MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2023, 19(6), em2270
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/13187

Theory of didactical suitability: An enlarged view of the quality of mathematics instruction

Juan D. Godino [1*], Carmen Batanero [1], María Burgos [1]

[1] Universidad de Granada, Granada, SPAIN

Received 09 February 2023 ▪ Accepted 04 April 2023

Abstract
Characterizing and measuring the quality of instruction is a matter of growing interest in mathematics education. Based on the notion of didactic suitability and the theoretical assumptions of the onto-semiotic approach, we develop an instrument to systematically analyze the different facets involved in a mathematics instruction process. We also explore the concordances and complementarities with instruments for measuring the quality of instruction. Thus, quantitative quality measurement efforts are complemented by another qualitative approach, focusing on the initiative and responsibility of teachers when they have to make decisions about their teaching practices. This reflective activity must be supported by specific instruments that reveal the complexity of the processes and the difficulty of achieving a balance between sometimes conflicting didactic principles.

Keywords: mathematics education, didactic suitability, instructional quality, onto-semiotic approach

INTRODUCTION

Several researchers have developed instruments to observe and measure the quality of instruction, either generic, content-specific, or a combination of both. Charalambous and Praetorius (2018) cite, among others, the projects elementary mathematics classroom observation form (Thompson & Davis, 2014), instructional quality assessment (IQA) (Matsumura et al. 2008), mathematical quality of instruction (MQI) (Learning Mathematics for Teaching Project, 2011), and mathematics-scan (M-Scan) (Walkowiak et al., 2014). Most of these works seek to provide valid and reliable information for decision-making on reform plans or teachers' accreditation.

Measuring instruction at scale is critical for providing information to the public about school quality, supporting better decision-making in districts, and directing attention toward excellent instructional practice (students' opportunity to learn) as a desired outcome (Matsumura et al., 2008, p. 297).

A distinctive feature of instructional quality studies is the selection of samples of classes, schools, and teachers, whose work is observed, together with samples of student productions, in an attempt to statistically relate some teaching variables to the students' learning. Classroom and student work observation protocols are constructed, making explicit criteria for the assignment of scores by external evaluators (Boston, 2012; Learning Mathematics for Teaching Project, 2011). The result is providing recommendations to improve instruction at the school or district level.

Usually, instructional quality measurement instruments assess limited aspects of instructional practices empirically associated with students' learning. Instrument feasibility and technical quality are pursued to ensure reliable use in assigning scores to classroom observations and students' productions. These measurement requirements may reduce the generalizability of results, as important aspects of instruction (e.g., misconceptions about mathematics or the role assigned to mathematical processes) may not be captured.

Although the assessment of a few well-chosen aspects of instruction can provide useful information to improve instruction, a comprehensive instrument can help becoming aware of the complexity of educational processes and identifying significant variables. In

✉ jgodino@ugr.es **(*Correspondence)** ✉ batanero@ugr.es ✉ mariaburgos@ugr.es

Contribution to the literature

- We provide a four-level structure (facets, components, subcomponents, and elements) to analyze the mathematics instruction processes.
- This structure is applied to develop an instrument (GASMIP), which is a system of value judgments to analyze didactic suitability (understood as an enlarged view of the quality of mathematics instruction).
- The GASMIP is compared with other instruments for measuring the quality of mathematics instruction.

addition, the optimization of teaching and learning processes often requires prioritizing some principles and leaving others in the background, considering the specific circumstances of the context and the students. Therefore, in this article, we aim to develop an instrument to systematically analyze the different facets and components involved in an instructional process. Teachers can use this tool to reflect on their practice and make informed decisions for its progressive improvement.

Although framed within the research on instructional quality, we adopt a broader point of view, supported by the theory of didactic suitability developed within onto-semiotic approach (OSA) to mathematical knowledge and instruction (Godino & Batanero, 1994; Godino et al., 2007). The issue is relevant because as Boston (2012, p. 77) states, "alternative measures of instructional quality are important and needed because they have the potential to go beyond simply measuring instruction and serve as a means of improving instruction". Considering the large amount of research being conducted on instructional quality and the measurement instruments constructed (Bostic et al., 2021; Charalambous & Praetorius, 2018), it is necessary to study the contribution of the new tool and its applications to mathematics teacher education.

In our research, we try to complement the efforts of quantitative measurement with a qualitative approach that focuses on the teachers' initiative and responsibility when they make decisions about their teaching practices. This reflective activity must be supported by specific instruments that reveal the complexity of the processes and the difficulty of achieving a balance between sometimes conflicting didactic principles. The results of this work have implications for research on teacher education, in particular for those interested in the teachers' reflection and decision-making about their practice (Karsenty & Arcavi, 2017; Tzur, 2001).

The article is structured in the following sections. First, we include a review of the literature on the quality of mathematics instruction; we highlight the article by Praetorius and Charanlambous (2018) for providing a measurement instrument that synthesizes 12 relevant papers on the topic. We then describe the theoretical framework, onto-semiotic approach to mathematical knowledge and instruction, and theory of didactic suitability on which we base the construction of the instrument for analyzing the quality of instruction. After describing the research questions and the method, the results develop the system of didactic suitability criteria for the different facets and components that characterize mathematical instructional processes (guide for the analysis of the suitability of mathematical instructional processes [GASMIP] [**Appendix**]). The discussion follows in which we identify the concordances and complementarities of GASMIP with the Praetorius and Charalambous (2018) instrument. To show the potential of the developed tool, we also refer to several research works, where implications of the use of didactic suitability in the training of mathematics teachers have been addressed. The article ends with sections on limitations and conclusions.

QUALITY OF MATHEMATICS INSTRUCTION AS A RESEARCH TOPIC

There is no consensus on how best to conceptualize and measure the quality of mathematics instruction, as it is evidenced by the proliferation of specific classroom observation tools (Berlin & Cohen, 2018). Such protocols emphasize different dimensions of instructional quality, and also raise the debate about the use of scales designed to measure the quality of teaching general academic content, and those specific to mathematics. While scales focusing on generic content include teacher feedback practices, and aspects of teaching relevant to various subjects, mathematics-specific scales incorporate elements characteristic of mathematics teaching, such as the use of multiple representations or the type of argumentation. The cited authors argue that important aspects of classroom development, such as emotional support, classroom organization, and student engagement, remain hidden in the specific instruments, so that mixed-type instruments may have greater validity in assessing the quality of instruction.

Some Specific Mathematics Frameworks

Below we summarize some specific proposals for assessing the quality of mathematics instruction, described by Praetorius and Charalambous (2018). Teaching for robust understanding (TRU) framework (Schoenfeld, 2013, 2018) emphasizes the experiences proposed to students that determine their learning, and it is mainly used to design and implement professional development activities. TRU model assumes principles of student-focused instruction and distinguishes five dimensions in instructional processes: mathematical content; cognitive demand; equitable access to content; agency, ownership and identity; and formative assessment to characterize the types of instruction that make students knowledgeable, flexible and resourceful thinkers, and problem solvers.

TRU involves a fundamental shift in perspective, from teacher-centered to student-centered. The key question is not: "Do I like what the teacher is doing?" It is: "What does instruction feel like, from the point of view of the student?" The teacher's actions are critically important, of course–but what really matters are the ways in which the students have meaningful opportunities to make sense of the content (Schoenfled, 2018, p. 494).

In *learning mathematics for teaching* project[1] , Hill et al. (2008) construct an instrument to and notice that measuring more satisfactorily quality of instruction would help educators to improve the teaching and learning. Learning Mathematics for Teaching Project (2011) describes the conceptual framework developed to identify and analyze the mathematical characteristics of classroom work, introducing MQI construct and developing a coding guide to assess various criteria. By ''mathematical quality of instruction,'' we mean only the nature of the mathematical content available to students during instruction. (Learning Mathematics for Teaching Project, 2011, p. 30). MQI framework includes six constructs and their corresponding scales: richness and development of the mathematics, responding to students, connecting classroom practice to mathematics, language, equity, and presence of mathematical errors.

Praetorius and Charalambous (2018) synthesize 12 articles included in a special issue of the *ZDM* journal devoted to instructional quality, each of which describes a theoretical framework and methodological tools, some focused on mathematics, others applicable to any content or with a mixed character. The authors recognize the multidimensional and complex nature of instruction, which explains the existence of different frameworks and observational instruments to assess the quality of instructional processes. They also note the differences, purposes, theoretical foundations, instructional aspects covered by each framework, the way in which they operationalize and measure quality, and issues related to the reliability and validity of the measures. They elaborate a global framework of aspects to be observed in the instructional processes (**Table 1**).

Table 1 synthesizes level I and level II categories of this model (Praetorius & Charalambous, 2018), which will serve as a basis for the revision and refinement of that proposed by TDS. In level I, the model distinguishes seven components, with between two and four observable characteristics each, that are part of level II of categories. Appendix D of that article includes more details for some of level II aspects, thus defining a level III of indicators in the quality structure. For example, in the aspect, presenting the content in mathematically accurate and correct ways (level II), three indicators are included:

1. presentation of concepts and procedures is mathematically accurate,
2. use of precise mathematical language and notation, and
3. lack of main mathematical errors.

