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Understanding the Algebraic Variable: Comparative Study of Mexican and Spanish Students

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Students' difficulty in learning and suitably understanding the concept of the algebraic variable has been studied with a number of tools and documented for several populations. Little research has been conducted, however, using the same tool to explore understanding of the notion among populations from different countries in an attempt to establish similarities and differences in levels of achievement and difficulties. This article discusses the findings of a survey applying a questionnaire designed for use with the 3UV model, a theoretical-methodological tool for analysing the understanding of algebraic variables and their various usages (as specific unknown, as general number, functional). Comparisons were drawn of the results of a survey of a population comprising 184 ninth-year students, 92 in Spain and 92 in Mexico, and 82 Mexican and 85 Spanish eleventh-year students, all from medium-lower income families and attending public schools. The results obtained revealed similarities and differences among the groups, providing evidence of the strengths and weaknesses of the education system of each country.

Keywords: algebra, variable, unknown, general number, function, STEM education

INTRODUCTION

In addition to providing grounds for determining how students from different cultures learn, intercultural comparative studies shed light on a number of mathematics teaching and learning issues that affect student performance in different countries. Prior comparative studies identified five general components that affect math students' performance and disposition: (1) social influence; (2) teachers' attitudes, values and beliefs; (3) students' perceptions and beliefs; (4) parents attitudes, values and beliefs; and (5) language (Bush, 2003). Other content-related factors less clearly identified but also affecting curricular development and assessment (Yore, Pimm & Tuan, 2007) would appear to have a less explicit impact on performance. One of the difficulties encountered in their analysis the

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identification of these factors is the want of suitable theoretical-methodological tools.

The present study focused on the algebraic variable, a key concept in the understanding of the natural and artificial world in which many temporary and permanent relationships between objects and circumstances unfold and where changes take place within systems of inter-related objects (PISA, 2012). With a fuller knowledge of these relationships, suitable mathematical models can be understood and used to describe and predict that world. These include algebraic and functional models that require students to create, interpret and translate symbolic and graphic representations of relationships into precise equations. The notion of the variable is consequently imperative to teaching and learning algebra and is both the basis for the transition from arithmetic to algebra and a concept required for the significant use of all advanced mathematics (Philipp, 1992).

Students' difficulties in suitably understanding the algebraic variable has been studied for decades with a number of tools and documented for several populations. In recent years, authors such as Malisani & Spagnolo (2009), Knuth et al. (2005), Pedersen (2013), Trigueros & Ursini (2003) and Ursini (2014) have addressed the question. Little research has been conducted, however, using the same tool to explore understanding of the concept among populations from different countries in an attempt to establish similarities and differences in achievements and difficulties. The present study applied a test-questionnaire adapted from one proposed by Trigueros & Ursini (2003) to reveal similarities and differences among Spanish and Mexican students in terms of their understanding of the algebraic variable. The questionnaire design was associated with the 3UV model, a theoretical-methodological tool developed by the authors to analyse the understanding of the notion. This test has been used with populations in different educational levels in Mexico (see for example Trigueros, Ursini &Lozano, 2000) and Italy (Ursini, 2014).

Spain and Mexico were chosen on the grounds of the historic ties extant between the two countries since the sixteenth century that have determined not only their use of the same language but the existence of many centuries-old cultural bonds. They are still very similar in many ways. Their school mathematics curricula cover very similar contents, for instance. Both countries are also keen on furthering STEM education, i.e., education that teaches skills in science, technology, engineering and mathematics. Successful STEM education is expected to enable science, mathematics and engineering/technology students to develop skills applicable in the real world. Skills in these STEM areas are acquiring an increasingly prominent role in basic

State of the literature

- The variable as a mathematical concept is highly complex and difficult to define, for its meaning varies depending on the context.
- The concept of algebraic variable is a key concept for the comprehension of the results of international assessments and for the furtherance of STEM education. In comparative studies factors must be identified and suitable theoretical-methodological tools are needed for their diagnosis.
- The different meanings of variable are related to different individual conceptions of algebra and abstraction levels. Sequencing those meanings in teaching and learning algebra and establishing models to integrate the various uses of the variable are open questions.

Contribution of this paper to the literature

- The 3UV Model offers an appropriate answer to algebraic problems and three uses of the variable and the aspects that characterise each. The results of this study could constitute a first step in the systematisation of a series of case studies, by combining theoretical approaches that afford a better understanding of empirical research.
- The 3UV Model-based tool was assessed empirically to validate its aptness for comparative studies.
- The results show dependence on the delivery of the curriculum and teaching approach in each context and the importance of characterising the dual processes in mathematical reasoning.

literacy in the knowledge economy[1]. Spain and Mexico also differ in essential ways, however, such as in teacher training and socio-economic level, the number of students per classroom, school infrastructure and access to technology, to mention only a few of the factors with a heavy impact on schooling. Another major difference lies in Mexican and Spanish students' performance in international assessments such as PISA, in which Spanish students score significantly higher on average than their Mexican counterparts, as discussed in a later paragraph.

According to Robitaille & Robeck (1996), the results of international comparisons have been used for different albeit complementary purposes: (1) to draw comparisons of student performance and the effects of certain factors in different countries; (2) to explain differences in achievement among different groups of students; (3) to help countries understand their own education systems by setting their strengths and weaknesses against the backdrop of those of other countries; and (4) to identify other countries' models and practices that may provide solutions to national problems. The present study addresses the third purpose identified by Robitaille & Robeck (1996), i.e., to help the countries involved (Spain and Mexico) to understand the strengths and weaknesses of their education systems on the grounds of the findings of a comparative study on the understanding of the variable in algebraic thought processes.

In spite of the limited number of students tested, the authors expect a study such as the one discussed here to deliver results that will help the participating countries to revisit their curricula, focusing on their strengths and weaknesses in the support provided students to develop the ability to think in algebraic terms.

PISA RESULTS FOR SPAIN AND MEXICO

Some of PISA 2012 results (Table 1), in particular as referred to the assessment's "change and relationships" content category, were taken as a reference to introduce and describe the contexts pertinent to the mathematical question chosen.

Table 1 reveals substantial differences (over 70 points) in the results between Spain's and Mexico's scores in mathematics and its content categories. While both countries' actual scores were lower than the OECD and EU means, their distribution across the four categories was observed to be similar to the mean for the two international organisations. Both countries exhibited better relative performance in the quantity and uncertainty and data categories than in change and relationships and space and shape, although there was room for improvement in all four.

