse existing literature on MSTs and mathematics education. I focus mainly on the benefits of MSTs on mathematical creativity and problem solving with a particular example of a multiple solution task in geometry. Thereafter, I change my focus to the South African context, indicating the state of research on MSTs and how can incorporating MSTs in mathematical activity benefits South African learners in particular.

Mathematical Giftedness and Creativity

There is a vast amount of literature on creativity; amongst others, is Hadamard (1945), who attempted to explain his conceptualisation of his mathematical theories. Despite numerous studies investigating creativity and mathematical creativity, a precise scientific definition of the construct seems to be lacking (Anderson, 1960; Mann, 2006). To describe creativity, Poincare (1948) used the metaphor of *choice*, while Anderson (1960) perceives creativity as reflecting original unique thought. For some, creativity is related to divergent thinking (Cropley, 1997; Guilford, 1967), while for others, it is the complex interrelationship between divergent and convergent thinking (Runco, 1993). Divergent thinking has been investigated in a plethora of studies concerned with mathematical originality. Testing divergent thinking through psychometric tests gives an idea of one's creative potential, but performing well in such tests does not guarantee creativity (Runco & Acar, 2012; Sriraman, 2005). Most divergent thinking tests focus on fluency – one's ability to provide numerous ideas on a specific subject – as the main measure of creativity (Runco & Acar, 2012). What seems to link to this conceptualisation of fluency is Polya's (1954) emphasis on utilising various heuristics in solving mathematical problems. Currently, the literature on mathematics education works with functional definitions of creativity (Ervynck, 1991; Sriraman, 2009; Torrance, 1966). The problem with obtaining a precise definition of creativity is explained clearly in the work of Sriraman (2009) and other related studies. Hence, in this article, that avenue is not taken. In all these working definitions of creativity, "originality" is central in showing creativity (Runco & Acar, 2012). Here, I conceptualise mathematical creativity as the exercise of producing multiple varied, novel solutions to mathematical problems.

Creativity has long been characterised as a trait of being gifted (Renzulli, 2011). Hence, teachers need to be aware of traits resembling giftedness in learners (Mhlolo, 2017). According to Renzulli (2011), giftedness occurs at the intersection of three traits: creativity, task management, and above-average ability. Giftedness is a multifaceted concept in which creativity is but one part of assessing giftedness in mathematics education. A clear explanation of the latter claim is articulated by Winner (2000), Sriraman (2005), Leikin (2010). Sriraman, Haavold, and Lee (2013), Leikin and Pitta-Pantazi (2013), and Leikin and Lev (2013), amongst others. Again, these studies are not deeply analysed because one can access any of these articles and acquaint oneself with the explanations; hence my choice of just highlighting these studies. While most studies conceptualise giftedness in terms of above-average ability and IQ tests (Leikin, 2007; Leikin & Lev, 2013; Pfeiffer & Blei, 2008), I adopt the view that creativity is an important trait of giftedness, one that can be natured through incorporating MSTs in mathematics education.

Early studies on creativity associated it with ingenuity possessed by rare individuals with exceptional thought processes (Weisberg, 1988) – a "quality of protoplasm" (Anderson, 1960, p. 10). However, studies indicated that creativity can be natured in all human beings with careful planning and implementation of material designed to facilitate the process (Silver, 1997; Sriraman, 2009). In particular, mathematical problem-solving is the main tool that can nurture mathematical creativity (Pehkonen, 1997; Silver, 1997). Hashimoto (1997) discusses methods of fostering mathematical creativity that are all based on problem-solving. However, one ought to recognise that problem-solving has been conceptualised differently in mathematics education research. Therefore, in this case, I focus on problem-solving of both clearly defined, routine, ill-posed and non-routine mathematical problems because I only aim to position the production of multiple solutions as important in the development of mathematical thinking. Furthermore, the requirement of multiple solutions positions an algorithmic problem as a non-routine problem that requires more insight into and understanding of the constituents of the problem (Leikin, 2010). Furthermore, Silver (1997) and Sriraman (2005) suggest that the incorporation of ill-structured mathematical problems can promote learners' production of multiple solutions, in turn, developing their mathematical creativity.

