The Use of Logical Implication as an Indicator of Understanding the Concept of Number Sequences

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Abstract

This paper aims to characterise an indicator of the development of the number sequence scheme among students at the level of Compulsory Secondary Education (14-16 years old students). To do so, we use a scheme development proposed by the APOS theory to characterise students' use of relations between mathematical elements when solving a mathematical task. We use a qualitative methodology and the data collection instruments are a written questionnaire and a semi-structured interview. In this work we show the questionnaire task that provides analytical expressions and ask students to determine which of them numbers sequences are. We find that students' use of logical implication when solving tasks related to number sequences is an indicator of the development of the scheme. This indicator helps to locate the transition mechanisms between the levels of development of the number sequence scheme. Moreover, our research shows that arithmetic and geometric progressions play a key role as an indicator of the development of the number sequence scheme.

Keywords: number sequences, arithmetic and geometric progressions, APOS theory, scheme, compulsory secondary education

INTRODUCTION

Several studies have underlined the importance of research into the concept of number sequences due to its implications for understanding other concepts in mathematical analysis. In their research into the concept of limits, Mamona-Downs (2001) and Roh (2008) stressed that a good grasp of the concept of sequences is fundamental for understanding the concept of limits. In relation to understanding of the concept of number series, studies by Bagni (2005), Codes and González-Martín (2017), and Codes et al. (2013) noted the relevance of research into the concept of number sequences, given that a number series is formally a sequence of partial additions. Due to the growth in technology, Weigand (2015) has recently argued that more attention should be given to number sequences in terms of recurrence relations, as these are prototypical discrete objects in maths.

Various researchers have analysed understanding of the concept of sequences from a range of distinct theoretical perspectives. Cañadas (2007) carried out a study on the inductive reasoning used by secondary-school students when solving tasks related to linear and quadratic sequences, which noted the presence of different modes of analytical (numerical, algebraic) and graphical (number lines and Cartesian planes) representation. Cañadas (2007) concludes that using different modes of representation when solving tasks helps to understand the concept of number sequences.

Furthermore, regarding the mode of numerical representation, research conducted by Djasuli et al. (2017) with a secondary-school student, where he is given a task in which he must find the general term of a number sequence from the first terms, concludes that this type of task is fundamental for the formal construction of arithmetic and geometric progressions.

Additionally, McDonald et al. (2000) research with university students into the type of mental constructions used by students for understanding the aforementioned concept indicated that they construct two different cognitive objects: an object comprising a list of numbers (Seqlist), and an object comprising a function whose domain belongs to the set of natural numbers (Seqfun),

Contribution to the literature

- The understanding of the number sequence concept is important by its implications for understanding other concepts in mathematical analysis and there are few studies of it.
- The methodological design proposed in this research can be used as a model of data-collection instruments when it is not possible to do an oral interview. This is a written semi-structured interview designed for each student on the basis of the responses given in the written questionnaire.
- The results related to logical relationships as indicators of understanding of a concept confirm and extend conclusions obtained on the development of schema of other concepts.

their study focusing on the latter. In her research with secondary-school students (16-19 years old students), Przenioslo (2006) divided the students' conceptions into two groups. The first group perceived sequences as a function, and the second group saw a sequence as being associated with an ordered set of numbers, in which a relation between the terms or a certain regularity must be present.

In contrast, Mor et al. (2006) observed in another study of secondary-school students that number sequences are intuitively considered to be recursive by these students. In other words, they are seen more as a relation between successive values of a sequence than as a relation between the values and their respective positions. In the paper of González et al. (2011) with university students, the authors noted that the relation between graphic and algebraic interpretation presented difficulties in relation to the concept of sequences.

The study presented is part of a more extensive research project focused on characterising understanding of the concept of number sequences. Specifically, we outline the identification of indicators for levels of understanding of the number sequences concept.

Looking at the issue in terms of the curriculum, the concept of number sequences appears in the second stage (14-16 years old students) of Compulsory Secondary Education in Spain, as part of the algebra module, in the following terms: "Study and analysis of number sequences. Arithmetic and geometric progressions. Recurrent sequences. Curiosity and interest in investigating the regularities, relations and properties that appear in sets of numbers". (Boletín Oficial del Estado [BOE] 5 of 5 January 2007, p. 756).