An analysis of the performance data for Spain and Mexico revealed that the proportion of students in the two highest brackets in the space and shape (8.6 % in Spain and 1.1 % in Mexico) and change and relationships (8.5 % in Spain and 1.1 % in Mexico) categories was clearly lower than in the OECD mean (13.4 and 14.4 %, respectively). In the other two categories, the percentage of students with the

Table 1. Pisa results for Spain and Mexico in mathematics and it content categories

ODCE Country	Mathematics	SE	Quantity	SE	Change and Relationship	SE	Uncertanty and data	SE	Space and Shape	SE
Spain	484.3	1.896	490.8	2.252	481.8	2.007	486.8	2.276	476.9	2.039
Mexico	413.3	1.353	413.6	1.495	404.8	1.634	413.0	1.227	412.5	1.621
EU Mean	489.0	0.530	492.1	0.568	487.6	0.610	486.8	0.540	484.2	0.592
ODCE Mean	494.0	0.492	495.1	0.517	492.6	0.556	493.1	0.499	489.6	0.544

[1] BusinessEurope (2012) Plugging the Skills Gap – The clock is ticking (science, technology and maths) [cited 16.01.2014]

Table 2. Pisa results for Madrid and Mexico City in Mathematics in mathematical and its content categories

Region	Mathematics	SE	Quantity	SE	Change and Relationships	SE	Uncertainty and Data	SE	Space and Shape	SE
Madrid	503.8	3.497	512.4	4.187	499.9	4.415	505.2	3.553	499.8	4.804
Mexico City	428	5.0	431	6.1	428	7.1	422	4.8	421	5.3

Table 3. Confidence interval in Pisa results in mathematics and the content category "Change and relationships" for Spain and Mexico

ODCE Country	Mathematics	SE	Confidence Interval	Change and Relationships	SE	Confidence Interval
Spain	484.3	1.896	(480.58, 488.01)	481.8	2.007	(477.86, 485.73)
Mexico	413.3	1.353	(410.64, 415.95)	404.8	1.634	(401. 59, 408.00)
EU mean	489.0	0.530	(487.96, 490.03)	487.6	0.610	(486.4, 488.79)
ODCE mean	494.0	0.492	(493.03, 494.96)	492.6	0.556	(491.51, 493.68)
Madrid	503.8	3.497	(496.94, 510.65)	499.9	4.415	(492.83, 508.55)
Mexico City	428	5	(418.2, 437.8)	428	7.1	(414.08, 441.91)

highest scores differed only slightly for Spain, although not for Mexico. The mean values for the OECD countries in quantity and uncertainty and data were 14 and 12.4 %, respectively, while for Spain they were 12.4 and 9.6 % and for Mexico 1.5 and 0.2 %. The inference is that in the highest brackets (5 and 6) of the mathematics scale, Spanish students exhibited greater difficulties in change and relationships and space and shape, categories where their skills need to be enhanced. Mexican students' scores showed that they had severe deficiencies in all categories.

The above pattern of category prevalence was not observed in all of Spain's regions or all Mexican states. Table 2 gives the data for the region of Madrid and Mexico City, where the students participating in the study were enrolled. In both cases, the scores were higher and more variable than in the national data.

The data were assumed to be normally distributed. On those grounds, the quartiles for the normal distribution, the sample mean and the standard error were used to calculate approximate 95% confidence intervals for the mean, which, as shown in Table 3, were indicative of significance.

While Madrilenian students had higher percentages in all categories than the national average, the mean values for Mexico City students was very similar to the percentages for the country overall. Focus was placed on the "Change and relationships" category in light of its relationship with the use of the algebraic variable.

These data clearly call for the identification of strengths and weaknesses in the use and understanding of the algebraic variable by lower and upper secondary school students in different countries (different cultural and academic environments), stressing similarities and differences. This information may help improve the way the notion is taught and learnt in mathematics curricula and in STEM education proposals.

BACKGROUND AND THEORETICAL FRAMEWORK

The concept of mathematical, and more specifically algebraic, variable afforded a theoretical framework for analysing the responses to the questionnaire. This was the king post around which the results were interpreted an analysed. Nonetheless, this study leaves certain questions unanswered, questions from which different analytical strategies could be posed, combining several theoretical approaches for a better understanding of the variables involved in this empirical research (Bikner-Ahsbahs & Prediger, 2014).

This section is consequently divided into three parts, the first two dealing with the variable as a concept and the third with networking theories

Contextual background

The variable as a mathematical concept is highly complex and difficult to define, for its meaning varies depending on the context (Schoenfeld & Arcavi, 1988; Kieran, 2007). Bardini, Radford and Sabena (2005) defined it as an algebraic object that can be replaced by a number. Wagner (1981) contended that mathematical variables acquire meaning when appearing in a given context. Therein lies a substantial obstacle for students when confronted with problems involving this notion.

Of the many studies revolving around the algebraic variable, some focus on seventh to twelfth year students (Stacey & MacGregor, 1997; Warren, 1999; Christou & Vosniadou, 2006; Tahir, Cavanah & Mitchelmore, 2009) while others analyse progress in learning the concept in different stages of education, from late secondary to early university education (Lozano, 1998; Trigueros & Ursini, 1999; Trigueros, Ursini & Lozano, 2000).

Comparative studies on early high school and university students' comprehension of the various meanings of variable (Lozano, 1998) have shown that students in the earliest years of secondary education prior to a formal introduction to algebra had a higher mean number of correct answers than first-year university students. Studies on higher education students' difficulties with the use of letters showed that they confounded dependent and independent variables when working with functional relationships (Rosnick, 1981) and contraposed range and domain (Arnold, 2004). Some of these difficulties were put down to their use of arithmetic procedures to solve algebraic problems, avoiding the application of algebraic processes (Ursini & Trigueros, 2006).

Usiskin (1998) pointed out that the various meanings attributed to variable are related to different ways of conceiving algebra (generalised arithmetic, problem solving, study of inter-quantity relationships and study of structures). In this study, the three fundamental meanings of variable in elementary algebra were regarded to be: specific unknown, general number and functional variable. Students therefore must acquire the ability to interpret the notion correctly in each problem and switch flexibly from one meaning to another wherever necessary. Authors such as Küchemann (1980) suggested that these three uses of variable reflect rising degrees of difficulty, contending that students understand the meaning of the use of symbols in algebra when they can work with a "letter as a variable". Students find it easier to work with the meaning "letter as specific unknown" than "letter as a general number", which is in turn more readily assimilated than "letter as variable". Other authors such as Ursini (1994) noted that this alleged order of difficulty in comprehension is not mirrored in learning; rather, students encounter serious problems in use at each educational level, for variables appear in situations with different degrees of complexity. After Küchemann's initial study, as Ursini & Trigueros (2010) noted, much research addressed the various uses of the algebraic variable, with most authors focusing on a single specific use and the related difficulties (Filloy & Rojano, 1989; Gascón, 1994; Bednarz & Dufour-Janvier, 1991; Reggiani, 1994; Ruíz, 1991; English & Sharry, 1996; Ursini, 1996; Dreyfuss & Hosch, 2004). They all reported that despite the number of mathematics courses taken, many students continued to find it hard to interpret the use of the variable analysed. Different types of difficulties have been identified, including the distinction between unknowns and general numbers, symbolisation of word problems and tendency to eschew manipulation, according to Herscovics & Linchevski (1991), Drouhard (1992) and Kieran (1984)).