The utilisation of MSTs has been studied from different perspectives, and here I give a glance at the outcomes of such studies on mathematical creativity. Levav-Waynberg and Leikin (2012a) incorporated MSTs in 11 classes comprising 229 students so as to measure development in their geometrical knowledge and creativity and compared the results with a control group. They used connectedness and correctness to measure learners' geometrical knowledge and fluency, flexibility and originality to measure learners' creativity. They found that the incorporation of MSTs improved learners' geometrical knowledge and creativity, influencing connectedness, fluency and flexibility without an observable change in learners' originality (Levav-

Waynberg & Leikin, 2012a). In the same project, Levav-Waynberg and Leikin (2012b) concluded that the MST geometry tool used in their study could be used to stratify learners according to their required level of instruction. A study conducted by Guberman and Leikin (2013) indicated that the incorporation of MSTs is effective in the development of pre-service mathematics teachers' systematic and craft modes. Their study presents evidence that both low-performing and high-performing pre-service mathematics teachers can benefit greatly from the incorporation of MSTs (Guberman & Leikin, 2013). MSTs present teachers with a window they can use to investigate and realise learners' mathematical potential at different complexity levels of mathematical problem-solving (Leikin, 2014). After a careful analysis of students working on MSTs, Schindler, Joklitschke, and Rott (2018) proposed that, in mathematics, creativity should be viewed as a sub-domain, hinting at the existence of geometry creativity or algebraic creativity. Hence, MSTs can be used not only to nuture learners' mathematical creativity and problem-solving (De Villiers, 2016, 2017; Pillay, 2017; Samson, 2015, 2017) but also to evaluate their creative ability and geometry knowledge (Leikin & Lev, 2013; Levav-Waynberg & Leikin, 2012b). Creativity has been recognised as one of the important skills to possess in the 21st century together with problem-solving (Mhlolo, 2017; Organization for Economic Co-operation and Development, 2018). Nurturing creativity as a source of innovation (Mhlolo, 2017) in learners should therefore be the goal of mathematics education in the 21st century. As a critical component of advanced mathematical thinking and mathematical discovery (Ervynck, 1991), nurturing mathematical creativity in learners through MSTs can enhance their mathematical thinking.

Developing Mathematical Thinking and Problem-solving

De Villiers and Heideman (2014) expand on Lockhart's (2009) lament that school mathematics is usually far from the activity of mathematicians. Their concluding remarks about Albert Einstein's quotes, though humorous, are indeed a reflection of the activity of mathematicians, and they directly relate to the activity of seeking multiple solutions to mathematical problems. De Villiers and Heideman stress the significance of exploration in mathematical activity (De Villiers & Heideman, 2014), an activity that is usually employed in seeking multiple solutions to mathematical problems (Santos-Trigo & Reyes-Rodriguez, 2016). They place explorations as important in the growth of mathematical thinking during problem-solving (Samson, 2015, 2017; Samson & Kroon, 2019). Problem-solving is further important not only in developing creative thinking in mathematics but also in mathematical thinking as a domain-specific type of thinking (Schoenfeld, 1985, 1992). The development of mathematical thinking in

schools is facilitated by the curriculum, but mainly by teachers in their respective classrooms in different ways. One approach involves the utilisation of MSTs in mathematical instruction. De Villiers (2016, 2017) asserts that the processes of reflecting during problem-solving can enhance one's mathematical thought and problem-solving ability. Problem-solvers' prior knowledge as a resource brought into the problem-solving scenario is useful because the production of multiple solutions in mathematics requires some degree of mathematical knowledge acquisition (Schindler et al., 2018; Schoenfeld, 1985). MSTs encourage learners to use different strategies to solve mathematical problems and prove to improve their success in future mathematical problem-solving (Erbas & Okur, 2012). Encouraging learners to pursue multiple solutions to mathematical problems can improve their effectiveness and flexibility in solving such problems (Daher, Tabaja-Kidan, & Gierdien, 2017; Levav-Waynberg & Leikin, 2012a, 2012b; Stanislaw & Krug, 2014) even though it does not always affect their performance in mathematics (Schindler et al., 2018; Schukajlow, Krug, & Rakoczy, 2015). Furthermore, utilising MSTs allows more potentially creative learners to present their creative solutions to problems (Levav-Waynberg & Leikin, 2012a; Sriraman & Dickman, 2017). Deductive reasoning and critical thinking skills are important in geometry problem-solving and in the 21st century, and the use of MSTs can improve these skills (Segal, Stupel, & Flores, 2017; Sriraman & Dickman, 2017). MSTs can also be used as a strategy to develop young mathematicians who can thrive in the fourth industrial revolution (4IR), relying on their ability to use different solution strategies to solve problems.