THEORETICAL FRAMEWORK

This section has been partitioned into two subsections: one providing number sequences notions, and the other concerning APOS theory and its application to this work.

Number Sequences Notions

Our research considers the concept of number sequences, in line with the definition of Stewart et al. (2007), as follows: a sequence is an infinite set of written numbers in a specific order, a_1, a_2, a_3 ..., a_n ,..., in which every member of the set has been labelled with a natural-number subscript, a_1 being the first and an being the general nth term; we will designate the sequence $\{a_n\}$.

There are various ways of constructing the terms of a number sequence, of which we will take into account the following: (i) through some kind of rule or formula that defines the nth term (the general term); (ii) through a set of instructions that indicates how to obtain a term using the preceding ones (through recurrence); or (iii) by providing a series of ordered, consecutive terms, one after another, as an infinite list of numbers (through extension).

Similarly, an arithmetic progression is a sequence of the type a, a+d ..., a+nd … in which the number "a" is the first term and d≠0 is the common difference between two consecutive terms (Stewart et al., 2007, p. 181). Finally, a geometric progression is a sequence of the type a, ar ..., ar^n ... in which the number "a" is the first term and r≠1 is the ratio of the progression (Stewart et al., 2007).

The APOS Theory

In this subsection, we will describe the APOS theoretical framework (Arnon et al., 2014; Dubinsky, 1991) that we have used in this research. The APOS theory explores how understanding of mathematical concepts is developed in terms of the construction of schemas, through the mechanism of reflective abstraction (Piaget & García, 1983). Within this model, a schema is defined:

> A Schema is a coherent collection of structures (Actions, Processes, Objects, and other Schemas) and connections established among those structures. It can be transformed into a static structure (Object) and/or used as a dynamic structure that assimilates other related Objects or Schemas, (Arnond et al. 2014, p. 25).

This way of conceptualising the development of a schema has been referenced in a range of research aiming to characterise the understanding of various mathematical concepts, such as derivative (Baker et al., 2000; Sánchez-Matamoros, 2004), linear transformations (Roa-Fuentes & Oktac, 2010), and limit (Valls et al., 2011).

In these studies, a schema is developed by progressing through three stages in a fixed order, and these are identified in terms of: mental structures of the concept, modes of representation, mathematical elements and the logical relations among them.

There are several mental structures - actions, processes, objects and schemas - which are organised in the genetic decomposition of a concept. The genetic decomposition is understood as "a structured set of mental constructs, which can describe how the concept is developed in the mind of the individual" (Asiala et al., 1996). Genetic decomposition of a mathematical concept provides a potential progression in the student's learning towards formation of the concept.

According to Arnon et al. (2014), a concept is first conceived as an action: an external transformation that has to be explicitly performed upon an object or objects that were conceived beforehand (as recipe). As the actions are repeated and the individual reflects on them, then the individual progresses from depending on external signals to gaining internal control over them and begins to see the concept as a process.

- Interiorisation is the mechanism that allow the shift from action to process.

- Reversal of a process is the ability to think about it in reverse, in the sense of breaking down the steps of the interiorised process, giving rise to a new process.

- Coordination of processes is the cognitive act of taking two or more processes and using them to construct a new process. In general, coordination may transform processes into processes.

- Encapsulation occurs when an individual applies an action or a process to another process; in other words, the individual sees a dynamic structure (process) as a static structure (object) to which actions may be applied. Once a process has been encapsulated in an object, it may be de-encapsulated, when necessary, and returned to its underlying process.

In light of our literature review (Cañadas, 2007; Duval, 2006; González et al., 2011) on modes of representation, we have taken into account the following: numerical representation, algebraic representation, representation in a line graph (representing number sequences as points on a number line) and representation in a Cartesian graph (representing number sequences as points on a Cartesian plane).