The consensus seems to be that when students have to recognise and symbolise general patterns or methods, their interpretation of variables depends heavily on the context of the problem, but when the symbol has a clear reference, they can interpret and symbolise it more readily (Chevallard, 1989; Gascón, 1993).

Many studies have shown that while students are not generally challenged by the relationship between variables,their joint variation poses difficulties (Kieran, 1992; English & Sharry, 1996, Ursini & Trigueros, 1997).

The algebraic variable as a multifarious entity has also been researched in connection with the obstacles encountered by students to view it as a global entity with several sides. That vision requires students to work with each usage separately while at the same time developing the flexibility to change from one use to another. Several authors (Usiskin, 1988; Warren, 1999; Bills, 2001; Ursini & Trigueros, 2001) found that students in different years of schooling had serious difficulties in interpreting the several roles that a variable can adopt in one and the same problem and in switching flexibly from one to another.

The following is a description of the 3UV model, the framework in which students' achievements and difficulties in connection with the algebraic variable were analysed in this study.

3UV model

The 3UV (three uses of variables) model (Trigueros & Ursini, 2003; Ursini et al., 2005) constituted the theoretical framework applied in this research The model arose out of an analysis of what is required to do standard algebra textbook exercises and problems. The analysis revealed that in elementary algebra courses, variables are assigned essentially three uses: specific unknown, general number and to symbolise functional relationships. A series of associated factors facing the algebra user when solving problems or doing exercises were also identified. These factors, which are related to different levels of abstraction, are synthesised below. The present authors believe that competence in solving algebraic problems calls for flexible handling of the three uses of variables and the aspects that characterise each (Ursini et al., 2005).

- The successful solution of problems and exercises involving an unknown requires:

U1 - recognising and identifying in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem;

U2 - interpreting the symbols that appear in equation, as representing specific values that can be determined by considering the given restrictions;

U3 - substituting to the variable the value or values that make the equation a true statement;

U4 - determining the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations;

U5 - symbolising the unknown quantities identified in a specific situation and use them to pose equations.

- The successful solution of problems and exercises that involve a general number requires:

G1 - recognising patterns, perceiving rules and methods in numeric sequences and in families of problems;

G2 – interpreting a symbol as representing a general, indeterminate entity that can assume any value;
G3 - deducing general rules and general methods by distinguishing the invariant aspects from the variable ones in sequences and families of problems;
G4 - manipulating (simplify, develop) general expressions;
G5 - symbolising general statements, rules or methods;

- The successful solution of problems and exercises that involve variables in a functional relationship requires:

F1 - recognising the correspondence between related variables independently of the representation used (tables, graphs, verbal problems or analytic expressions);
F2 - determining the values of the dependent variable given the value of the independent one;
F3 - determining the values of the independent variable given the value of the dependent one;
F4 - recognising the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions);
F5 - determining the range of variation of one variable given the domain of the other one;
F6 - symbolising a functional relation based on the analysis of the data of a problem.

While aspect U4 (determination of the value of the unknown) is implicit in aspects F2 and F3, they are not equivalent, for to be able to determine the values of one variable from the values of another calls first for assigning a value to one of the variables to convert an expression involving a functional relationship into an equation.

The 3UV model has proven to be a useful tool in designing student activities and planning and structuring teaching strategies (Montes, 2003); designing diagnostic tools (Ursini & Trigueros, 1997); analysing the use of variables in textbooks (Benítez, 2004); and diagnosing students' (Ursini & Trigueros, 1997) and teachers' (Juárez, 2001) conception of variables.

Theory integration

Be it said again that the purpose of this study was to reveal similarities and differences in Spanish and Mexican students' understanding of the algebraic variable and their ability to work with this concept. Further to the third purpose set out by Robitaille & Robeck (1996), the intention was to furnish stakeholders in the two countries with information that would enable them to better understand their own education systems and reveal their strengths and weaknesses. The findings would identify the elements of algebra teaching in each country that favour understanding of the algebraic variable. This type of information may suggest possible solutions to the problems detected. While the explanation for the similarities and differences of the concept of variable between Spanish and Mexican students' understanding was not one of the objectives of this study, the authors are aware of the need to enlarge the scope of the research to that end. That would involve a dual reading of the data from the perspective of the differences and a broader source base, as well as the incorporation of theoretical approaches attuned to the curriculum and its delivery by teachers, stressing the social and cultural environment in which teaching and learning take place (Remillard, 2005). The relationship between the prescribed curriculum and its delivery would also have to be studied, bearing in mind the interactive relationship between teachers' knowledge and the demands of the curriculum envisaged. That inter-relationship can only be understood in the specific context in which it occurs. Such a perspective might be combined with theories that take into consideration pre-existing cognitive conditions in algebra learning and the dual process model (Leron & Hassan 2006; Chi, 2000). The results of this study could constitute a first step in the systematisation of a series of case studies, by combining theoretical approaches that afford a better understanding of empirical research (Bikner-Ahsbahs & Prediger, 2014).

Conceptual transitions and the factors that involve working memory and intuitive reasoning, for instance, should be applied to guide students in learning algebra and developing advanced skills. Leron & Hassan (2006) described the requirements for learning mathematics in terms of dual process, the model initially proposed by Kahneman & Tversky to understand the limits of human rationality in problem solving and decision making (Stanovich & West, 2000; Kahneman & Frederick, 2005). That model postulates that two relatively independent areas of the brain and their respective systems play complementary roles in learning, problem solving and decision making. The present findings on common difficulties in both contexts could be explained on the grounds of the ways the curriculum is delivered, although for the interpretation of those findings a distinction might also be made between two types of cognitive reasoning: rapid processes involving no conscious deliberation and slower and more reflective processes. A substantial proportion of mathematical education studies have (explicitly or implicitly) addressed the relationships between intuitive and analytical thinking (Fischbein, 1987; Stavy &Tirosh, 2000; Vinner, 1997). Conceptual mathematical errors have been attributed in several studies to the divide between students' intuition and the formal thinking required in mathematics (Leron & Hazzan, 2006, 2009; Gómez-Chacón et al., 2014; Gillard, et al., 2009).