Both level II and level III indicators are described in observational terms in the development of a lesson whose manifestation is valued as positive. That is, it is implicitly assumed that the "content should be presented in a mathematically precise and correct way". Consequently, all these statements can be formulated as norms or criteria that should be followed in the implementation of lessons, that is, as value judgments whose rationality should be explicit. In some cases, such as in the example, the justification for these norms is obvious: teaching should not spread errors. But the issue of the precision of mathematical content requires some nuances when mathematical objects can have different meanings, and they can be expressed in more or less formal ways, according to the students' ages. These ontological, semiotic, and cognitive assumptions are characteristic of OSA as indicated in the following section.

THEORETICAL FRAMEWORK

Onto-Semiotic Approach to Mathematical Knowledge

OSA to mathematical knowledge and instruction is a theoretical system that includes various tools to address the research problems posed by mathematics teaching and learning (Godino et al., 2007; Godino et al., 2019; Font et al., 2013). The aim of this theoretical framework is to address in an articulated manner the epistemological, ontological, semiotic-cognitive, and educational problems involved in teaching and learning mathematics. An anthropological (Wittgenstein, 1953) and pragmatist (Peirce, 1958) view of mathematics is assumed; therefore, the activity of problem solving is the central element in the construction of mathematical knowledge.

Two specific tools have been introduced in OSA to analyze the epistemic (content) and cognitive (learning) facets of instructional processes: meaning[2] and onto-semiotic configuration of practices, objects, and processes. Mathematical practice–any action performed

[1] Available at: www.sitemaker.umich.edu/lmt

[2] In OSA holistic view on meaning (Godino et al., 2021), general semiotic theories (Hjelmslev, 1943; Peirce, 1958; Wittgenstein, 1953), pragmatic and referential positions on meaning and sense in mathematics education are articulated.

Table 1. Aspects for observing and measuring instructional quality (model by Praetorius & Charalambous, 2018, p. 546)

Level I	Level II
I. Classroom and time management	Behavior management Time management
II. Content selection and presentation	Selecting mathematically worthwhile and developmentally appropriate content Motivating content Presenting content in a structured way Presenting content in mathematically accurate and correct ways
III. Cognitive activation	Potential for cognitive activation through: (a) teacher's selection of challenging tasks, which respond to students' cognitive level (b) teacher's use of mathematically rich practices Teacher facilitation of students' cognitive activity Teacher supports students' meta-cognitive learning from cognitively activating tasks
IV. Practicing	Teacher supports students solidify their procedural knowledge/skills Teacher procedural remediation of students' difficulties and errors in practicing
V. Formative assessment	Assessment is aligned with learning objectives Teacher regularly checks for understanding Quality of feedback for students Teacher capitalizes on formative assessment information to guide next instructional steps
VI. Socio-emotional support	Teacher-student relationships Student-student relationships
VII. Cutting-across instructional aspects aiming to maximize student learning	Forming an environment that nurtures productive habits (e.g., agency, ownership/autonomous learning, identity, and perseverance) Differentiation and adaptation Enhancing participation and active engagement of all students

to solve a problem, communicate, or generalize its solution–constitutes the starting point for analyzing mathematical activity in OSA. Consequently, the meaning of a mathematical object is the systems of operative and discursive practices performed by a person (personal meaning), or shared within an institution (institutional meaning), to solve a problem-situation (Godino et al., 2007). In mathematical practices (**Figure 1**), the primary mathematical objects, problem-situations, languages, concepts, propositions, procedures, and arguments, related to each other emerge and intervene through the respective mathematical processes of problematization, communication, definition, enunciation, algorithmization, and argumentation (Godino et al., 2007). Other more general processes (mega processes) such as problem solving, modeling, establishing connections between objects and meanings are also considered.

Characterizing the different meanings of the objects, and being aware of their plurality and relativity, helps constructing a global meaning that serves as a reference for the analysis of the mathematical instruction.

Didactical Suitability and its Structure

The comparative analysis of theoretical frameworks developed to characterize and evaluate the quality of instruction must consider four aspects (Charalambous & Praetorius, 2018):

1. specific approach taken to develop the framework,
2. its purpose (the "why"),
3. framework conceptualization (the "what"), and
4. its operationalization and measurement (the "how").

In this section we describe these aspects for theory of didactical suitability (TDS), which adopts a global approach to encompass the different dimensions involved in the teaching and learning of mathematics.

The didactical suitability of an instructional process is the degree to which such a process (or a part of it) meets certain characteristics that qualify it as optimal or adequate to achieve the adaptation between the students' personal meanings (learning) and the intended or implemented institutional meanings (teaching), considering the circumstances and available resources (environment). These institutional meanings are also representative of the global reference meaning.

This definition describes the conditions required in an instructional process to be attributed the value of suitability, which is initially linked to the adequacy of the coupling between teaching and learning and the implementation of rich mathematics, considering the multiple factors involved. From here one can go on to state an overall criterion (principle) of didactic suitability:

> Students should be helped to learn the intended mathematics, being representative of their overall meaning and considering the personal, contextual, and temporal circumstances.

This global criterion of suitability incorporates social values of mathematics education, such as avoiding

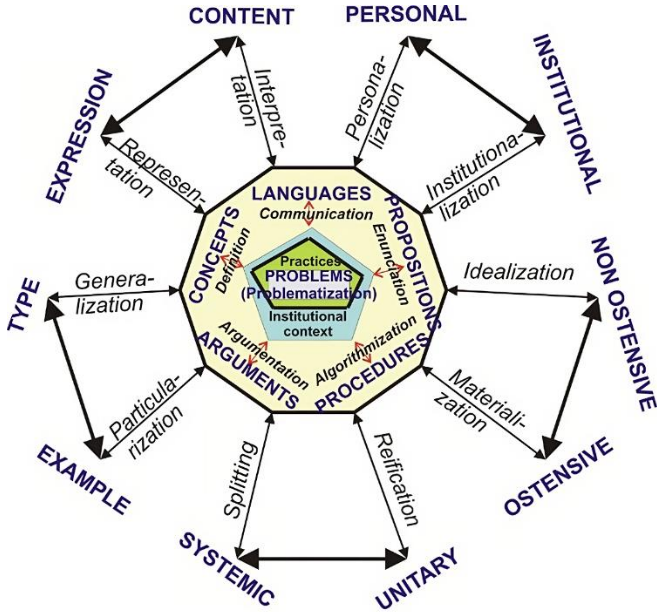

Figure 1. Configuration of practices, objects, and processes (Godino, 2014, p. 23)

school failure and making efficient use of available resources.

Structure of System of Categories of Suitability Criteria

TDS proposes six dimensions or facets, which define a level I of analysis:

1. **Epistemic facet:** Institutional meaning planned or implemented for a given mathematical content (problems, procedures, concepts, properties, language, and arguments).
2. **Ecological facet:** Relations of the content with other subjects and with the social, political and economic settings that support and condition teaching and learning.
3. **Mediational facet:** Material and technological resources available for teaching, possible use, and time allocated to the instruction.
4. **Interactional facet:** Organization of classroom discourse and interactions between teacher and students, considering students' learning difficulties and the negotiation of meanings.
5. **Cognitive facet:** Students' levels of development, understanding and mathematical competence (personal meanings), difficulties and errors in the intended content.
6. **Affective facet:** Students' emotions, attitudes, beliefs, values, interests, and needs regarding the content.

The actions and resources used in the epistemic, ecological, interactional, and mediational facets are aimed at students' learning in which both cognitive and affective aspects are contemplated.

There are also interactions between the different facets since the educational-instructional processes take place within recursive and open social systems and is based on the interpretation and negotiation of meanings, as well as on values.

Education can be characterized as an open recursive semiotic system. It is a semiotic system because the exchanges between teachers and students are not exchanges at the level of physical force but at the level of meaning. The system operates as a recursive system because teachers and students act upon the basis of their interpretations and understandings. Educational systems are generally open because they interact with their environments (Biesta, 2010, p. 497).

Didactic suitability requires the coherent articulation of six partial suitabilities related to the facets described. These can be refined from the components provided by the various tools elaborated in OSA. Thus, for example, epistemic suitability refers to the degree to which the institutional meanings of the content and the configurations of objects and processes implemented represent the reference global meaning, considering the contextual and personal circumstances of the subjects involved. Cognitive suitability refers to the degree to which the learning aims are an achievable cognitive challenge for the students, and the personal meanings achieved match the planned institutional meanings, considering their personal and contextual circumstances. The joint optimization of the partial suitabilities may be conflicting in a specific context and circumstances:

> which leads, first, to treating the suitability criteria jointly (and not as independent criteria …) and second, to questioning or relativizing the validity of a given criterion in a specific context, which leads to giving different relative weights to each criterion depending on the context (Breda et al., 2018, p. 265).