In this study, we will consider the following mathematical elements (E) in relation to the concept of sequences as numerical lists:

(1) E1 Number sequence (as a list): a sequence of real numbers arranged in an order. In other words, for each natural number n, a real number exists.

(2) E2 Terms of a number sequence: are defined as the elements of a number sequence. The place that each occupies is determined by its position, denoted by a subscript which is a natural number.

(3) E3 General term of a number sequence: is defined as the term whose value is known on the basis of its position - its subscript - and which is denoted by "a_n" (n being a natural number).

(4) E4 Arithmetic progression: a number sequence in which each term is obtained from the preceding one by adding a fixed quantity to it, which we call the difference.

(5) E5 General term of an arithmetic progression: a_1 being the first term and d the difference between consecutive terms.

(6) E6 Geometric progression: a number sequence in which each term is obtained from the preceding one by multiplying it by a fixed quantity, which we call the ratio.

(7) E7 General term of a geometric progression: a_1 being the first term and r the ratio of the progression.

(8) E8 Recurrent number sequence: a number sequence is recurrent if it is defined by a law of recurrence - a relation between one term and those that precede it.

(9) E9 Number sequence by extension: a number sequence is defined by extension when a series of terms that follow on from it are given.

(10) E10 Increasing number sequence: a number sequence is described as increasing when each term is lower or equal to the following term.

(11) E11 Decreasing number sequence: a number sequence is described as decreasing when each term is greater or equal to the following term.

The logical relations considered in our research are:

- Logical conjunction: is the relation that is established between mathematical elements when they are used jointly to make inferences.

When, for example, terms of the number sequence are obtained by using an algebraic expression, and when any value may be obtained on the basis of that position, one can define it as a number sequence. In other words, the general term of a number sequence (E3) and the terms of a number sequence (E2) are used in conjunction to infer that the baseline algebraic expression is a number sequence.

- Logical implication: [A→B] Implication is a structure in which a mathematical element is a logical consequence of another or others. This means that if the mathematical element on the left is verified, the mathematical element on the right is also verified.

An example is the relation that exists between progressions and number sequences: arithmetic or

geometric progression (E4 or E6) → number sequences (E1).

- Counter-reciprocal: is the relation that is established between a logical implication and its counter-reciprocal [(A→B) ↔ (not B→not A)].

For example, numerical sequence as a list (E1) and terminus of a sequence (E2) are related through logical implication (E1→E2) and by the ratio of their counter-reciprocal (not E2 → not E1), i.e. if there is no term then there is no sequence.

As mentioned earlier in the discussion of APOS theory, genetic decomposition of a mathematical concept provides a potential progression in a student's learning towards formation of a concept. In our work, with respect to the concept of number sequence, we have specifically explored the following genetic decomposition (Bajo-Benito et al., 2019):

Prerequisites to the Genetic Decomposition of number sequence

The concepts required for the construction of a number sequence schema are: algebraic expressions and the numerical value of algebraic expressions, as objects; and graphical representations of points on a number line and on a Cartesian plane, as a process.

Genetic decomposition of number sequence

(1) The action of calculating terms of a number sequence on the basis of the position they occupy.

(2) Interiorisation of the action of calculating terms of a number sequence on the basis of the position they occupy, as a process, reflecting on the results obtained through repetition of the action of calculating different terms of the number sequence, substituting them for the general term.

(3) Reversal of the process constructed in point 2, to obtain the position occupied by a specific term of the number sequence.

(4) Coordination of the process constructed in point 2 in the different modes of representation.

(5) Encapsulation of process 4 as an object, upon which to carry out actions or processes to study the overall properties involving all the terms of the sequence ($a_n=[a_1, a_2,…]$).

(6) De-encapsulation of object 5 as a process, whereby the complete sequence and some of its specific terms may be considered. For example, in situations of comparing number sequences, trends, etc.

In this study, the levels of development of the schema for number sequences as numerical lists are characterised as follows:

Intra level

Characterised by the use of mathematical elements in isolation, in a certain mode of representation, without establishing relations. An individual at this level of development of a schema focuses on individual actions, processes and objects without relating them to others.