Authors such as Stanovich & West (2000) refer to these dual processes as "System 1" and "System 2". System 1 has been characterised as unconscious, associative, instantaneous and unrelated to individual working memory (WM) or fluid intelligence. This system, which human beings largely share with other animals, affords individuals speedy access to responses which are often valid but at times may lead to error. System 2 is regarded as conscious, slow, controlled and associated with individual WM and fluid intelligence. The performance of System 2 involves overriding System 1 and depends on intellectual capability as well as individuals' disposition and mental styles of individuals.

In line with this perspective of information processing, algebra teaching should not only build S1 models and make them accessible through classroom tasks, but also foster the control and awareness afforded by S2. Learning should therefore be structured to minimise the cognitive burden in situations in which both S1 and S2 processing are required.

STUDY AND METHODOLOGY

Context and objectives

In this study, conducted in Spain and Mexico, 9th- (third year secondary) and 11th- (sixth semester of sixth year secondary and first year of baccalaureate, respectively) year students in both countries responded to two slightly different versions of a questionnaire containing questions about the three uses of the algebraic variable (unknown, general number and related variable). The samples were very uniform: 92 ninth-year students each in Spain and Mexico and 85 Mexican and 82 Spanish eleventh-year students, all from medium-lower income families and attending public school.

The object of the study was the students' understanding of the different uses of variable and an analysis of any improvements from the earlier to the later stage of secondary education. The research addressed the following questions.

What do students learn about the algebraic variable in middle and late secondary education? What errors are most commonly committed? Are the errors committed and difficulties encountered similar or different in the two countries?

Methodology and questionnaires

The information was collected using two slightly different versions of the same questionnaire. The 9th-year version, questionnaire 1, had 47 questions, while questionnaire 2, answered by the 11th-year students, had 53. All were adapted from a prior 65-question test[2] (Trigueros & Ursini, 2004). The structure and content of the questionnaires were very similar. With the exception of two very elementary questions, 7a.1 and 7a.2, questionnaire 1 was included in its entirety in questionnaire 2, although in some cases with more complex sub-questions to make the study more exhaustive and better suited to 11th-year level mathematics.

The students' replies to the questionnaires were analysed in a number of stages. The first reviewed each student's capacity to work with each use and the variable factors implicit in each question. To that end, the items were grouped by variable use and each group of items was analysed separately. The students were not requested at any time to follow a given method or to use any specific language. The most advanced students were allowed to use their knowledge of other areas of mathematics, but not obliged to do so. In the second stage, the number of students able to satisfactorily deal with each item was determined and the respective percentages computed (right, wrong and blank answers). This procedure ensured a comparison of the results by group and country and served as a basis for judging students' ability to integrate and differentiate among variable uses and factors. The 3UV model was used to analyse the replies and identify achievements (over 50 % of right answers) and difficulties (under 50 % of right answers) and the type of errors committed.

Lastly, a qualitative analysis was conducted of each student's response to the questionnaire as a whole to identify patterns and illustrate and validate the preceding analysis with examples.

RESULTS

The results for all the students from both countries are given in Tables 4 and 5 below, which show the percentages of right, wrong and blank answers per item, grouped by variable meaning.

Using the data in Tables 4 and 5, in conjunction with the 3UV model, identified the skills shared by Mexican and Spanish students (questions correctly answered by over 50 % of students) could be identified. Similarly, the items with under 50 % of correct answers were defined as areas in which they shared difficulties. An analysis of the errors committed, in turn, showed whether they were of the same or a different nature.

The skills attained and difficulties encountered by Spanish and Mexican students in the three uses of the variable, further to a 3UV model-based analysis of the

[2] The questionnaires used had been previously adapted by Sonia Ursini and Luciana Zuccheri for a joint Italian-Mexican project implemented by the Universitá di Trieste's CIRD and the Cinvestav's DME, which included a comparative study on the understanding of the algebraic variable by Italian and Mexican students.

information in Tables 4 and 5 and of the errors committed, follows in sections 5.1 (9th-year) and 5.2 (11th-year).

Ninth-year Spanish and Mexican students' achievements and difficulties

Use of the variable as an unknown (8 items)

The abilities in the use of the variable as an unknown developed by 9th-year students in both countries included:

- symbolising a word problem in which the unknown appeared only once (item 1.a, "Re-write the following in mathematical language (don't do the actual calculation): 'an unknown number multiplied by 13 equals 127'")
- solving an equation in which the unknown appeared only once (item 3c, "4+x=2").

The difficulties identified in both countries included manipulating quadratic expressions (item 3.b), interpreting letters as unknowns in expressions including quadratic terms where the problem involved finding only the number of values represented by the letter, not the actual value (item 4.h), and deriving an equation from a word problem (item 5.b). Over 50 % of students in both countries, for instance, were unable to solve the equation "(x + 3) 2 = 36" (item 3.b). Most gave "x=3" as the sole solution, while some committed various types of errors when attempting to solve the binomial. In answer to the question "How many values can be adopted by the variable in the equation $4 + x^2 = x (x + 1)$?" they replied "2,

Table 4. Percentage of right (2), wrong (1) and blank (0) answer, 9Th-year students

9TH year students

Variable as unknown

	1.a		1.b		3.a		3.b		3.c		4.b		4.h		15.b	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	91	100	45	58	14	64	2	48	67	84	24	63	14	28	26	38
1	8	0	54	37	51	26	80	43	20	10	50	25	53	49	70	55
0	1	0	1	5	35	10	18	9	13	6	26	12	33	23	4	7

Variable as general number

	1.d		2.a		4.a		4.c		4.d		4.e		5		6	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	14	28	45	84	25	26	30	35	26	24	21	22	27	37	70	53
1	83	70	46	9	51	68	48	49	45	52	50	63	65	50	20	14
0	3	2	9	7	24	6	22	16	29	24	29	15	8	13	10	33

	7.a		7.a.1		7.a.2		7.b		7.c		7.d		8		18	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	78	73	76	68	73	67	16	28	90	91	89	90	3	7	58	61
1	17	20	21	21	24	22	55	30	8	5	9	7	86	61	25	15
0	5	7	3	11	3	11	29	42	2	4	2	3	11	32	17	24

Variable in functional relationship

	1.c		9.a		9.b		9.c		10		11.a		11.b		12.a		12.b	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	60	65	85	82	42	59	63	76	75	66	70	39	43	14	50	57	51	61
1	39	33	13	13	54	36	34	16	18	26	24	28	33	32	45	32	41	25
0	1	2	2	5	3	5	3	8	7	8	7	33	24	54	5	11	8	14