A second level of analysis (**Figure 2**) is determined by each facet components, some being applicable to any discipline, and others specific to mathematics. For the epistemic, mediational and cognitive facets, it is possible to propose a level III of analysis distinguishing subcomponents determined by the elements that characterize mathematical knowledge according to OSA. When the instructional process being analyzed refers to specific contents, for example, probability, it is possible to define a level IV in the epistemic, mediational and cognitive facets considering specific aspects to the teaching and learning of such content (Beltrán-Pellicer et al., 2018).

The category system in **Figure 2** structures the didactic suitability criteria. Comparing **Figure 2** with **Table 1**, we observe the disparity of expressions used to describe didactic suitability and instructional quality.

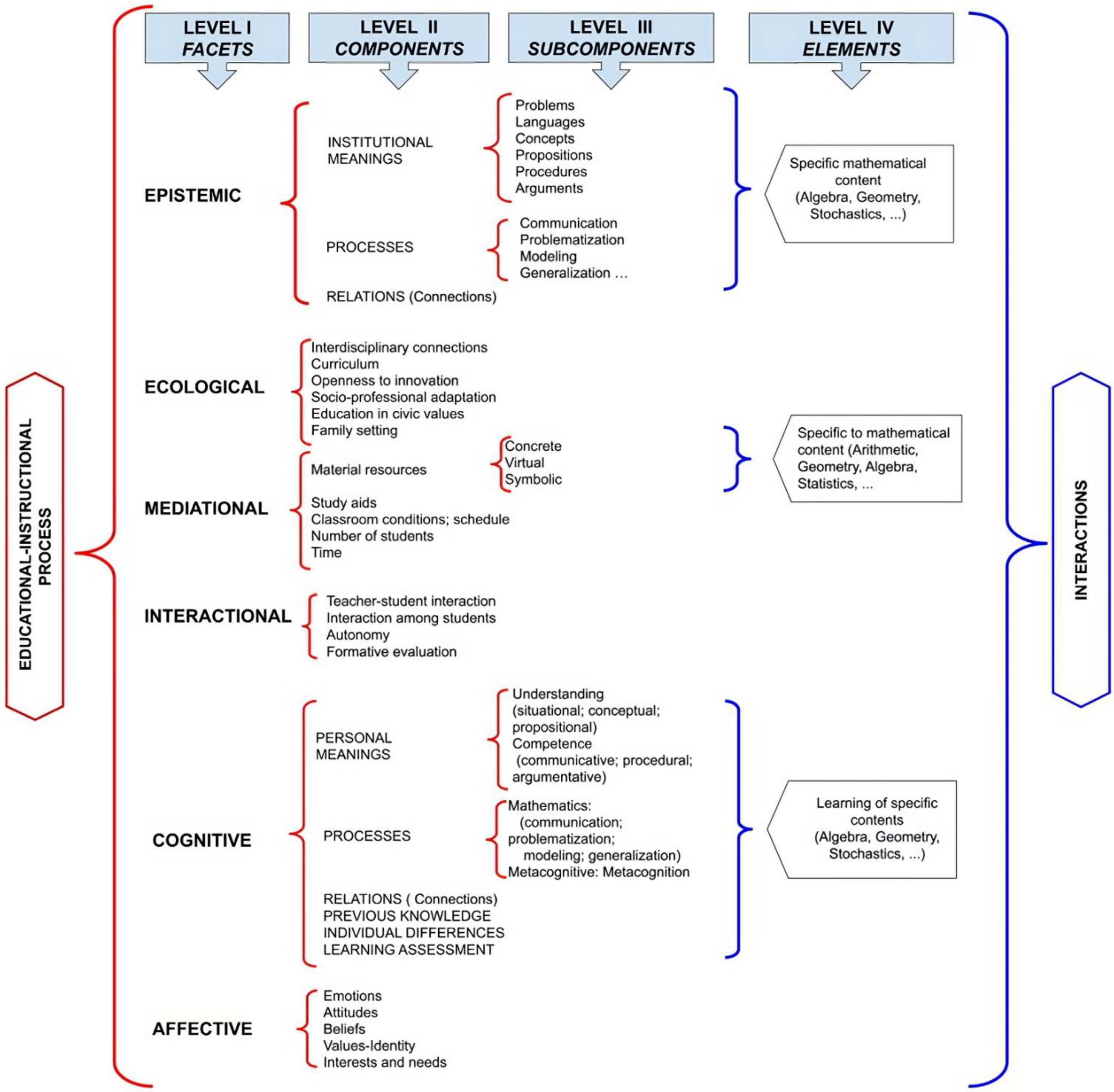

Figure 2. Facets and components of an educational-instructional process (Adapted from Godino et al., 2021, p. 10)

This diversity of expressions was noted by Praetorius and Charambolus (2018) in their analysis of 12 instruments for measuring instructional quality. Nevertheless, it can be observed that the seven aspects of level I in **Table 1** are more or less explicitly covered in the epistemic, mediational, interactional, cognitive, and affective facets of TDS model.

RESEARCH QUESTIONS AND METHODS

The main purpose of studies on the quality of instruction is to provide valid and reliable information to educational authorities to make decisions on reform plans or teacher accreditation and selection processes. The instruments designed for this purpose are applied to samples of classes to assess the quality (results) of learning, or to samples of teachers to assess the quality of teaching (their mathematical or didactic knowledge) and compare and rank them. The need for informed decisions imposes demands on the reliability of measurements, which requires focusing on objectively observable traits and developing rigorous assessment procedures.

The construct of instructional suitability developed within OSA proposes an expanded view of instructional quality by attempting to articulate its different facets and components. Its main objective is to support teachers' self-evaluation processes to identify aspects for progressive improvement of their practice. The quantitative approach with which quality studies are usually approached is complemented by an interpretative vision of the local optimization of educational processes. In this article, we pose the following research question:

How an instrument to analyze instructional quality should consider the various facets and components of mathematics teaching and learning processes?

To answer this question, we develop GASMIP included. The discussion on its relevance and necessity leads us to raise another question derived from the previous one:

What are concordances and complementarities of GASMIP instrument with other quality measurement instruments?

Since GASMIP is conceived as a resource for teachers' reflection on their practice, a research program follows focused on answering the following questions:

How does TDS contribute to analyze the teaching practice and to understand such practice?

What kind of training actions should be designed and implemented to train teachers in the use of GASMIP?

How change the practice of teachers using GASMIP?

The first question is addressed theoretically by applying the structure of facets and components of an instructional process and the notion of didactic suitability criteria proposed by OSA. The rationality of the criteria formulated, understood as value judgments, is based on OSA assumptions about mathematics, its teaching and learning, and their concordances with other educational theories. The second question is answered by selecting and analyzing a representative article on instructional quality (Pretorious & Charalambous, 2018) and comparing its quality indicators with the didactic suitability criteria. Although we do not study the remaining issues in this paper, we will refer to some studies using TDS in mathematics teacher education.

RESULTS

In this section we propose the general suitability criteria for the different facets, justifying their rationality in the assumptions of OSA framework and the concordances with other mathematics education theories. These criteria are also related to the aspects considered by other models on quality of instruction. The complete system of didactic suitability criteria for the facets and components configures the instrument GASMIP.

These suitability criteria are understood as principles that should be followed to attain a suitable instructional process. In previous works (Godino, 2013; Breda et al., 2018) suitability indicators–understood as features that should be observed in a suitable instructional process–were formulated for these components. To assign a greater or smaller degree of suitability, it would be necessary to develop rubrics with rules for assigning numerical values to the degree of compliance of each indicator. This quantitative orientation in the assessment of suitability has not been developed in TDS, since the main use of GASMIP instrument is the teacher professional development, and not comparing the quality of samples of lessons or teaching actions.

Criteria Related to Characteristics of Mathematical Content (Epistemic Facet)

In TDS, it is essential to assess the quality of the content that is taught and learned, and therefore the epistemic facet plays a prominent role We state the general criterion of epistemic suitability in the following terms:

> The system of partial institutional meanings of the content and the configurations of objects and processes linked to each meaning, implemented throughout the instructional process, should be articulated, be representative of the reference global meaning and consider the contextual and personal circumstances of the subjects involved.

The implemented mathematical content must meet some characteristics to achieve epistemic suitability, i.e., to be rich, optimal or adequate mathematics, which depend on the students' contextual and personal circumstances (ecological and cognitive facet). The onto-semiotic model of mathematical knowledge provides elements to characterize such mathematics, as developed in the different components and subcomponents of the epistemic facet (Font et al., 2013). The meanings of a content (e.g., the concept of natural number, fraction, etc.) is understood in a pragmatic way as the system of operative, discursive, and normative practices carried out to solve a type of problem-situations. A specific instructional process takes place in a particular environment and is carried out in a usually limited interval of time, so it is inevitable to select some partial meanings of the object in question and, therefore, the configurations of objects and processes associated with the selected meanings, but globally (throughout education) the set of meanings should be representative[3].