The Significance of MSTs in Geometry Education

Challenging learners mathematically is at the core of developing their mathematical thinking and equipping them with various 21st-century mathematical skills. Euclidean geometry is one area in which most South African learners and teachers struggle the most (Alex, 2019; Alex & Mammen, 2014; Naidoo & Kapofu, 2020; Stols, Van Putten, & Howie, 2010; Ubah & Bansilal, 2019). However, it is an area in which MSTs and mathematical challenge are always present (Leikin, 2014; Stupel & Ben-Chaim, 2017). Due to the dire situation of poor performance in geometry-related tasks and teachers struggling with teaching geometry successfully, different measures have been taken to remedy the situation in South Africa, for example, the use of a geometry software (Bayaga, Mthethwa, Bosse, & Williams, 2019; De Villiers, 2004b; Stols et al., 2015; Umugiraneza, Bansilal, & North, 2018), but in most cases, schools are under resourced and teachers are not proficient in using geometry software for teaching. Other attempts include providing teachers with professional development opportunities – like the Wits Maths Connect Secondary (WMCS) project – with aims

to improve the quality of mathematics teachers (Pournara, Hodgen, Adler, & Pillay, 2015). However, the situation has not improved significantly. Recent studies and analyses of national marking guidelines of Grade 12 Paper 2 in South Africa have shown that multiple solutions exist in Euclidean geometry problems, some of which are not initially evident to others. As such, geometry allow for the production of different proofs and solutions – standard, elegant and proofs without words (Sigler, Segal, & Stupel, 2016). Another potent approach to geometry teaching is the incorporation of dynamic geometry software (DGS) (such as geometers` Sketchpad, Cabrii or GeoGebra) which allows for explorations, conjecturing and proofs (De Villiers, 2004a, 2004b). Exploring geometry conjectures in a DGS may lead to different conclusion, some correct, some not, and some a new discovery, and proving these conjectures may be the central activity of producing multiple solutions. Producing multiple solutions to geometry problem in the classroom may invoke excitement and wonder in learners instead of producing just one which can leave them in awe (Stupel & Ben-Chaim, 2017). In this section I want to present different solutions to a particular geometry problem to support that Euclidean geometry is a goldmine of multiple solutions. I do not want to make universal conclusions about this phenomenon but to awaken the need to nurture mathematical creativity in our learners through geometry MSTs. In doing so, I will use a problem (*Problem 1.7*) I came across in a book by Chen (2016, p. 5) titled *Euclidean geometry in mathematical Olympiads*.

THE PROBLEM

Let O and H denote the circumcentre and orthocentre of an acute triangle ABC respectively. Show that $\angle BAH = \angle CAO$.
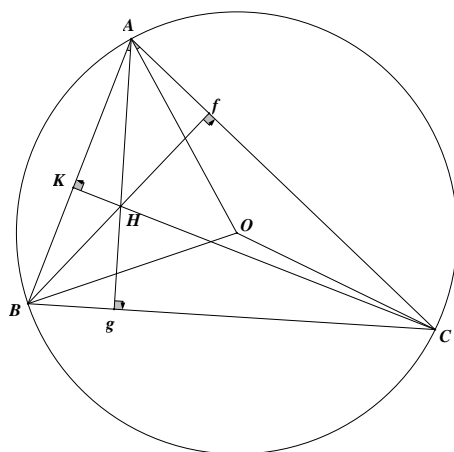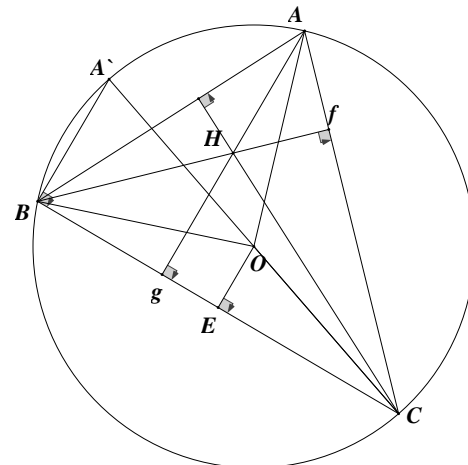
The Problem