	13		14.a		14.b		16.a		16.b		16.d		17.a		17.b		17.c	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	4	9	22	5	76	52	1	1	50	59	34	15	1	22	0	3	4	5
1	93	90	67	60	15	25	96	92	33	30	33	26	49	39	47	24	32	15
0	2	1	11	35	9	23	3	7	17	11	34	59	50	39	53	73	64	80

	19.a		19.b		19.c		19.d	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	13	35	2	29	40	58	42	47
1	74	49	82	50	42	13	38	24
0	13	16	16	21	18	29	20	29

Table 5. Percentage of right (2), wrong (1) and blank (0) answer, 11th-year students

11Th -year students

Variable as unknown

	1.a		1.b		3.a		3.b		3.c		3.d		4.b		4.f		4.h	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	98	98	83	85	94	85	18	71	93	98	12	83	71	90	43	68	52	53
1	2	2	15	11	2	13	81	28	7	1	79	9	23	6	35	21	33	39
0	0	0	2	4	4	2	1	1	0	1	9	8	6	4	22	11	15	8

	15.a		15.b	
	Mx	Sp	Mx	Sp
2	7	9	52	56
1	74	50	43	35
0	18	41	5	9

Variable as general number

	1.d		2.a		2.b		4.a		4.c		4.d		4.e		4.g		5	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	44	26	84	94	76	92	62	49	72	62	62	60	57	55	55	21	72	64
1	52	67	13	6	23	8	33	49	21	34	34	32	32	40	34	71	23	32
0	4	7	2	0	1	0	5	2	7	4	4	8	11	5	11	8	5	4

	6		7.a		7.b		7.c		7.d		7.e		8		18	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	74	79	96	84	67	55	100	95	96	91	21	9	56	31	74	74
1	20	6	4	13	29	21	0	4	4	8	63	45	39	49	13	10
0	6	15	0	3	4	24	0	1	0	1	16	46	5	20	12	16

Variable in a functional relationship

	1.c		9.a		9.b		9.c		10		11.a		11.b		12.a		12.b	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	93	87	89	87	61	75	80	91	88	94	93	71	80	40	44	21	43	46
1	7	8	9	12	37	24	15	4	10	5	6	15	15	29	54	78	54	48
0	0	5	2	1	2	1	5	5	2	1	1	14	5	31	2	1	4	6

	13		14.a		14.b		16.a		16.f		17.a		17.b		17.c	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	4	9	24	33	89	75	0	4	12	2	41	32	17	12	37	19
1	94	90	65	53	7	18	100	90	73	62	48	37	57	17	28	7
0	2	12	11	14	4	7	0	6	15	36	11	31	26	71	35	74

	19.a		19.b		19.c		19.d	
	Mx	Sp	Mx	Sp	Mx	Sp	Mx	Sp
2	62	68	40	56	79	68	54	57
1	32	18	51	28	10	16	35	27
0	6	14	9	16	11	16	11	16

because it is a second degree equation". Both cases suggest that the students replied automatically, with no clear idea of the meaning of equations. The replies to item 15.b revealed students' difficulty in deriving an equation from a word problem. The most frequent answer in both countries was "x+15=41", an indication, on the one hand, of the ability to symbolise the unknown and an attempt to write out the equation, and on the other, the difficulty in relating the unknown to the problem data to derive an equation from which the problem could be solved (this item explicitly asked to derive the equation, not to solve the problem posed).

Two of the most prominent differences between Spanish and Mexican students' skills and difficulties were that over 50 % of Spanish students could:

- symbolise a word problem in which the unknown appeared twice and parentheses were needed (item 1.b)
- operate with linear expressions with several unknowns on both sides of the equation (item 3.a)
- interpret the variable when it appeared as an unknown several times on both sides of a linear expression (item 4.b).

In contrast, over 50 % of Mexican students encountered difficulties in these items, denoting inabilities to manipulate, symbolise or interpret expressions containing an unknown. Manipulation difficulties, for instance, were revealed in most students' answers to items 1.b and 3.a. Some of the replies to item 1.b ("Rewrite in mathematical language: 'an unknown multiplied by the sum of that number plus 12 equals 6'") were: "x(x+x)+12=6", "x+x+12=6" or "x•x+12=6", revealing that over 50 % of Mexican students experienced difficulties in symbolising problems involving manipulation, despite their ability to symbolise the unknown. The most frequent answer to item 3.a, "Calculate the values that can be adopted by the letter in the following equation: '13x + 27 - 2x = 30 + 5x'" was "x=2". When asked to solve the equation, they clearly interpreted the variable correctly and sought its value, although the right answer to the item was elusive due to their manipulative limitations. When asked "How many values can the letter a adopt in '3+a+a=a+10'?" (item 4.b), the most frequent reply was "3", denoting the scant manipulation skills acquired and students' attempt to avoid using them, in addition to their mistaken belief that a given letter in an equation can represent more than one value. All these errors and difficulties have been reported in the literature for several decades (from Küchemann's (1980) and Booth's (1982) pioneering studies). That they continue to be detected to a greater or lesser extent despite the research conducted and recommendations put forward for over thirty years should give food for thought.

Use of the variable as a general number (16 items)

Both similarities and differences among Mexican and Spanish students were also found in the use of variables as general numbers.

The results revealed that at the end of their 9th-year, students could only recognise very simple rules and apply them to specific cases only (items 7.a, 7.a1 and 7.a2). They were adept at solving problems such as follows, for instance (See Table 6):

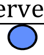
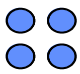
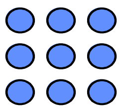
The many difficulties encountered in both countries referred essentially to the ability to:

- symbolise an open expression given in word form (item 1.d) or, deduced from the data in a problem (item 5), or symbolise a rule deriving from a generalisation (items 7.b, 8)
- interpret variables as general numbers in open or tautological expressions (4.a, 4.c, 4.d, 4.e).

The erroneous answers were also similar. For instance, the most frequent answer in both countries to item 1.d ("Re-write in mathematical language, 'an unknown number divided by 5 and the result added to 7'"), was "x/5 =y+7". That attested to a difficulty to regard x/5 as an object of mathematical operation and hence the need for another variable to represent the result of such operation.

Table 6. Problem number of points

Observe and do the following:

	**Figure 1**	Number of points: 1
	**Figure 2**	Number of points: 4
	**Figure 3**	Number of points: 9

7.a Given the pattern of succession in these figures, how many points would Figure 4 have?
7.a.1 Draw Figure 5 and write in the number of points.
7.a.2 Draw Figure 6 and write in the number of points.