Inter level

Characterised by the correct use of mathematical elements in certain modes of representation and establishing logical relations between mathematical elements that are in the same mode of representation. This level is characterised by the construction of relations and transformations between the processes and objects that make up the schema.

Trans level

At this level, there is an expansion in the repertoire of logical relations between the mathematical elements employed. "Synthesis" of the modes of representation occurs. This leads to construction of the mathematical structure. It is at this level that the student reflects on the connections and relations developed at the previous level, and that new structures appear. Through the synthesis of these relations, the student is made aware of the transformations that occur in the schema and constructs a new structure. The schema develops coherence at this level, demonstrated by an individual's ability to recognise the relations that are included in the schema, in order to reflect upon the explicit structure of the schema, and thereby think about what content in the schema is suitable for solving a problem.

At the inter and trans levels of the three stages, the student reorganises the knowledge acquired in the preceding level. A student's progression from one level to the next includes an expansion in their repertoire of mathematical elements, and the construction of new forms of relations or transformations between the mathematical elements used by the students to solve a problem (Table 1).

In this study, we address the following question:

Is the logical implication an indicator that can help characterise the transition between the different levels of understanding of the number sequences schema in students in the last two years of Compulsory Secondary Education (14-16 years old students)?

METHODOLOGY

Participants

The participants in this research are 105 students from the last two years of Compulsory Secondary Education (14-16 years old students) in Spain. These students have been coded with the first two codes representing the course and the following codes

Table 1. Characteristics of the levels of the numerical sequence scheme

Level	Characteristics
Intra	• The same element can be used correctly in certain tasks and incorrectly in others linked to a representation mode.
	• No logical relationships between elements are established.
	• They show a mental structure action of the numerical sequence concept.
Inter	• Use of the logical conjunction, logical implication and counter-reciprocal ratios but in some cases with errors.
	○ Use, with errors, of the logical implication ratio in its negative form between sequences and progressions.
	○ Use, with errors, of the reciprocal ratio of the general term element and sequence.
	• They show a mental structure process of the numerical sequence concept.
Trans	• Use of logical ratios: logical conjunction, logical implication, and counter-reciprocal correctly
	• They show a mental structure object of the numerical sequence concept.
	• They use all modes of representation (numerical, algebraic, graphical-linear and graphical-cartesian), and are able to move from one representation mode to another without difficulty.

Task

Given the following algebraic expressions, identify which of them are number sequences. Justify each answer.

$$a)\ a_n = \frac{1}{5-n} \qquad b)\ a_n = \frac{1}{n^2+1} \qquad c)\ a_n = \sqrt{1-n}$$

$$d)\ a_n = 3n-2 \qquad e)\ a_1 = 1, a_2 = 3 \atop a_n = a_{n-1}+a_{n-2} \qquad f)\ 16, 8, 4, 2, 1,\ldots$$

Figure 1. Questionnaire task

representing the student within the course (3b1, 3b2, 3b22...).

These students had been introduced, for the first time, to the concept of number sequences, in accordance with the official state curriculum (BOE, 2015):

> Investigation of regularities, relations and properties that appear in sets of numbers. Expressions using algebraic language. Number sequences. Recurrent sequences. Arithmetic and geometric progressions. (BOE 3 of 3 January 2015, Section I, p. 392).

Data-Collection Instruments

The data-collection instruments used were a questionnaire with four tasks (Bajo-Benito et al., 2019) and a written, semi-structured interview designed for each student on the basis of the responses given in the questionnaire, so as to explore in greater depth those responses which had not been explained.

The design of the questionnaire tasks took into account the genetic decomposition of number sequences, literature review (for instance, González et al., 2011; Przenioslo, 2006; Stewart et al., 2007), mental structures, the mathematical elements that form part of the concept of number sequences (listed in the introduction), the relations that may be established between them and the various modes of representation.

In this study, we will focus on one questionnaire task (Figure 1).

Participants responded to the questionnaire in one hour of class, and the semi-structured interview was completed a couple of weeks later. We will now describe the task of the questionnaire (Figure 1).