Learners who have acquired the definition of the circumcentre can use isosceles triangles and the angle at the centre theorem to prove the result. This problem is equivalent to showing that $\angle ABH = \angle CBO$.

Multiple Solutions/Proofs for this Problem

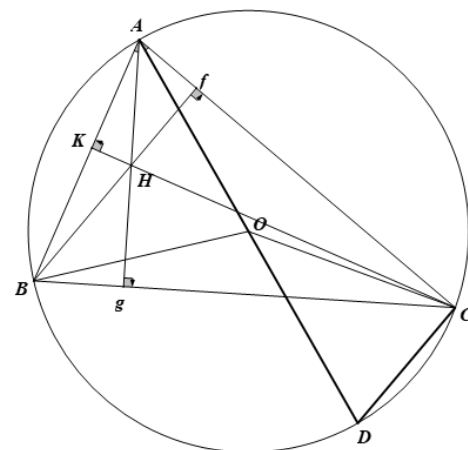### Solution 1: Obtained through exploration in Geometer's Sketchpad

Extend CO to meet the circumcircle at A`. Then $\angle BA`C = \angle BAC = x$, since they are inscribed in chord BC. Hence, from similarity $[\Delta BA`C$ III $\Delta EOC]$ (or corresponding angles) $\angle EOC = \angle A = x$. Therefore, since ΔOBC is an isosceles, $\angle OBC = 90° - x = \angle OCB$. But $\angle ABf = 90° - x$, since $\angle ABF = 90°$. Thus, $\angle ABH = \angle ABF = 90° - x = \angle OBE = \angle OBC$. QED. Similarly, we can prove that $\angle ABH = \angle CAO$ and $\angle BCH = \angle ACO$, and thus $\angle BAH = \angle CAO$.

Solution 2

From the diagram in Figure 1, $\angle BOC = 2\angle A$, since the angle at the centre of a circle is twice the angle at the circumference. In ΔBOC, $\angle BOC + \angle OBC + \angle OCB = 180°$ (sum of angles in a triangle), but $\angle OBC = \angle OCB$ (angles opposite equal sides of a triangle), $\therefore 2\angle A + 2\angle OBC = 180°$, which simplifies to $2(\angle A + \angle OBC) = 180°, \therefore \angle OBC = 90° - \angle A$. Then in ΔABf, $90° + \angle A + \angle ABH = 180°$ (sum of angles in a triangle), making $\angle ABH$ the subject of the equation, we have $\angle ABH = 180° - 90° - \angle A$, $\therefore \angle ABH = 90° - \angle A = \angle OBC$. Similarly, we can show that $\angle BAH = \angle CAO$. QED.

Solution 3

Another approach that involves a construction would be when one extends AO to D on the circumference and

join CD. Then, $\angle ACD = 90°$ because of the angle in a semicircle. If we let $\angle CAO = x$, then $\angle ADC = 90° - x$, because of the sum of angles in ΔADC, $\angle ABC = 90° - x$, since it is also subtended by AC which subtends $\angle ADC$. Lastly, we can conclude that $\angle BAH = x$, since $\angle AgB = 90°$, the sum of angles in ΔABg, implying that $\angle CAO = \angle BAH$.

The reader is challenged to seek even more solutions to this problem as they exist.