With this general criterion, it is accepted that there is not only one "good mathematics", but several possibilities, since for each content there are various "correct" meanings that vary in their generality,

[3] The requirement that meanings, objects, and processes implemented be representative of intended institutional meaning implies that there should be no mathematical errors in the teacher's planning and presentations. For this reason, "absence of errors" in epistemic facet is not included as a level II component, as some models do, e.g., MQI (Breda et al., 2018; Hill et al., 2011). Absence of epistemic conflicts is contemplated as criteria related to subcomponents definitions, propositions, and procedures (level III).

formalization[4]. Consequently, the optimization of learning should be adapted to the context, subjects and circumstances.

In OSA anthropological vision, mathematics is an activity of people and a system of cultural objects emerging from it, and hence, problem solving is fundamental in the instructional processes. This is reflected in the general criterion and in the criteria linked to the components: meanings (contextualization through situations-problems understandable to students), relationships or connections between meanings, objects (situations-problems) and processes (problematization).

The epistemic suitability criteria (**Table 1A** and **Table 1B** of **Appendix**) are consistent with the principles assumed by theory of didactic situations in mathematics (Brousseau, 1997), and with realistic mathematics education (RME) (Van den Heuvel-Panhuizen & Wijers, 2005), based on Freudenthal's (1983, 1991) didactic phenomenology. In these theories, as well as in curricular proposals (such as NCTM, 2000), solving, communicating, and generalizing solutions to problem-situations is the means to contextualize and generate mathematical ideas. RME principles of activity and reality support the consideration of epistemic suitability criteria. For Freudenthal (1991) "there is no mathematics without mathematization", an activity that can consists of solving environment problems, or problems of reorganization of mathematical knowledge itself.

A central point to assure a high epistemic suitability is, therefore, the selection and adaptation of rich problem-situations or tasks. However, although problem situations constitute a central element, OSA assumes that high epistemic suitability also requires attention to the various representations or means of expression (consistent with Duval's, 1995, 2006 publications), definitions, procedures, propositions, as well as the arguments associated with them. Such tasks should provide students with diverse ways of approaching the problems, involve diverse representations, and require students to conjecture, interpret, and justify solutions (Hanna & de Villiers, 2012).

Attention should also be paid to connecting different parts of the mathematical content, as well as different types of objects and processes. Mathematics is an integrated field of study. "In a coherent curriculum, mathematical ideas are related and built on each other" (NCTM, 2000, p. 14). This position agrees with the "principle of Interconnectedness" of RME: Blocks of mathematical content (numeration and calculus, algebra, geometry, ...) cannot be treated as separate entities. Problem situations should include interrelated mathematical content. Moreover, solving problems from rich contexts often means applying a wide range of mathematical tools and knowledge.

Criteria for Ecological Facet

Ecological suitability describes the degree to which a training plan or action for learning mathematics is appropriate within the setting in which it is used, that is, everything outside the classroom, such as society, school, pedagogy, and didactics of mathematics, which condition the classroom activity. The instructional process develops into an educational context that sets goals and values for the citizens' and professionals' education. These goals and values are interpreted and specified within the educational project of the center or department that coordinates the action of the different teachers involved. The teacher does not work in isolation but is part of a community of study and inquiry that provides useful knowledge about good mathematical and didactic practices that should be known and considered.

These considerations lead to formulate a general criterion of ecological suitability in the following terms:

> The educational-instructional process should be in accordance with the educational project of the center and society, considering the conditioning factors of the context in which it is developed, as well as innovations based on educational research.

Critical mathematics education (Skovsmose, 2012) provides ideas to ensure that mathematics education enables citizens to be an active part of a democratic society. Beyond each person's individual mathematical learning, it is necessary to reflect on the collective consequences of this learning in the society. At school, mathematical practices can exert an enormous influence in two completely opposite directions: on the one hand, mathematics reduced to mere routine calculations can reinforce passive and complacent attitudes and, on the other hand, mathematics in its broadest sense can develop critical and alternative thinking.

An aspect that influences the richness of mathematical content is included as a component of the ecological facet, which is the connection between the various blocks of content and disciplinary subjects. This is also related to epistemic suitability, as well as to other parts of a cross-cutting nature, and whose responsibility corresponds not only to the teacher, but also to other agents. This is the case of the curriculum component, which should consider the results of research in mathematics education, consider the social and professional training of students and education in values. The family setting is also mentioned as a

[4] Examples of the reconstruction of the global meaning of some mathematical objects are described in Batanero and Díaz (2007), Burgos and Godino (2020), and Wilhelmi et al. (2007).

conditioning factor of learning since there is significant research evidence on the influence of the family setting on educational achievement. "Yet in most cases we would find it undesirable to take children away from their parents simply to improve their chances of educational success somewhere down the line" (Biesta, 2010, p. 501). This observation shows the complexity of achieving an axiological balance in educational-instructional processes.

Criteria for Mediational Facet

The mediational facet includes resources of various types that condition and support the teaching and learning of mathematics. In addition to the concrete and technological material resources, such as calculators and computers, we consider the study aids (textbooks, activity notebooks, educational videos, ...), the number of students assigned to the teacher, the timetable in which the classes take place, the classroom material conditions, as well as the total time assigned to the study and its distribution. As a general criterion of mediational suitability, we indicate:

> Adequate resources should be available for the optimal development of the teaching and learning process.

In recent decades, there has been a broad consensus in mathematics education on the use of manipulative materials and virtual resources as a support for teaching and learning, considering that they allow "concretizing and visualizing" mathematical concepts. "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 24). This professional organization also considers calculators and other technological tools, such as algebraic calculus systems, dynamic geometry software, applets, spreadsheets, and interactive presentation devices, to be vital components of high-quality mathematics education.

But there are also studies (e.g., McNeil & Jarvin, 2007; Uttal et al. 1997) that take a more critical approach to the use of manipulatives. Uttal et al. (1997) consider that the sharp distinction between concrete and symbolic forms of mathematical expression is not useful. There is no guarantee that students will make the necessary connections between manipulatives and more traditional mathematical expressions, since ultimately the manipulative is intended to represent something different, i.e., it is also a symbol.

> A concrete manipulative may be interesting to young children, but this is not sufficient to advance their knowledge of mathematics or concepts. To learn mathematics from manipulatives, children need to perceive and comprehend relations between the manipulatives and other forms of mathematical expression (Uttal et al. 1997, p. 38).

In OSA, the relationships between material representations and visualizations (ostensive objects) of mathematical concepts, propositions and procedures are complex, since these (non-ostensive objects) have a regulative nature and should not be confused with their representations (Font et al., 2013; Godino et al., 2007). For example, the rational number "one third" can be referred to, and participate in mathematical practices, by the symbolic expression 1/3. It can also be represented by a pie chart in which the unit disk is divided into three equal parts and one of the portions is set aside, which is said to be one-third of the unit whole. But any fraction equal to 1/3 also represents the rational one-third. Progress in mathematical understanding thus requires distinguishing the mathematical object from its ostensive representations (whether visual or manipulative), which materialize the mathematical object in an iconic or indexical way.

It is also necessary to recognize the different efficiency of symbolic representations versus iconic and indexical ones for the processes of calculation, generalization, and demonstration. Mathematical activity is usually carried out with the support of means of expression and calculation whose nature can be tangible or manipulative (abacus, geoboard, ...), visual-diagrammatic (Cartesian graphs, probabilistic simulators, ...), or with alphanumeric symbolic means. Any of these means of expression is dialectically related to non-ostensive mathematical objects. They regulate the development of operative and discursive mathematical practices, whose purpose is to answer intra or extra (related to the physical or social environment) mathematical problems.

From these onto-semiotic postulates we derive a criterion of suitability specific to the use of material resources in mathematical instruction:

> Mathematical objects (regulative, non-ostensive) should be distinguished from their respective concrete, visual or symbolic representations in mathematical and didactic practices.

Moreover, it is necessary to consider the dialectic between the configurations of objects and processes based on the use of manipulative-visual resources and the analytical configurations, based on symbolic means of representation. Synergic relationships are established between these two types of configurations and are often intertwined in mathematical practices. Configurations based on concrete and visual representations play a key role, not only in school mathematical work, but in the generation of conjectures, induction and explanation, while analytical configurations are essential in the processes of generalization, calculation and justification (proof). From this derives another specific criterion of mediational suitability:

> The use of configurations of objects and processes based on alphanumeric representations should be articulated with those based on concrete representations to progressively enhance the processes of generalization, calculation, and mathematical demonstration.