Task provides analytical expressions (which are numerical and algebraic) and asks students to determine which of them numbers sequences are. This requires, in itemsb) and d), the student to have a mental structure action of the concept of number sequences, by using the following mathematical elements in conjunction ("logical conjunction" relation): term (E2), sequence as a list (E1) and general term (E3). This is because, in solving said sections, they should obtain specific terms of certain algebraic expressions (giving values to position "n", n=1, n=2, etc.). Additionally, identifying that said algebraic expression is a number sequence requires students to have a mental structures process of the concept of number sequences, given that they should understand that it is possible to obtain the infinite values that make up the number sequence.

Items a) and c) require the student to engage in coordination of processes in the concept of number sequences by using the counter-reciprocal logical relation of number sequences as a list (E1) and terms of a sequence (E2) (not E2→ not E1). The student will thereby indicate that these are not number sequences: in item a) because there is no term corresponding to n=5

b) It is not a number sequence because when we apply the general term, the sequence is not progressively added to, multiplied, subtracted or divided by a fixed number.

Figure 2. Response of student 3b13 to item b of the questionnaire task

and in item c) because only the term corresponding to n=1 is present.

Item e), which is defined by recurrent relation, requires the student to have a mental structure action to calculate the terms on the basis of the first term (law of recurrence (E8)), and item f), as a geometric progression (E6 and E7) in numerical form by extension (E9), requires the student to have a mental structures action to identify the terms. Further, in order to identify that the expressions given in items e) and f) are number sequences, students are required to have amental structures process of the concept of number sequences, as they must understand that it is possible to obtain the infinite values that make up the sequence: in section e), through the law of recurrence, and in item f) they should identify that it is a geometric progression with a ½ ratio (E6) and determine the infinite values from the general term (E7).

Analysis Method

The analysis focused on identifying the mathematical elements, the logical relations established between them and the mental structures of the concept of number sequences that were exhibited in the responses of the students when solving the tasks.

The analysis was performed by considering the data from the questionnaire in conjunction with those from the written, semi-structured interview. A feature of the analysis method used is that the semi-structured interview was carried out in the days following the completion of the questionnaire, with the objective of clarifying the problem-solving process carried out by the student in the questionnaire. To this end, the questionnaire responses provided by the students were examined before carrying out the semi-structured interview, in order to adapt the semi-structured interview to the responses given by the students in the questionnaire. The semi-structured interview was thereby customised for each student, enabling us to expand the information gathered.

We will now illustrate how this analysis method was carried out, using one student as an example.

In student 3b13's solution to item b) of task of the questionnaire, the student considered a_n not to be a sequence, because the student established a relation of equivalence between number sequences and progression (E1, E4 and E5), as can be seen in Figure 2.

However, when we asked the student 3b13 about this in the written, semi-structured interview, the student thought about it and responded:

> Question: Could you explain the response given in item b: Why is it not a sequence?
>
> 3b13: It is a sequence because, although you do not multiply, add, subtract or divide by a fixed number, it follows a set pattern.

In the written, semi-structured interview, the student makes use of the general term (E3) provided in the text to respond correctly that, although it is not a progression (neither an arithmetic nor a geometric one: "you do not multiply, add, subtract or divide by a fixed number"), it is a number sequence because "it follows a set pattern" (referring to the general term of the sequence provided in the text).

The analytical procedure was performed by considering the responses given to the task in the questionnaire in conjunction with the written, semi-structured interview, for each of the students. We were thereby able to characterise understanding of the concept of sequences through the use students made of logical relation when solving the task.

Based on our joint analysis of the questionnaire and the written, semi-structured interview, we can therefore conclude that student 3b13 made correct use of the mathematical elements related to progressions and the general term of a sequence, and of the relations established between progressions and sequences.