THE STATE OF MSTs IN SOUTH AFRICAN MATHEMATICS EDUCATION

Opportunities Given to Learners

Most South African learners do not explore multiple solutions to mathematical problems because their teachers lack the expertise of developing MSTs and they are unaware of the significance of MSTs in developing learners' mathematical thought. Hence, many South African learners usually do not achieve near their true mathematical potential because their teachers do not give them adequate support to explore multiple solutions in their classrooms (Donohue & Borman, 2014; Hoth et al., 2017; Mhlolo, 2017; Oswald & de Villiers, 2013). After obtaining a solution to a certain problem, South Africa learners usually end there and do not reflect on the solution or look for alternative solutions. If learners can be enculturated into always reflecting and looking for alternative solutions to mathematical problems, their competency in solving mathematics problems would improve. This means that, as these learners grow up, their desire to always reflect on solutions to mathematical problems increase, and unconsciously, they find themselves looking for multiple solutions to mathematical problems in a self-directed way. As learners are not always forced to find multiple solutions to mathematical problems, the decision to do so becomes their initiative to enhance their competency in solving mathematical problems, because they would be diagnosing their learning, reflecting on which approach is the best and would attempt to utilise different strategies to arrive at the same answer (Knowles, 1975). South African learners should possess mathematical content knowledge of definitions and algorithms for routine problem-solving because it is a necessary requirement for one to display one's mathematical creativity (Leikin & Lev, 2013). Hence, South African teachers need to transform this ritualistic discourse participation to explorative ones that can support the development of creativity by using MSTs. Though it is argued that classroom practice of the gifted should be different from that of regular classrooms (Leikin, 2010) and that different learners have different instructional needs (Levav-Waynberg & Leikin, 2012b), in the South African context, teaching methods advocated for teaching gifted learners are perfect for learners considered not gifted (Mhlolo, 2017).

Research Conducted on MSTs

So far, the significance of MSTs in mathematics and specifically geometry education in general have been discussed. In this section, the nature of research done on MSTs in the South African context is discussed. Though there is a dearth of literature on the significance of MSTs in South African mathematics education, multiple solutions manifest themselves in learners' solutions as observed in their scripts. Specifically, geometry education questions usually end up with more than one solution, different in both approach and structure. Much of the research on MSTs in South Africa is theoretical: the research focuses mainly on showing that a certain national examination problem (Samson, 2015, 2017; Samson & Kroon, 2019), Olympiad problem (Christison, 2019; De Villiers, 2016, 2017), or a competition problem (Pillay, 2017) have multiple solutions. Though South African researchers collaborate in global empirical studies investigating the effects of MSTs on learners (Daher et al., 2017) and there is a haphazard appearance of the phrase *multiple solutions* in empirical studies in South Africa (Mhlolo, 2017; Wessels, 2014), there is a lack of empirical studies on the role of MSTs in the South African mathematics education. I am currently not aware of any empirical study in the South African context on the role of MSTs in both teacher and learner education. This raises red flags about both South African teachers' and learners' competency in mathematical problem-solving, because not being engaged in the activity of producing multiple solutions to mathematical problems might mean that they are likely not able solve mathematical problems in various ways. The incorporation of MSTs in both teacher and learner education is critical in the development of their mathematical knowledge (Schukajlow et al., 2015; Stanislaw & Krug, 2014) and the ability to solve problems in multiple ways is a necessary skill for the 21st century (Gleason, 2018; Organization for Economic Co-operation and Development, 2018). Hence, South African mathematics educators, curriculum designers and other stakeholders should start investigating the possibility of incorporating MSTs in mathematics teaching and learning.

The Significance of MSTs in Teachers and CAPS

Despite the above-highlighted research on the benefits of MSTs in mathematics education, the use of MSTs in supporting teachers' development of content knowledge and problem solving abilities for competent teaching of mathematics in the South African context is poorly researched. An example of an empirical study on MSTs is that of Mhlolo (2017), who investigated whether South African teachers were able to recognise and support the development of mildly gifted learners'