Bartolini and Martignone (2020) distinguish between concrete and virtual manipulatives. The former are physical artifacts that can be manipulated by the students and offer tangible experiences in school mathematical activity, while the latter are digitally manipulated and contribute visual experiences. But alphanumeric symbols, which are part of the epistemic facet category language, are also "manipulated", that is, they are processed and translated between different registers (Duval, 2006). Uttal et al. (1997) remark that articulating the use of these means of symbolic expression with material resources leads us to distinguish three subcomponents in the category of material resources: concrete manipulatives, virtual and symbolic tools (**Figure 2**). There is a wide variety of the three categories of devices depending on the mathematical content to be taught: arithmetic (abacus, rulers, fraction wall, ...), geometry (geoboard, GeoGebra, ...), statistics (simulators, graphers, ...), algebra (algebraic balance, ...). These devices form a level IV of analysis for the mediational facet material resources component.

Criteria for Interactional Facet

Although there is a debate between the knowledge-transmitting school and the knowledge-constructing school models, there is a current tendency in favor of the latter. "The constructivist learning framework is a foundation for current reform mathematics in grades K-12. Many prospective teachers across the United States are being trained that this is the way students learn best" (Andrew, 2007, p. 157). This preference for student-centered didactic models is visible in the curricular orientations of various countries, which adopt constructivist or social-constructivist theoretical frameworks, as in the NCTM:

> Students learn more and learn better when they can take control of their learning by defining their goals and monitoring their progress. When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere (NCTM, 2000, p. 20).

Likewise, educational research attributes great importance to discourse, dialogue, and conversation in the classroom:

> The nature of mathematical discourse is a central feature of classroom practice. If we take seriously that teachers need opportunities to learn from their practice, developing mathematical conversations allows teachers to continually learn from their students. Mathematical conversations that center on students' ideas can provide teachers a window into students' thinking in ways that students' individual work cannot do alone (Franke et al., 2007, p. 237).

These tendencies justify the proposal in the TDS of the following general criterion for the interactional suitability:

> Interaction patterns should serve to identify potential semiotic conflicts, put adequate means for their resolution, favor progressive autonomy in learning and develop students' communicative competences.

In **Table 3** in **Appendix** we include suitability criteria linked to the interactions between the teacher and the students and among the students themselves. Considering widely assumed principles of socio-constructivist learning (Ernest, 1998), moments in which students take responsibility for learning are positively valued. However, becoming aware of the onto-semiotic complexity of mathematical knowledge, this constructivist principle of learning is nuanced in the TDS by the following specific interactional criterion (Godino et al., 2020):

> The teacher-student interaction should be adapted considering the moments of the study process, by applying a dialogic-collaborative format in the first encounter with the content and attributing autonomy to the student in the periods of exercise and application.

Accepting autonomy in learning is an essential feature of theory of didactic situations in mathematics (Brousseau, 1997) in which the situations of action, communication and validation are conceived as didactic moments of the study processes, that is, situations in which the students are protagonists in the construction of the intended knowledge. Likewise, RME assumes a principle of interaction, according to which the teaching of mathematics is considered a social activity. Interaction among students and between students and the teacher can lead each student to reflect on the others' input, thus achieve higher levels of understanding. Rather than being recipients of ready-made mathematics, students are active participants in the teaching-learning process in which they themselves develop tools and understandings and share their experiences. Explicit negotiation, intervention, discussion, cooperation, and evaluation are essential elements in a constructive learning process in which the learner's informal approaches are used as a platform to reach formal methods. In this interactive instruction, learners are encouraged to explain, justify, agree and disagree, question alternatives and reflect (Van den Heuvel-Panhuizen & Wijers, 2005, p. 290).

One of Freudenthal's (1991) fundamental principles for mathematics education is that students should be given a "guided opportunity" to "reinvent" mathematics. This implies that, in RME, both teachers and educational programs have a pivotal role in how students acquire knowledge. They direct the learning process, but not in a fixed way by showing what students have to learn, which contradicts the activity principle and leads to false understandings. On the contrary, students need space and tools to build mathematical knowledge on their own. To achieve this desired state, teachers have to provide students with a learning environment in which the construction process can emerge.

Making decisions about the progression of study, both on the part of the teacher and the students, requires the implementation of assessment of formative learning observation and survey procedures.

Criteria Related to Learning Characteristics (Cognitive Aspect)

In OSA it is assumed that learning entails students' appropriation of planned institutional meanings, which implies that they recognize and interconnect the objects involved in the mathematical practices that determine them. The progressive coupling between the students' initial personal meanings and the planned or effectively implemented institutional meanings is accomplished through their participation in the community of practices generated in the class. This leads to introducing the general criterion of cognitive suitability in the following terms:

> Learning objectives should pose an achievable cognitive challenge for students, considering their personal and contextual circumstances. In addition, the personal meanings achieved by students should be consistent with the planned institutional meanings. Assessment of learning should serve to improve the instructional process.

Cognitive suitability is attributed to an instructional process as a gradable trait linked to the achievement of learning objectives that demand attainable effort and in accordance with rich mathematics adapted to personal and contextual circumstances. This general criterion of cognitive suitability is inspired by the concept of the zone of proximal development (Vygotsky, 1934), so that the learning objectives should involve the development of valuable mathematical knowledge and skills that imply an attainable effort with the support of the teacher and peers, considering previous knowledge and individual capabilities (principle of equity). Relational learning and understanding of institutional meanings are assumed. The evaluation of the learning should consider the students' personal characteristics and the different levels of understanding and competence they can reach. **Table 5B** in **Appendix** displays specific criteria for the cognitive facet components: students' establishment of relationships or connections, competence to implement mathematical and metacognitive processes, consideration of prior knowledge and individual differences.

Three of the six principles formulated by NCTM (2000) on mathematics education are related to cognitive suitability. The principle of equity states "excellence in mathematics education requires equity, high expectations, and strong support for all students." It requires that reasonable and appropriate accommodations be made, and that motivating content be included to promote access and success for all students. The learning principle assumes that "students should learn mathematics by understanding it, actively constructing new knowledge from their experiences and prior knowledge." Likewise, the assessment principle establishes that "Assessment should support relevant mathematics learning and provide useful information to both teachers and students."

Criteria for Affective Facet

Solving any mathematical problem is associated with an affective situation for the subject involved, who brings into play not only his/her knowledge to give an answer to the problem, but also emotions, attitudes, beliefs, and values that condition his/her answer. Affective processes are usually considered as psychological entities, describing more or less stable states or mental traits, or dispositions for action of the subjects. However, from the didactic point of view, reaching affective states that interact positively with the cognitive domain must be considered by the educational authorities and by the teacher (Gómez-Chacón, 2000) whose work is conditioned by institutional affective norms.

Judging the amount of affective suitability of the process in question is based on the degree of involvement, interest, motivation, self-esteem and disposition of the students. Beliefs about mathematics and the study of mathematics also influence learning and therefore need to be considered. The general criterion of affective suitability is introduced in the following terms:

> The instructional process should seek to engage students (interest, motivation, self-esteem) and

consider their beliefs about mathematics and their learning.

Affective suitability is attributed to an instructional process as a gradable characteristic dependent on features of the realm of emotions, beliefs and attitudes that are promoted and manifested in it. **Table 6** in **Appendix** formulates suitability criteria for the different components of this facet, which are not exclusive to mathematics instruction, that is, they have a general character. These criteria are in line with principles assumed by several investigations on the interactions between the cognitive and affective domains in mathematical learning (Beltrán-Pellicer & Godino, 2020; Gómez-Chacón, 2000; McLeod, 1992).

Interactions Among Facets

In the previous sections, suitability criteria have been described for the six facets involved in an educational-instructional process. As indicated in **Figure 2**, these facets are not independent, there are interactions among them. Thus, for example, the use of a technological resource may determine that certain types of problems and the corresponding configurations of objects and processes are addressed, leading to new forms of representation, argumentation, generalization, etc. The forms of interaction between teacher and students, interest and motivation, and ultimately learning, can also be affected.

Godino (2013, p. 127) includes some suitability criteria related to interactions among facets, formulated in terms of indicators, not as value judgments. For example, an interaction between the epistemic and ecological facet is stated as: "The curriculum proposes to study problems from varied contexts such as school, daily life and work". This indicator can be formulated as a criterion: "The curriculum should propose to study problems ...", which in turn is assuming a value attributable to a greater or lesser extent to the instructional process: it is positive that the curriculum proposes the study of problems in various contexts. The same approach can be made with the remaining indicators of interactions between facets.

DISCUSSION

To analyze the differences between the suitability and instructional quality models, we have developed **Table 1** in which we have projected the instructional elements considered by the quality model developed by Praetorius and Charalambous (2018) (**Table 1**) onto the structure of facets, components, and subcomponents of the suitability model (**Figure 2**). We also considered information from Appendix D of the Praetorius and Charalambous's article in which the quality descriptors are extended to level III of the subcomponents. We indicate below the concordances found.