RESULTS

The logical implication relation between mathematical elements, when students in Compulsory Secondary Education solve a task on number sequences, can be considered as an indicator of understanding the scheme of that concept. On the one hand, there are those students who make correct use of this relation, both to confirm and to deny—that is, if "A" is verified then "B" is verified (if A → B, affirmative form), but "A" not being verified does not imply that "B" is not verified (no A ≠> no B, negative form), evidencing a flexible use of this relation. These students are at the trans level of

development of the number sequence scheme. On the other hand, there are those students who, while making correct use of some relations (e.g. conjunction logic), still use the logical implication relation incorrectly when solving the task requires them to make use of that relation in its negative form. These students are at the inter level of development of the number sequence scheme. Therefore, the use of this indicator shows the transition between levels of understanding of the concept of number sequences.

More specifically, these results show the coordination of processes in the concept of number sequences by using the logical implication relation that is established between the mathematical elements of number sequence (E1) and arithmetic (E4) or geometric (E6) progression, that is, progression (arithmetic or geometric) implies number sequence [(E4 or E6) → E1]; however, no progression (arithmetic or geometric) does not imply no sequence, [no E4 ≠> no E1 or no E6 ≠> no E1].

Next, we present two sections. In the first one, we show a correct use of this logical implication, and in the second one, we show an incorrect use.

Correct Use of the Logical Implication Relation

Evidence of the correct use of this relation is found in the answer to the task in items d) (E4 → E1) and f) (E6→ E1) in an affirmative form, that is, if the arithmetic (E4) or geometric (E6) progression element is verified, it implies that the number sequence element (E1) is verified. The negative form is used in item e) (no E4 ≠> no E1 or no E6 ≠> no E1), i.e., if the mathematical element of arithmetic progression (no E4) or geometric progression (no E6) is not verified, it does not imply that the mathematical element of number sequence (E1) is not verified.

As an example of the correct use of this relation, we look at the student 3b14 who, when solving different items of the proposed task, shows the coordination of processes in the concept of number sequences by using of this relation of logical implication between sequences and progressions correctly, both in its affirmative form [(E4 or E6) → E1] and in its negative form [no E4 ≠> no E1 or no E6 ≠> no E1].

Thus, the student in item d), by the way of answering, makes use of the logical implication relation between sequence and arithmetic progression in the affirmative form, E4 →E1:

> Question: Why have not you done item d)?
>
> 3b14: I forgot, but it is a number sequence since the terms follow each other and go in threes with respect to the previous one.

And, in item e) of the written semi-structured interview, it can be inferred that the student makes correct use of the logical implication in the negative form, since he/she answers that it is a number sequence, even if it is not a progression. That is, not verifying arithmetic progression (not E4) or geometric progression (not E6) does not imply that it is not a number sequence (E1):

> Question: Is it a number sequence or not?
>
> 3b14: Yes, it is a number sequence because it follows a rule by which I can calculate any value.

Moreover, it can be inferred from these answers that the student is using a mental structure process of the concept of number sequence, since the student 3b14 considers that it is possible to obtain the infinite values that make up the number sequence through the recurrence relation given by the formula. This can be considered a manifestation of the internalisation of the action to calculate specific terms of the sequence.

Item f) confirms our hypothesis, since the student uses the implication relation between geometric progression (E6) and number sequence in affirmative form, that is, E6 →E1, when they answer: "It is a progression because the terms follow each other, specifically it is a G.P. [geometric progression] and the ratio is 1/2" (Figure 3). From this answer, it can be inferred that this student views progressions as a special case of number sequences.

Furthermore, at different points in the written semi-structured interview, this student proves that they differentiate number sequences and progressions:

> Question: Explain the difference between number sequence and progression.
>
> 3b14: A progression is a sequence of numbers where to find its terms you have to add or multiply, while a number sequence follows a rule by which its terms are found.

Incorrect Use of the Logical Implication Relation

Evidence of the incorrect use of this relation can be found in the answer to the task in items a), b), and e), where the student 3b4 states no E4 → no E1 or no E6 → no E1, that is, if the mathematical element of arithmetic progression (no E4) or geometric progression (no E6) is not verified, it implies that the mathematical element of number sequence (E1) is not verified, thus showing incorrect use of the logical implication relation in the negative form.