creative potential. Four teachers' classroom practice was observed over a week, and the data were analysed based on their representational fluency and their responses to mildly gifted learners' creative ideas (Mhlolo, 2017). The results of the latter study showed that, in South Africa, regular classrooms might not be conducive to the nurturing of mildly gifted learners' creative potential. MSTs are effective in the professional development of pre-service mathematics teachers' (PMTs) and in-service mathematics teachers' knowledge of mathematics, pedagogy and their competence in solving mathematical problems (Chapman, 2013; Guberman & Leikin, 2013; Leikin, 2007, 2011; Leikin & Levav-Waynberg, 2008; Stupel & Ben-Chaim, 2017). After engaging in a course on MSTs practice, in-service teachers began to give novel multiple solutions to mathematical problems and their creativity increased (Guberman & Leikin, 2013; Stupel & Ben-Chaim, 2017). MSTs put teachers in a better position to not only reproduce solutions from books or previous experiences but to produce their innovative solution (Guberman & Leikin, 2013), a practice they can nurture in their learners. Teaching through MSTs is not only effective in developing learners' mathematical thinking but also in improving their problem-solving competency (Lynch & Star, 2014; Schoenfeld, 1983; Stanislaw & Krug, 2014). Hence, teachers should organise their mathematical classroom practices such that they incorporate MSTs and train their learners to always reflect on solutions to determine if there is a better, more elegant solution than the one they already found (Santos-Trigo & Reyes-Rodriguez, 2016).

Recently, South Africa has developed a teaching and learning framework that aims to teach mathematics for understanding (Department of Basic Education, 2019). One of the key principles in this document is to ensure *quality* mathematics teaching in the classroom to maximise the achievement of the CAPS outcomes and performance in the subject. It uses reflections from countries that perform higher in international tests to guide teachers on how they can support learners' mathematical development to match international standards. The framework states that teachers should strive to:

- develop learners' strategic competence – the ability to formulate, represent and *decide on appropriate strategies* to solve mathematical problems;
- provide *multiple and varied opportunities* for learners to develop their mathematical reasoning skills – the capacity for logical thought, reflection, explanation and justification (Department of Basic Education, 2019, p. 13).

However, if teachers strive to give learners opportunities to *decide on appropriate strategies* they would use to solve mathematical problems, learners need to have more than one problem-solving strategy at their disposal. The current regular teaching in South African schools cannot equip learners with enough problem-solving strategies because it focuses on rote learning. Furthermore, providing learners with *multiple and varied opportunities* to develop their mathematical thinking cannot be achieved through rote learning in South African classrooms; a learning environment characterised by exploration and reflection is required. Some of these teacher competencies envisaged by the CAPS supporting document are currently difficult to achieve, because CAPS is "structured" and does not afford teachers a chance to use their expertise to teach their learners effectively in their respective contexts. It is challenging to obtain fluidity, as teachers are given work schedules with deadlines for the completion of topics, and in certain situations, details of class and homework activities are offered to teachers to give to learners. Amongst other conclusions, Mhlolo (2017) and Julie and Gierdien (2020) mentioned that South African teachers are usually caught between teaching for mathematical competency and meeting curriculum demands.

CONCLUSIONS AND RECOMMENDATIONS

Most South African classrooms focus on training learners to pass tests and examinations without sound understanding. Some thinking moves[2] related to mathematical problem-solving are usually neglected and not nurtured in the performance-based South African mathematics education curriculum. The exercise of finding multiple solutions to mathematical problems is one of these thinking moves. This theoretical article is an attempt to answer the following question: What is the significance of MSTs in mathematics education, and why is it relevant for South African mathematics education to make the exercise of producing multiple solutions accessible to learners? In this exhaustive literature study, I explicated ideas that would help in answering the above question. Before summarising these ideas to answer the above-stated question, the reader should bear in mind that the subsequent conclusions – though they may be of global importance – need to be viewed in the context of South African mathematics education. Furthermore, above I have synthesized the available evidence of the benefits of incorporating mathematics education, the deficiency in this research within South African mathematics education, and I occasionally linked my discussion with the context of South Africa. In

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[2] Here I use the phrase "thinking moves" to refer to the shifts in mathematical thinking. These thinking shifts are related to beliefs about mathematical problem solving, for example, one may have the belief that to solve mathematical problems certain procedures need to be followed and if these procedures are not followed then the problem cannot be solved. This also include beliefs about what constitutes as a mathematical problem and a mathematical solution to a problem.

this conclusion I mainly aim to provide recommendations towards the second part of my proposed question, why is it relevant for South African mathematics education to make the exercise of producing multiple solutions accessible to learners?