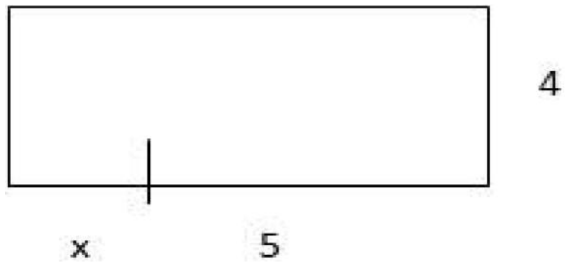


Figure 1. Perimeter

While that procedure may be valid, summing 7 to the new variable generated, i-e-, answering with the expression “y+7”, students’ failure to do so revealed once more the difficulty that accepting open expressions (x/5+7 or y+7) as solutions posed for them.

The wide variety of incorrect answers to item 5, in turn, which called for an expression to represent the perimeter of the following figure (Figure 1);provided greater insight into student misconceptions when manipulating variables. Students frequently regarded x+x to be equal to x2, for instance, 5+x to equal 5x or 5x + 5x to be equivalent to 5 x2.

The replies to item 7.b (“If we keep on drawing, how many points would Figure m have?”) revealed the difficulty in generalising or expressing a generalisation when manipulating variables . In this case, the most frequent replies in both countries were: “it can’t be done”, “m has no value”; “x”; “m”; “13x13”; “infinite.”

Item 8 read:

“Given the first two equations, complete the last one.”

1+2+3=(4x3)/2

1+2+3+4=(5x4)/2

.

.

1 + 2 + 3 + 4 + ... + n =

The most frequent answers to this item in both countries were “(5x6)/2” and “(8x9)/2”. This shows that if students deduce the rule, they can apply it to specific cases, but are unable to move beyond the specific to a level of abstraction that would enable them to reflect on the process involved and generalise their actions in algebraic language. They could be said to have implicit knowledge but to be unable to make it explicit and applicable in general cases. The difficulty involved in algebraic symbolisation has been detected by a number of researchers (Heffernan & Koedinger, 1997; Koedinger & Anderson, 1998). Moreover, the findings showed that students in both countries encountered difficulties in interpreting variables as general numbers, which entails reflecting on the meaning of the variable in a given expression rather than simply calculating its possible values. For instance, when asked how many values, and not which values, may be adopted by the letter in the expression "x+2=2+x", the most frequent reply in Mexico was “2” and in Spain “1” or “none”. The Mexican students’ most frequent reply to the question “How many values may be adopted by the letter in ‘3 + a + a + a + 10’?” (item 4.e) was “3" while Spaniards’ was “1”. These answers revealed that most students in both countries failed to understand the question and were unaccustomed to reflection, although the Spanish students at least appeared to know that a variable in any given expression must always represent the same value, a concept not mastered by most of the Mexican students.

Lastly, the difference between the two countries in achievement and difficulty was found to lie in the ability to manipulate linear expressions (item 2.1: “Simplify ‘a+5a-3a’ to an equivalent expression”). While most Spanish students were able to do so, most of their Mexican counterparts experienced difficulty due to a number of misconceptions, which confirmed the findings on the use of variables as unknowns. Most Mexican students answered this question as “2a” or “$5a^2$-3a”, denoting their lack of command of exponents and coefficients.

Use of variables in a functional relationship (23 items)

Similarities were likewise found in students' responses in the two countries to the questions on functional variables. These were the questions with the lowest percentage of right answers and the highest of blank answers.

The shared achievements included the ability to:

- translate a very simple verbally expressed functional relationship to algebraic language (item 1.c “Re-write in mathematical language (don’t perform the calculations): ‘an unknown number is equal to 6 plus another unknown number ’”)
- work with relationships, recognise variation, determine an interval from a given linear expression, but only for very simple expressions (items 9.a, 9.c, 10, 12.a, 12.b, 16b, 16c) (by way of example, item 10 read: “If ‘y=7+x’, what happens to y when the value of x rises?”)
- recognise and apply a very simple functional relationship (item 14.b “In a platform scales used in a market, the tray drops 4 centimetres per kilogramme. If the tray drops 10 centimetres, how much does the merchandise weigh?”).

The difficulties detected in both countries were as follows:

- Students were unable to symbolise a simple functional relationship in data given in tables or word problems. The most frequent replies to item 14.a (“In a platform scales used in a market, the tray drops 4 centimetres per kilogramme. Find the relationship between the weight of the merchandise and the the drop in the tray’s position.”) were “1 kg = 4 cm” in Mexico and “4x” or “x+4” in Spain.
- Nor could they determine the range of variation when manipulation was involved (items 17b., 17.c). The most frequent replies to item 17.b (“Given the expression ‘40 - 15x - 3y = 17y - 5x’. Between wich values must be x, if we want the value of y to be between 1 and 5? ”) were “2”, “3” and “4” or “1” and “3”.
- Perceiving the joint variation based on tabled data (item 16.a), graphs (items 19.a, 19.b) or analytical expressions (item 13) was another area of difficulty. For instance, the most frequent replies to item 13 (“Observe the following expressions: ‘n + 2’ or ‘2·n’ Which is bigger?Explain your answer”) were: “2n, except if n=2”.
- Items 19.a and 19.b read: “Given this graph (Figure 2) between which values of x, the values of y increase? Give approximate answers. (the most frequent answer was “0 to 9”). “Between wchich values of x, the values ofe ydecrease?” Give approximate answer.”(the most frequent answer was “10 to 20”).

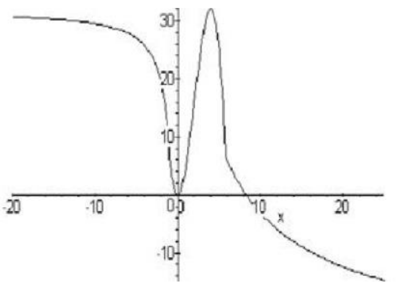


Figure 2. Graph

Table 7. Number of photocopies and prize

Number of photocopies	Prize in $
5	6.25
10	12.50
15	18.75
18	22.50
	27.50
23	

Inter-country differences were again identified in students' achievement and difficulty. Most Mexican students could recognise and apply a functional relationship (item 11.a): "To facilitate his work, an employee began to draw up the following table. Finish it for him. (Table 7)"

The wide variety of attempts made by Spanish students to reply to this item revealed their difficulty in perceiving such a relationship.

These findings showed that at the end of ninth year in both countries, most students had acquired only an elementary and at best precarious understanding of the algebraic variable, with many uncertainties and misinterpretations. Most of them clung to specific cases and numerical calculations and were unable to de-contextualise or generalise the knowledge acquired or broach the more abstract thought that characterises algebraic thinking. They were scantly used to reflecting on their actions, although able to perform them when told to do so.