However, this same student demonstrates the coordination of processes in the concept of number sequences by using of the implications (E4 → E1) in item d) and of (E6→ E1) in item f) correctly; i.e., if the arithmetic (E4) or geometric (E6) progression element is verified, it implies that the number sequence element (E1) is verified.

f) It is a progression because the numbers follow one another, specifically it is a geometric progression and the ratio is 1/2

Figure 3. Response of student 3b14 to item b) of the questionnaire task

Evidence of incorrect use of this relation is found in a student who makes correct use of it in the affirmative form but makes incorrect use of it in the negative form, as shown below.

Thus, in the answers to item d), student 3b4 uses a mental structure action to obtain the first three terms of the algebraic expression [$a_n=3n-2$], and answering the question of whether it is number sequence in affirmative form with: "Yes, d is 3", referring to the fact that it is an arithmetic progression (E4) of difference 3; the student then concludes that it is a number sequence (E1), i.e. [E4 → E1], demonstrating a correct use of the affirmative form of the logical implication between arithmetic progression and number sequence.

The written semi-structured interview confirms student 3b4's use when he/she is asked about the answer given in the questionnaire.

Question: Justify the answer to item d)

3b4: It is a sequence because it is an arithmetic progression.

E: Justify why in paragraph e) you say it is not a sequence.

3b4: It is not a sequence because there is no matching number, i.e., it is fixed, to find the next one.

E: Do you know of any sequence that is not a progression?

3b4: There can be a sequence that is not a progression if there is no r, i.e., a sequence of numbers that you invent, any number one after another.

E: So, for you, what is a sequence?

3b4: A series of numbers one after the other.

E: Then there is no reason or difference, is there?

3b4: Of course, there doesn't have to be.

E: Then you contradict yourself in the previous section, don't you?

3b4: No.

This correct use of logical implication in an affirmative form is also evident in this student through the geometric progressions, so in his/her answers to item f), which shows a number sequence given by extension (E9) [16, 8, 4, 2,...], he/she again answer: "Yes, [referring to the fact that it is a number sequence (E1)] it is a geometric progression(E6)and the ratio is 2", that is, [E6 → E1].

Since the ratio of the geometric progression is not 2, as the student answers, but 1/2, the student is asked to clarify his/her answer in the written semi-structured interview:

Question: For item f), why do you say that it is a number sequence?

3b4: Because the values decrease when it is a geometric progression when divided by two.

Based on this answer, we can infer that the student recognises that the ratio of the geometric progression is 1/2, as he/she write that it is a decreasing progression when divided by two, and that it is a number sequence (E1) because it is a geometric progression (E6). This shows a correct use of the affirmative form of the logical implication between geometric progression and number sequence.

The answers given by student 3b4 in these sections are not enough to evidence a correct use of the logical implication between progressions and number sequences; for this, we need proof that the student is making correct use of the logical implication in negative form [no E4 ≠> no E1 or no E6 ≠> no E1].

To evidence this fact, we look at how such a logical implication relation in the negative form is used incorrectly by this student when he/she states that, as it is not an arithmetic (E4) or geometric (E6) progression, it implies that it is not number sequence, as we see below.

Thus, in item a), in which the algebraic expression an = 1/(5-n) is presented, the student correctly uses the
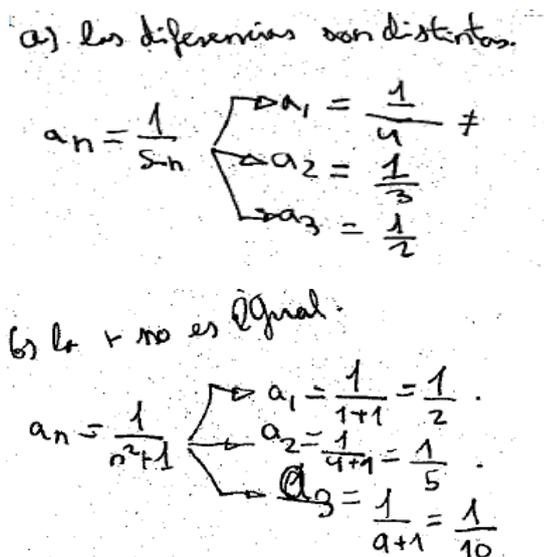
Figure 4. Response of student 3b14 to item b) of the questionnaire task

elements general term (E3) and term (E2) to determine if it is a number sequence, jointly through the "conjunction logic", demonstrating a mental structure action to obtain the first three terms of the expression.