Comparison of Instructional Elements of Suitability and Quality

Table 1 (Praetorius & Charalambous, 2018) includes aspects dealing with the epistemic facet, although they are mixed with other facets. Thus, for the component content selection and presentation (level I) the indicator selecting mathematically worthwhile and developmentally appropriate content (level II) is included. The absence of epistemic conflicts or errors in the content is mentioned with the indicator presenting the content in mathematically accurate and correct ways (level II). However, also in cognitive activation (level I), the indicator (b) teacher's use of mathematically rich practices (level II) is included. This point (which corresponds to several criteria in **Table 1B** in **Appendix**), is developed in Appendix D with the following level III indicators: linking and connecting representations/ concepts, explanations and justifications, multiple solutions and approaches, patterns and generalizations, problem solving and modeling, proof, which refer to the epistemic facet.

In the model of Praetorius and Charalambous (2018) or in MQI (Learning Mathematics for Teaching Project, 2011) we do not find indicators of quality that correspond to the ecological facet proposed by TDS, given that usually the focus of attention of studies on instructional quality are behaviors within the classroom. This is a distinctive feature in the approach of the notion of didactic suitability with respect to quality of instruction: the educational setting as a conditioning and supporting factor of the instructional activity.

In **Table 1** we find some indicators related to the mediational facet, which are part of classroom and time management, namely, the organization of the physical space and resources, and the appropriate allocation of time to different lesson elements based on main learning goals.

Table 1 describes the types of interaction patterns that are considered preferable in Praetorius and Charalambous (2018) model to promote learnings in various aspect components: Presenting the content (in a motivated, structured, accurate and correct manner); cognitive activation (teacher facilitation of students' cognitive and metacognitive activity); formative assessment (teacher regularly checks for understanding; quality of feedback for students); cutting-across instructional aspects (enhancing participation and the active engagement of all learners). A certain commitment towards student-centered teaching is observed without forgetting the teacher's role in the presentation of content.

Table 1 (Praetorius & Charalambous, 2018) includes different aspects related to the cognitive facet, mainly in cognitive activation (level I), which is concretized with indicators (level II) linked to the challenge and assurance of achievement by students, as well as the inclusion of metacognitive tasks. Also, in practicing (level I), support is provided to students in the consolidation of knowledge and procedural skills, as well as in the correction of their mistakes. The evaluation of learning is contemplated in formative assessment, facet of level I, while in the case of TDS it has been included as evaluation of learning, which also includes aspects of summative evaluation, in order to consider the degree of achievement of planned learning. The attention to prior knowledge and individual differences in TDS is considered in Praetorius and Charalambous' (2018) model through the differentiation and adaptation of the component cutting-across instructional aspects aiming to maximize student learning.

Table 1 (Praetorious & Charalambous, 2018) includes social-emotional support, with two components (level II), teacher-student relationships and student-student relationships. Appendix D of that article details them, as follows:

- teacher-student relationships (e.g., mutual respect, fair treatment, politeness, interest in students' personal situations, responsiveness) and
- student-student relationships (e.g., respect, positive competition, lack of negative competition, students not being negative/sarcastic toward each other).

Also included in the content selection and presentation facet is motivating the content. Other facets of affectivity such as forming an environment that nurtures productive habits (e.g., agency, ownership/autonomous learning, identity, perseverance) are included in cutting-across instructional aspects.

Further detail of components related to the affective facet, in particular beliefs about mathematics and its teaching, is included in **Table 6** in **Appendix**.

Implications of TDS for Mathematics Teacher Education

Several empirical research works have been conducted employing the didactic suitability criteria to analyze lesson design and teaching practices in mathematics teacher education courses (Breda et al., 2017; Garcés et al., 2021). These investigations reveal the usefulness of TDS to recognize that the value judgments issued by teachers on the quality of their practices correspond to TDS criteria in some facets, but do not mention systematically all its facets and components. Consequently, experiences have been designed for teachers to know and appropriate the suitability tool, as this will allow them deeper, argued, and systematic reflection (Burgos et al., 2020; Giacomone et al., 2018). GASMIP instrument can be applied to the analysis of instructional processes of other teachers in the context of specific training actions, in self-training processes by teaching teams involved in action research processes, or in the analysis of curricular materials, such as textbooks (Castillo & Burgos, 2022a, 2022b) and educational videos (Beltrán-Pellicer et al., 2018).

Limitations and Future Research

GASMIP instrument has been formulated as a system of didactic criteria or principles whose application would make it possible to attribute suitability to a mathematical instruction process. These are value judgments whose rationality has been made explicit before. However, it is considered necessary to go deeper into the concordances and complementarities with other educational theories and quality measurement models. It will be necessary to identify the educational principles underlying the indicators and rubrics of the quality measurement instruments and compare them with those proposed in the TDS. For example, to compare with the instrument developed by Praetorius and Charalambous (2018) (**Table 1**), it would be necessary to carry out a documentary study of the bibliography these authors used to substantiate the indicators they propose. Each of the 12 IQA models they used to build their synthesis instrument is based on a set of theoretical underpinnings, which refer to 11 aspects:

(a) educational effectiveness research,
(b) learning theories/conceptualizations,
(c) teaching theories/conceptualizations,
(d) research in mathematics education,
(e) research on classroom discourse,
(f) instructional triangle,
(g) conceptualizations of teacher knowledge,
(h) motivational theories/ conceptualizations,
(i) conceptualizations of the task potential,
(j) social-interactive theories/conceptualizations, and
(k) developmental theories/conceptualizations.

This diversity of theoretical aspects considered in studies on the quality of instruction indicates the field complexity. We refer the reader to Praetorius and Charalamous (2018) to deepen the methodology followed by these authors in their comparative study.

Although GASMIP is mainly a tool for analysis and reflection by the teachers themselves, which is why it has been formulated in terms of didactic criteria, developing empirical indicators and rubrics that allow assigning quantitative assessments to the suitability in circumstances, where this would be possible and useful is still needed.

Considering the great amount of research being done on the problems of teaching and learning specific mathematical content, such as statistics, geometry, algebra, etc., it would be possible and necessary to develop versions of GASMIP specific to such contents.

CONCLUSIONS

This paper has presented TDS as a tool for analyzing mathematical instructional processes and as an aid for teachers to reflect on their practice. Likewise, the notion of suitability has been compared with that of instructional quality, showing that they have related but different objectives. It assumes a broad view of instructional processes, not limited to the teaching activity and the learning activity, but which also considers the context and the other educational agents involved. A structure is proposed for the system of suitability criteria in which four levels of aspects are distinguished, revealing the complexity of the measurement of the constructs suitability and quality of instruction. This leads to an interpretative approach oriented to the elaboration of GASMIP, which can be used as a support tool in teachers' professional development.

OSA theoretical foundations and tools make it possible to identify aspects of the epistemic, cognitive, and mediational facets corresponding to level II and level III of the suitability criteria structure, as well as to formulate these criteria, which helps to make their rationality explicit and to establish connections with other models. The notion of meaning (pragmatic and referential) of mathematical objects, their relativity with respect to contexts of use and institutional frameworks, as well as the recognition of the types of objects and processes involved in the different meanings, provide essential aspects to characterize what can be understood as "good mathematics" (institutional point of view), and "good learning" (personal point of view). The application of OSA assumptions also reveals the complexity of objects and processes involved in mathematical practices, which should be reflected in the types of interaction formats in the classroom, especially when it is the students' first encounter with some new content.

The holistic perspective adopted by TDS assumes that to optimize an educational-instructional process it is not enough to implement good teaching; it is also necessary to value learning, content and the affective and social environment as good. Measuring the degree of suitability of such a large number of factors is so complex and costly that it possibly explains why instruments for measuring the quality of instruction focus on partial aspects. The suitability construct seeks to address this complexity by identifying the various facets and components, formulating suitability criteria to guide the work of the educational agents involved.

Author contributions: JDG: writing–original draft and supervision; **MB:** project administration and funding acquisition; **JDG & CB:** conceptualization and formal analysis; **CB & MB:** writing–review and editing. All authors have agreed with the results and conclusions.

Funding: This study was supported by the Grant PID2019-105601GB-I00 funded by MCIN/AEI/10.13039/501100011033.