Progress in understanding the algebraic variable in different stages of education in Spain and Mexico - 11th-year students

The analysis of the 11th-year students' replies to the questionnaire revealed substantial progress in both countries in their understanding of the three uses of algebraic variables. Certain errors and difficulties remained, however, and as for the 9th-year students, similarities and differences were found between the two countries.

An essential difference between Mexican and Spanish students was the greater progress from the 9th to the 11th year in the former. Eleventh-year Mexican students had surmounted some of the manipulation- (items 1.b, 2.a, 3.a) and variable interpretation- (items 4.a, 4.b, 4.c, 4.d, 4.h, and 4.e) related difficulties detected in the 9th-year.

The percentage of students leaving questions blank declined in both countries, but especially in Mexico. Like their 9th-year compatriots, larger percentages of Spanish 11th-year students tended to leave more questions blank than Mexican pupils. The questions with the highest percentage of blank answers on the questionnaires filled in by Spaniards were 17.b (71 %) and 17.c (74 %), while for the Mexican students they were 17.c (35 %) and 4.f (22 %). Given the exploratory nature of this study on the similarities and differences in understanding the algebraic variable, the students were not interviewed. For that reason, no evidence was available from which to draw explanations for these differences between Spanish and Mexican students for, while the latter tended to reply more assiduously, they also exhibited higher percentages of error in some questions. Question 17.5, for instance, was answered erroneously by 57% of the Mexican students and only 17 % of the Spaniards. The questions answered incorrectly by the highest percentage of Spanish students were 4.g (71 %), 12.a (78 %), 12 (90 %) and 16.a (90 %), and by the Mexicans 3.b (81 %), 3.d (79 %), 13 (94 %) and 16.a (100 %).

The findings for each variable use are discussed below.

Use of the variable as an unknown (11 items)

Both Mexican and Spanish students showed progress in their understanding of variables as unknowns in two areas where their 9th-year schoolmates experienced difficulties:

- interpreting unknowns in apparently quadratic equations (item 4.h)
- deriving an equation to solve a very simple problem (item 15.b).

The difficulties shared included identifying an unknown in a slightly complex problem and using it to pose an equation. This was visible in the replies to item 15.a (not included on the 9th-year questionnaire), "The area of this figure is 27. What is the area of the shaded square?" (Figure 3).

The most frequent answers in Mexico were "27-6x=A" and "$(x-3)^2 = 27$", while the Spanish students' numerous attempts failed to produce an equation.

Two differences in the two groups' achievements were detected. Most Spanish students could:

- manipulate quadratic expressions (items 3.b and 3.d, while 9th-year students could not)
- interpret the variable as an unknown in a more complex quadratic expression (item 4.f, not on the 9th-year questionnaire).

Most Mexican students encountered difficulties with both these items. The most frequent reply to item 3.b "Write the values the letter that may be adopted by the letter in '$(x + 3)^2 = 36$'" was "3" (as among the 9th-year students), for the vast majority were unaware of the double sign on square roots. Mexican students replying to item 4.f "How many values can be adopted by the letter in the following expression?"

$$\frac{x}{x^2-4} = 3$$

responded most frequently "one"; or, despite finding two values when solving the equation, they failed to note that the variable could adopt two values.

Use of variable as a general number (17 items)

Overall, students in both countries performed better than the 9th-year students in:

- interpreting variables as general numbers in open expressions (items 4.a, 4.b, 4.c, 4.d and 4.e)
- deducing and symbolising very simple rules.

Progress in the ability to interpret variables as general numbers was greater among the Mexican students.

Both groups continued to find difficulty in symbolising open verbal expressions (item 1.d) or a rule that governed a pattern (7.c). Here no material improvement over 9th-year performance was recorded. For instance, as on the 9th-year questionnaires, the most common reply to item 1.d "an unknown number divided by 5 and the result added to 7" was x/5=y+7 in both countries. While students could

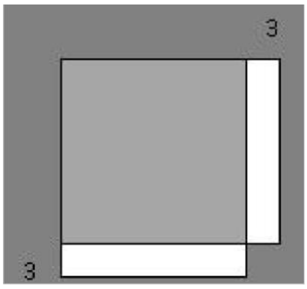


Figure 3. The shaded square

generalise a simple pattern and give it algebraic form (item 7.b), they could not deduce a more complex pattern (item 7.e "How many points would you add to convert Figure m into the next figure?"). The most frequent replies to item 7.e in both countries were: "2m-1", "m" and "$(m+1)^2$".

The findings also showed some differences in achievements and difficulties around the use of variables as general numbers. Most Mexican students, for instance, could:

- interpret variables as general numbers in tautologies (item 4.g)
- symbolise a rule involving the use of parentheses (two-step algebraic expression) (item 8).

Most Spanish students experienced difficulties in this respect. The most frequent answer given by Spanish students to item 4.g, for instance, "How many values can the letter adopt in $(x + 1)^2 = x^2 + 2x + 1$" were: "2" or "1" (item not on the 9th-year questionnaire).

Spanish students' difficulties in symbolising a two-step algebraic expression can be gleaned from their answers to item 8:

"Given the first two equations, complete the last one."

1+2+3=(4x3)/2

1+2+3+4=(5x4)/2

.

.

1 + 2 + 3 + 4 + ... + n =

The most frequent answers were "(n+n+1)/2", "5xn/2", "(n.n+1)/2" and "5.6/2", showing that, like the 9th-year students, they could perceive the rule and apply it to a numerical example. Nonetheless, some, identifying the general number, unsuccessfully attempted to use it to derive an algebraic expression for a general rule.

Summing up, 11th-year were better than 9th-year students at answering questions about the meaning of general numbers or that entailed deducing and symbolising very simple rules; such progress was greater among Mexicans than among Spaniards. The difficulty in symbolising open expressions and non-trivial patterns persisted, however, particularly among Spanish students.

Use of variables in a functional relationship (25 items)

While the percentage of correct replies to questions involving variables in functional relationships grew, most of the questions with over 50 % of right answers were the same as observed for the 9th-year students. Analogously, most of the questions that were answered correctly by fewer than 50 % of the 9th-year students continued to have less than 50 % right answers among their 11th-year counterparts. The inference is that certain properties of variables in functional relationships are not stressed as much as necessary and most students are unable to correct the errors and misconceptions acquired when first introduced to this use of the variable. This was true in both countries, in which similarities were detected in the type of wrong answers given.

Most 11th-year students continued to exhibit the same difficulties as in 9th year, particularly in determining ranges of variation (lists of whole numbers were accepted as correct answers on the 9th- but not on the 11th-year questionnaires, even when they lay within the right range). In both countries, over 50 % of students had difficulties in calculating the range of one variable given the range of another (items 21.a, 12.b, 16.e, 16.f, 17.b and 17.c). For instance, the most frequent replies to items 12.a ("Consider the following expression y = 3 + x. If we want the values of y to be bigger than 3 but smaller than 10, which values can x take?") and 12.b ("Consider the following expression y = 3 + x. If the values of x are between 8 and 15, which are the values corresponding to y? ") consisted merely in lists of whole numbers.