However, although student 3b4 answers that it is not a number sequence, the justification given shows that he/she incorrectly uses the negative form of the logical implication [no arithmetic progression implies no number sequence (no E4 → no E1)], since he/she does not consider it to be a number sequence because "the differences are different" (Figure 4). This fact is corroborated in the written semi-structured interview, when we asked:

> Question: For item a), what does it mean that the differences are different and therefore it is not a number sequence?
>
> 3b4: It is not a number sequence because the differences are different...

In item b), student 3b4 proceeds in the same way as in item a), linking the incorrect use of the negative form of logical implication to geometric progressions [no geometric progression implies no number sequence (no E6 → no E1)], stating that "r is not equal" (Figure 4). This fact is corroborated in the written semi-structured interview, when we asked:

> Question: For item b), what does it mean that r is not equal and therefore it is not a number sequence?
>
> 3b4: It is not a number sequence because the numbers do not follow each other, nor is each number multiplied by the same one to get the next one.

Then, this student's answers to a) and b) show the incorrect use of the logical implication in its negative form, that is, if it is not a progression (arithmetic or geometric) then it is not a number sequence.

DISCUSSION AND CONCLUSIONS

The fact that the questionnaire task requires using the logical implication relation in its negative form allowed us to identify an indicator of the transition between the inter and trans levels in developing the understanding of the concept of number sequences. This can be considered a manifestation of the process encapsulation mechanism for the purpose of the concept of sequence as a numerical list.

These results corroborate those obtained in previous studies such as that of Djasuli et al. (2017), which show the importance of arithmetic and geometric progression tasks for the understanding of the concept of number sequences. Furthermore, our research shows that this type of task facilitates the transition between the levels of development of the number sequence scheme, since arithmetic and geometric progressions play a key role as an indicator of the development of this scheme.

In addition, regarding the use of logical implication, the results of our research are in line with previous research in relation to other mathematical concepts such as derivative function (Fuentealba et al., 2017, 2019a, 2019b; Sánchez-Matamoros, 2004; Sanchez-Matamoros et al., 2008, 2013). These studies showed how the use of the logical relations conjunction logic, counter-reciprocal and logical equivalence are indicators of the development of the derivative function concept scheme. Through our research, we were able to confirm how, in addition to these logical relations, the logical implication relation is also an indicator of the development of the number sequence concept scheme. This fact allows us to

consider that the students' use of logical relations when solving tasks is indicator of the development of the mathematical concept scheme.

Our research complements the results obtained in the McDonald et al. (2000) research. These authors focus on identifying the mechanisms of transition between the levels of development of the schema for sequences as functions (Seqfunc as defined in the McDonald et al. (2000) research) among university students. In our paper, we show the transition mechanisms of the construction of the concept of sequence as a numerical list (Seqlist as defined in the McDonald et al. (2000) research) among compulsory secondary-school students, complementing the transition mechanisms of the two ways of conceiving the concept of sequence (Seqlist and seqfunc).

Przenioslo (2006) research on secondary-school students' understanding of a sequence as an ordered set of numbers indicates that they had notions that were far removed from the meaning of the concept. This is because they only treated those characterised by some regularity (arithmetic or geometric progression) as sequences. In this sense, our results are in line with Przenioslo's research, since students characterised at the inter level of development of the scheme in our research linked sequences and progressions through logical equivalence, which led them to consider number sequences as only arithmetic or geometric progressions.

Studys by Cañadas (2007) and González et al. (2011) indicate that using different modes of representation when solving tasks helps in the understanding of the concept of number sequences. We have found that those students who use logical relations correctly solve tasks by translating between different modes of representation.

Since this work has been conducted with students in Compulsory Secondary Education (14-16 years old), one future line of research is to carry out research with students at pre-university or university levels to explore the development of the understanding of this concept, including new tasks in the data collection instruments.

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