Ethical statement: Authors stated that ethical permission was not required for the study because humans and animals were not used. However, ethical guidelines were followed throughout the study.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Andrew, L. (2007). Comparison of teacher educators' instructional methods with the constructivist ideal. *The Teacher Educator, 42*(3), 157-184. https://doi.org/10.1080/08878730709555401

Bartolini, M. G., & Martignone, F. (2020). Manipulatives in mathematics education. In S. Lerman (Ed.) *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_93

Batanero, C., & Díaz, C. (2007). Meaning and understanding of mathematics. The case of probability. In J. P. Van Bendegen, & K. François (Eds.), *Philosophical dimensions in mathematics education* (pp. 107-128). Springer. https://doi.org/10.1007/978-0-387-71575-9_6

Beltrán-Pellicer, P., & Godino, J. D. (2020). An onto-semiotic approach to the analysis of the affective domain in mathematics education. *Cambridge Journal of Education, 50*(1), 1-20. https://doi.org/10.1080/0305764X.2019.1623175

Beltrán-Pellicer, P., Giacomone, B., & Burgos, M. (2018). Los vídeos educativos en línea desde las didácticas específicas: El caso de las matemáticas [Online educational videos from specific didactics: The case of mathematics]. *Cultura y Educación, 30*(4), 633-662. https://doi.org/10.1080/11356405.2018.1524651

Berlin, R., & Cohen, J. (2018). Understanding instructional quality through a relational lens. *ZDM, 50*, 367-379. https://doi.org/10.1007/s11858-018-0940-6

Biesta, G. J. J. (2010). Why 'what works' still won't work: from evidence-based education to value-based education. *Studies in Philosophy and Education, 29*, 491-503. https://doi.org/10.1007/s11217-010-9191-x

Bostic, J., Lesseig, K., Sherman, M., & Boston, M. (2021). Classroom observation and mathematics education research. *Journal of Mathematics Teacher Education, 24*, 5-31. https://doi.org/10.1007/s10857-019-09445-0

Boston, M. D. (2012). Assessing instructional quality in mathematics. *Elementary School Journal, 113*, 76-104. https://doi.org/10.1086/666387

Breda, A., Font, V., & Pino-Fan, L. (2018). Criterios valorativos y normativos en la didáctica de las matemáticas: El caso del constructo idoneidad didáctica [Evaluative and normative criteria in the didactics of mathematics: The case of the didactic suitability construct]. *Bolema: Boletim de Educação Matemática, 32*(60), 255-278. https://doi.org/10.1590/1980-4415v32n60a13

Breda, A., Pino-Fan, L., & Font, V. (2017). Meta didactic-mathematical knowledge of teachers: Criteria for the reflection and assessment on teaching practice. *EURASIA Journal of Mathematics Science and Technology Education, 13*(6), 1893-1918. https://doi.org/10.12973/eurasia.2017.01207a

Brousseau, B. (1997). *Theory of didactical situations in mathematics*. Kluwer Academic Publishing.

Burgos, M., & Godino, J. D. (2020). Modelo ontosemiótico de referencia de la proporcionalidad. Implicaciones para la planificación curricular en primaria y secundaria [Reference onto-semiotic model of proportionality. Implications for curriculum planning in primary and secondary]. *AIEM, 18*, 1-20. https://doi.org/10.35763/aiem.v0i18.255

Burgos, M., Beltrán-Pellicer, P., & Godino, J. D. (2020). The issue of didactical suitability in mathematics educational videos: Experience of analysis with prospective primary school teachers. *Revista Española de Pedagogía, 78*(275), 27-49. https://doi.org/10.22550/REP78-1-2020-07

Castillo, M. J., & Burgos, M. (2022a). Developing reflective competence in prospective mathematics teachers by analyzing textbooks lessons. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(6), em2121. https://doi.org/10.29333/ejmste/12092

Castillo, M. J., & Burgos, M. (2022b). Reflexiones de futuros maestros sobre la idoneidad didáctica y modo de uso de una lección de libro de texto [Reflections of future teachers on the didactic suitability and mode of use of a textbook lesson]. *Bolema: Boletim de Educação Matemática, 36*, 555-579. https://doi.org/10.1590/1980-4415v36n72a25

Charalambous, C. Y., & Praetorius, A. K. (2018). Studying instructional quality in mathematics through different lenses: In search of common ground. *ZDM, 50*, 355-366. https://doi.org/10.1007/s11858-018-0914-8

Duval, R. (1995). *Sémiosis et pensée: Registres sémiotiques et apprentissages intellectuels* [*Semiosis and thought: Semiotic registers and intellectual learning*]. Peter Lang.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131. https://doi.org/10.1007/s10649-006-0400-z

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. SUNY.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*, 97-124. https://doi.org/10.1007/s10649-012-9411-0

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). NCTM & IAP.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education*. Springer Science & Business Media.

Garcés, W., Font, V., & Morales-Maure, L. (2021). Criteria that guide the professor's practice to explain mathematics at basic sciences courses in engineering degrees in Perú: A case study. *Acta Scientiae, 23*(3), 1-33. https://doi.org/10.17648/acta.scientiae.6389

Giacomone, B., Godino, J. D., & Beltrán-Pellicer, P. (2018). Desarrollo de la competencia de análisis de la idoneidad didáctica en futuros profesores de matemáticas [Development of the didactic suitability analysis competence in future mathematics teachers]. *Educação e Pesquisa, 44*, e172011. https://doi.org/10.1590/s1678-4634201844172011

Godino, J. D. (2013). Indicadores de la idoneidad didáctica de procesos de enseñanza y aprendizaje de las matemáticas [Indicators of the didactic suitability of mathematics teaching and learning processes]. *Cuadernos de Investigación y Formación en Educación Matemática, 11*, 111-132.

Godino, J. D. (2014). *Síntesis del enfoque ontosemiótico del conocimiento y la instrucción matemáticos: motivación, supuestos y herramientas teóricas* [Synthesis of the ontosemiotic approach to mathematical knowledge and instruction: motivation, assumptions and theoretical tools]. http://enfoqueontosemitico.ugr.es/documentos/sintesis_EOS_2abril2016.pdf

Godino, J. D. Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM, 39*(1-2), 127-135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos [Institutional and personal meaning of mathematical objects]. *Recherches en Didactique des Mathématiques, 14*(3), 325-355.

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: Implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 37-42.

Godino, J. D., Batanero, C., Burgos, M., & Gea, M. M. (2021). Una perspectiva ontosemiótica de los problemas y métodos de investigación en educación matemática. [An ontosemiotic perspective of research problems and methods in mathematics education]. *Revemop, 3*, e202107. https://doi.org/10.33532/revemop.e202107

Godino, J. D., Burgos, M., & Gea, M. (2021). Analyzing theories of meaning in mathematics education from the onto-semiotic approach. *International Journal of Mathematical Education in Science and Technology, 53*(10), 2609-2636. https://doi.org/10.1080/0020739X.2021.1896042

Godino, J. D., Burgos, M., & Wilhelmi, M. R. (2020). Papel de las situaciones adidácticas en el aprendizaje matemático. Una mirada crítica desde el enfoque ontosemiótico [Role of didactic situations in mathematical learning. A critical look from the ontosemiotic approach]. *Enseñanza de las Ciencias, 38*(1), 147-164. https://doi.org/10.5565/rev/ensciencias.2906

Gómez-Chacón, I. M. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics, 43*, 149-168. https://doi.org/10.1023/A:1017518812079

Hanna, G., & de Villiers, M. (Eds.) (2012). *Proof and proving in mathematics education*. Springer. https://doi.org/10.1007/978-94-007-2129-6

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511. https://doi.org/10.1080/07370000802177235

Hjelmslev, L. (1943). *Prolegomena to a theory of language.* The University of Wisconsin Press.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education, 20*, 433-455. https://doi.org/10.1007/s10857-017-9379-x

Learning Mathematics for Teaching Project. (2011). Measuring the mathematical quality of instruction. *Journal of Mathematics Teacher Education, 14*(1), 25-47. https://doi.org/10.1007/s10857-010-9140-1

Matsumura, L. C., Garnier, H., Slater, S. C., & Boston, M. (2008). Toward measuring instructional interactions 'at scale'. *Educational Assessment, 13*(4), 267-300. https://doi.org/10.1080/10627190802602541

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-598). Macmillan.

McNeil, N. M., & Jarvin, L. (2007). When theory don't add up: disentangling the manipulatives debate. *Theory into Practice, 46*(4), 309-316. https://doi.org/10.1080/00405840701593899

NCTM. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Peirce, C. S. (1958). *Collected papers of Charles Sanders Peirce*. Harvard University Press.

Praetorius, A. K., & Charalambous, C. Y. (2018). Classroom observation frameworks for studying instructional quality: looking back and looking forward. *ZDM, 50*, 535-553. https://doi.org/10.1007/s11858-018-0946-0

Schoenfeld, A. (2018). Video analyses for research and professional development: the teaching for robust understanding (TRU) framework. *ZDM,* 50, 491-506. https://doi.org/10.1007/s11858-017-0908-y

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM, 45*, 607-621. https://doi.org/10.1007/s11858-012-0483-1

Skovsmose, O. (2012). *An invitation to critical mathematics education*. Springer Science & Business Media. https://doi.org/10.1007/978-94-6091-442-3

Thompson, C. J., & Davis, S. B. (2014). Classroom observation data and instruction in primary mathematics education: Improving design and rigor. *Mathematics Education Research Journal, 26*(2), 301-323. https://doi.org/10.1007/s13394-013-0099-y

Tzur, R. (2001). Becoming a mathematics teacher-educator: Conceptualizing the terrain through self-reflective analysis. *Journal of Mathematics Teacher Education, 4*, 259-283. https://doi.org/10.1023/A:1013314009952

Uttal, D. H., Scudder, K. V., & Deloache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology, 18*, 37-54.https://doi.org/10.1016/S0193-3973(97)90013-7

Van den Heuvel-Panhuizen, M., & Wijers, M. (2005). Mathematics standards and curricula in the Netherlands. *ZDM, 37*(4), 287-306. https://doi.org/10.1007/BF02655816

Vygotsky, L. S. (1934). *El desarrollo de los procesos psicológicos superiors* [*The development of higher psychological processes*]. Crítica-Grijalbo.