Nor was any improvement seen in students' perception of the joint variation of two variables in a given expression (item 13) or a table of values (item 16.a). The vast majority replied to item 13 ("Observe the following expressions: n + 2 and 2 x n Which is bigger? Explain your answer "2·n because it is a multiplication" and some added "there may be exceptions for n=1 or n=2".

The data also revealed differences in achievements and difficulties between the two countries.

- Most 11th-year Mexican students could symbolise a functional relationship from data in a table (item 11.a) but were unable to perceive the variation when analysing a graph (item 19.b), committing the same errors as in 9th-year.
- Conversely, most 11th-year Spanish students could symbolise variation when analysing a graph (item 19.b), but were unable to perceive a functional relationship from data in a table (item 11.a), committing the same errors as in 9th-year.

As in 9th-year, the worst results for the whole questionnaire were obtained when joint variation of two variables had to be determined from a table of values (16.a) or two expressions involving the same variable (item 13), a clear indication that these questions are not suitably addressed in mathematics teaching at these levels.

The notion of functional variable continued to pose many problems for 11th-year Spanish and Mexican students. While some improvement was observed over 9th-year, it was not as great as would be expected after several years of studying algebra.

DISCUSSION AND CONCLUSIONS

This comparative study shows that, most 9th-year students in both countries had acquired only an elementary and precarious understanding of variables, fraught with many uncertainties and misconceptions.

Most of them clung to specific cases and numerical calculations and were unable to de-contextualise and generalise the knowledge acquired, or broach the more abstract thought that characterises algebra. They were scantly used to reflecting on their actions, although able to perform them when told to do so.

Some progress was observed in 11th-year students, although several of the 9th-year difficulties persisted. Students in both countries found it difficult to interpret and distinguish different uses of variables and their meanings or to use them satisfactorily in the context of the problem to be solved.

The greatest difficulties in both countries lay in symbolisation and the perception of joint variation (shown in tables or graphs). Nonetheless, substantial differences were identified between the two groups. Spanish students obtained better results with unknowns and manipulation, while Mexican students performed better with general numbers and functional relationships. Spanish students were better at solving simple equations and the Mexicans at recognising and applying functional relationships.

The errors and difficulties observed in the students participating in this study are far from new in this area of research: rather, they have been reported for decades, while no substantial overall improvement has been perceived. This comparison of the achievements and difficulties in students from different countries such as Mexico and Spain, which showed that different difficulties were surmounted in each, attests to the fact that improvement depends essentially on the weight accorded to the various areas in the curriculum. That in turn indicates that better understanding can be acquired if the teaching of certain topics is suitably planned. In Mexico, for instance, greater stress should be placed on manipulation, whereas in Spain work with functional variables should be reinforced. In both countries students need support to develop the ability to reflect, evolve from specific to general, engage in abstraction, generalise and learn to express themselves in mathematical language, in addition to acquiring a grasp of joint variation. The present findings provide greater insight into these phenomena. What distinguishes the two populations? The reply to that question would call for a careful socio-cultural analysis of the teaching and learning contexts in the two countries, bearing in mind factors such as participants' socio-economic and cultural differences, their prior knowledge, teacher training, the social projection of the mathematics prevailing in each environment, the predominant approaches to teaching and the differences between the curriculum envisaged by education authorities and the one delivered by teachers in the classroom.

What common developments can be observed in both populations that appear to be independent of the cultural context and curricular training? An interpretation of the present findings would appear to require (in keeping with the theory of dual processes discussed in section 3) distinguishing between the two types of cognitive reasoning: rapid processes devoid of conscious deliberation and reflective and slower processes.

The results for the two countries concurred for items 4a, 4b and others (see Results). In these items, students' main task was interpretation, which required distancing themselves from calculation (as a direct action) and reflecting on the meaning of the algebraic variable. According to the findings, students with the lowest aptitude for cognitive reflection tended to make an initial wrong and "intuitive" choice. This difficulty was attributed to the fact that to answer correctly, participants had to first inhibit their initial "intuitive" and superficial answer to engage in deeper and more reflective thought.

These findings show that in addition to the construction of and conscious operation with symbolic and semantic representations in the use of algebraic variables, reasoning is frequently affected by unconscious processes that lead to error. Hence the importance of characterising such dual processes. The understanding of individuals' deductive capacities, based on the development of the explicit and analytical semantic processes characteristic of System 2, must be stressed, along with the enhancement of the capacity to inhibit replies originating in the superficial, heuristic and tactical processes characteristic of System 1 (see Stanovich & West, 2000; Gómez-Chacón et al., 2014). The present authors believe that this dual perspective of mathematical reasoning is directly applicable to schooling, for it directs teachers' attention to two basic educational aims: to further in-depth understanding of mathematical concepts and to inhibit superficial processes and strategies that lead to error.

The dual process framework affords more methodological tools than discussed in the present study (Gillard et al., 2009a). While here the initial aim was to analyse the algebraic variable as conceipt, the role of the dual process theory of reasoning applied to mathematics education merits specific mention. As a growing field of study that addresses the least understood areas of mathematical reasoning, it paves the way for a study of the intuitive nature of erroneous reasoning in algebraic thought.

Understanding each country's strengths and weaknesses can help institutions implement their curricula. A good understanding of algebra is essential for students to learn mathematics and apply them to the real world (MacGregor, 2004; Stacey & Chick, 2004), an objective pursued in today's STEM education. Among the problems acquired in learning algebra is acquiring a full understanding of the algebraic variable (e. g. Malisani & Spagnolo, 2009). One of the primary aims of basic education should be for a majority of students to develop the ability to reflect on the meaning of a variable in a given problem, use variables to model situations and decide independently when and how to use them, for such capacities are indispensable to rise to the challenges posed by today's professions. Such mathematical concepts are not only useful and basic tools for understanding mathematics per se, but also have a bearing on learning as a whole and help develop the ability to generalise and think abstractly.

Lastly, in the context of furthering STEM education, to which both the Spanish and Mexican systems are committed, this study provides empirical evidence of the need to combine the use of cognitive science based on studies of the dynamic infrastructure of the mind (Singer, 2009) in teaching and learning concepts that lie on the border between disciplines.

This entails identifying predominant concepts, in addition to the ones identified in connection with the algebraic variable, in science, technology, engineering and mathematics, and their inter-relationships with the main cognitive operations.

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