Children go to school so that teachers can facilitate their holistic development, train them to be productive individuals and to become innovators so that they can one day support their families. In the teachers' possession, children's minds are malleable, ready to be shaped into various objects through appropriate teaching methods (Kilpatrick, 1985). Since the development of children is generational, it is argued that mathematics education should allow learners to develop and be competent in mathematics-specific skills that would allow them to flourish in the 21st century. I regard the incorporation of MSTs in mathematics education as promising in developing several 21st-century skills, such as solving problems in different ways, creativity, innovation, collaboration, and higher order thinking. Given the benefits of incorporating MSTs to enhance mathematical thinking, problem-solving and performance, it suffices to begin investigating this dormant field in the South African context. To transform classrooms into meaningful learning environments and to teach mathematics as a creative enterprise, teachers are encouraged to design rich problems containing multiple solutions and support learners in finding these multiple solutions (or proofs). The incorporation of MSTs further presents a window for teachers to spot those learners who might be creative in producing multiple solutions to mathematical problems and to train them to compete in mathematics Olympiads with their global counterparts.

Comparing his arguments with those of Hamming (1980), Jeremy Kilpatrick (1981) provided reasons as to why research in mathematics education is not effective – something that is still visible nowadays. He mentioned the lack of ingenuity in adapting and linking our research tools or instruments to the problems faced in real classrooms. Kilpatrick (1981) lamented that most mathematics education research is not effective because it reveals little implications for classroom practice. Encouraging learners to produce multiple solutions to mathematics problems can engage them in reflective activity (Dewey, 1933), which can innovate their individual and country`s social progress (Schon, 1983). It has been observed that children benefit easily through studying behaviour modelled by influential adults, like their teachers (Cropley, 1997). It is in their nature to learn from adults from an early age. In the light of this, there is a dire need for teacher training institutions to develop mathematics teachers who can design and produce multiple solutions to mathematical problems. This behaviour should further be modelled in the classroom where learners can see the significance of producing multiple solutions to mathematical problems.

Currently, I am aware of legitimate concerns in South African education. One particular concern is that teachers are required to employ time-consuming innovative teaching strategies under strict and imposed time frames. In this way, teachers are required to balance high-quality teaching with curriculum coverage according to the CAPS mandate, which leaves them in a bewildered state between teaching for understanding and teaching for learners to pass the national examinations (Julie & Gierdien, 2020; Mhlolo, 2017). This concern, and others not mentioned here, point to the significance of including curriculum designers and other stakeholders in designing a mathematics curriculum that incorporates MSTs. This is because the benefits of incorporating MSTs in mathematics education outweigh the detriments of not doing so.

The propositions above stem from decades of empirical and theoretical research on mathematics education discussed in this article. These propositions are mainly framed by the growing demand for education to train learners to acquire certain skills that would allow them to function as productive citizens in 21st-century bombarded by digitisation and automation. Furthermore, the propositions are formulated to improve the dire situation of poor performance in South African mathematics education, which is currently not preparing South African learners for the 21st century and the workplace. Despite the perennial poor performance in both national and international tests, South African mathematics education has not adapted to improve this situation and meet global standards. In this article, I entered a field dormant in South African research to articulate the importance of MSTs in 21st-century mathematics education and to provide reasons as to why South African mathematics education must adopt an approach that focuses on engaging learners in MSTs. Within this field, two other important issues were also raised: (a) teacher training institutions need to train mathematics teachers who would be competent in formulating, producing and facilitating the teaching and learning of mathematics using MSTs; and (b) South African mathematics educational researchers need to conduct empirical studies to investigate the role of MSTs in both teacher and learner education. It is hoped that this article is sufficient to ignite a local debate on the suitability of MSTs in the context of South Africa and the suitability of MSTs in training South African learners to be successful in the 21st century.

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