Walkowiak, T. A., Berry, R. Q., Meyer, J. P., Rimm-Kaufman, S. E., & Ottmar, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*(1), 109128. https://doi.org/10.1007/s10649-013-9499-x

Wilhelmi, M. R., Godino, J. D., & Lacasta, E. (2007). Configuraciones epistémicas asociadas a la noción de igualdad de números reales [Epistemic configurations associated with the notion of equality of real numbers]. *Recherches en Didactique des Mathematiques, 27*(1), 77-120.

Wittgenstein, L. (1953). *Philosophical investigations.* The MacMillan Company.

APPENDIX: GASMIP

Table 1A. Suitability criteria for epistemic facet and its components

General criterion of epistemic facet	Component-specific criteria
The partial institutional meanings of the content and the configurations of objects and processes linked to each meaning, implemented throughout the instructional process, should be articulated, representative of the reference global meaning and take into account the contextual and personal circumstances of the subjects involved.	Institutional meanings
	– Selecting partial meanings whose study is adapted to contextual and students' personal circumstances, contextualizing them by means of understandable problem-situations. – Considering a representative sample of primary objects involved in mathematical activity (situations, languages, concepts, properties, procedures, and arguments) involved in partial meanings of content
	Processes
	– Considering diversity of processes from which objects involved in mathematical practices emerge (problematization, representation, definition, generalization, modeling, …).
	Relations (connections)
	– Relating partial meanings to each other and objects involved in corresponding practices, as well as to content of other topics that student already knows.

Table 1B. Suitability criteria for level III subcomponents of epistemic facet

Subcomponents	Specific criteria
Problem-situations	– Present a representative and articulated sample of contextualization, exercise and application situations and problem generation (problematization).
Languages	– Use a representative sample of different modes of mathematical expression (verbal, graphic, symbolic, ...), translations and conversions between them. – Adapt the level of language to the target children. – Propose situations of mathematical expression and interpretation.
Rules (concepts, propositions, and procedures)	– Propose definitions and procedures that are clear, correct and adapted to the educational level to which they are addressed. – Correctly present fundamental statements and procedures of topic for given educational level. – Propose situations, where students have to generate or negotiate definitions, propositions or procedures.
Arguments	– Propose explanations, proofs, and demonstrations that are correct and appropriate to the educational level to which they are addressed. – Promote situations where the student has to argue.

Note. To have high epistemic suitability, task design in instructional process should have these characteristics

Table 2. Suitability criteria for ecological facet and its components

General criterion of ecological facet	Component-specific criteria
The educational-instructional process should agree with the educational project of the center and society, considering the conditioning factors of the setting in which it is developed, and innovations based on educational research.	Interdisciplinary connections
	– Relating the contents with other intra and interdisciplinary contents.
	Curriculum
	– Proposing the progressive and articulated study of the various partial meanings of mathematical contents at different educational levels, by graduating the generality and formalization with which these meanings are approached.
	Openness to innovation
	– Implementing innovations that are based on research and recognized best practices. – Integrating use of new technologies (calculators, computers, ICT, etc.) in educational project.
	Socio-professional and cultural adaptation
	– Ensuring that the educational-instructional process as a whole contributes to the socio-professional growth of the students.
	Education in civic values
	– Including in the design and implementation of the educational-instructional process the education of students in democratic values and critical thinking.
	Family setting
	– Stimulating and supporting, as possible, the student's learning outside of school and his or her development as a person.

Table 3. Suitability criteria for interactional facet and its components

General criterion of interactional facet	Component-specific criteria
Interaction patterns should help identify potential semiotic conflicts, to put adequate means for their resolution, to favor progressive autonomy in learning and develop students' communicative competences.	Teacher-students' interactions
	– Adapting the interaction modes considering the moments of the study process, applying a dialogic-collaborative format in the first encounter with the content and attributing autonomy to the student in exercise and application. – Making adequate presentation of the topic (clear and well-organized presentation, not speaking too fast, emphasizing the key concepts of the topic, etc.). – Recognizing and resolving student conflicts (appropriate questions and answers are asked, etc.). – Seeking consensus based on the best argument. – Using a variety of rhetorical and argumentative devices to engage and capture the students' attention. – Facilitating the inclusion of students in the dynamics of the class. Encouraging the participation and active engagement of all students.
	Interactions among students
	– Encouraging dialogue and communication among students. – Enhancing inclusion in the group and avoid exclusion.
	Autonomy
	– Providing times when students take responsibility for the study (pose questions and present solutions; explore examples and counterexamples to investigate and conjecture; use a variety of tools to reason, make connections, solve problems, and communicate).
	Formative assessment
	– Systematically observing students' cognitive progress and use the information obtained to take decisions about the development of instruction.

Table 4. Criteria of suitability for the mediational facet and its components

General criterion of the mediational facet	Component-specific criteria
Adequate resources should be available for the optimal development of the teaching and learning process.	Material resources (concrete, virtual, and symbolic)
	– Distinguishing mathematical objects (regulative, non-ostensive) from their respective concrete, visual or symbolic representations in mathematical and didactic practices. – Articulating the use of configurations of objects and processes based on alphanumeric representations with those based on concrete representations to progressively enhance the processes of generalization, calculation and mathematical proof.
	Study aides (textbooks, exercise books, educational videos, …)
	– Making critical and reflective use of curricular materials (textbooks or activity worksheets in physical or virtual format, etc.) or educational videos, deciding when and how to use them to support the study process.
	Number of students, schedule, classroom conditions
	– Optimizing the number of students to provide personalized attention. – Adapting the classroom and the distribution of students to facilitate interactions. – To provide a schedule of class sessions that favors attention and commitment of students.
	Time (collective teaching/tutoring; learning time)
	– Assigning adequate time (face-to-face and non-face-to-face) for the intended teaching. – Assigning adequate time to the most important contents of the subject and to those that are more difficult to understand.

Table 5A. Suitability criteria for the cognitive facet and its components

General criterion of the cognitive facet	Component-specific criteria
Learning objectives should pose an achievable cognitive challenge for students, considering their personal and contextual circumstances. In addition, the personal meanings achieved by students should be consistent with the planned institutional meanings. Assessment of learning should serve to improve the instructional process.	Personal meanings
	– Promoting the understanding of problem-situations, representations, concepts and properties. – Developing communicative, procedural, and argumentative competence.
	Processes
	– Promoting the development of the student's competence to implement content-specific mathematical processes (modeling, generalization, problem posing and solving, proof, representation, ...) and metacognitive processes (reflection on one's own mathematical thought processes).
	Relations (connections)
	– Promoting relational learning, so that students are able to understand and relate the different meanings included in the teaching process and the objects involved.
	Previous knowledge
	– Considering the previous knowledge that students have in order to address the study of the intended content.
	Individual differences
	– Supporting students' learning by considering their individual differences in prior knowledge, learning styles, and levels of understanding and competence.
	Learning assessment
	– Regularly checking learning progress to enable instructional decisions for improvement (formative assessment).

Table 5B. Suitability criteria for level III subcomponents of the cognitive facet

Subcomponents	Specific criteria
Situational understanding	– Promoting and evaluating the correct resolution of problem-situations and learning tasks that pose an achievable challenge to students.
Communicative competence	– Promoting and assessing communicative competence with different modes of correct mathematical expression.
Conceptual and propositional understanding; procedural competence	– Promoting and assessing conceptual and propositional understanding. – Promoting and assessing correct procedural competence.
Argumentative competence	– Promoting and evaluating argumentative competence.

Table 6. Suitability criteria for the affective facet and its components

General criterion of the affective facet	Component-specific criteria
Instructional process should achieve the highest possible degree of students' involvement (interest, motivation, self-esteem) and consider their beliefs about mathematics and its learning.	Emotions
	– Designing situations for the identification and discussion of emotions in order to avoid rejection, phobia or fear of mathematics. – Highlighting the aesthetic and precision qualities of mathematics.
	Attitudes
	– Promoting that student assumes responsibility for learning, trying to complete tasks with perseverance, both that require personal inquiry as well as reception and retention of knowledge. – Favoring argumentation in situations of equality; argument is valued in itself and not by person who voices it.
	Beliefs
	– Identifying students' beliefs about mathematics and its teaching that may condition learning and take them into account in the instructional process.
	Values-identity
	– Promoting self-esteem so that students feel capable of contributing conjectures and solutions to the problems posed, relying on mathematical arguments to convince others of the validity of their assertions, thus building a positive mathematical identity.
	Interests and needs
	– Proposing tasks that are of interest to the students and that are within their reach. – Proposing situations that permit assessment of usefulness of mathematics in daily and professional life.