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University teachers' didactic-mathematical knowledge for teaching the effect of coefficient *b* on the quadratic function

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Abstract

Literature shows a tendency to relegate the role of coefficient *b* to second place in the teaching of the quadratic function. We report an experience with Chilean university teachers, who designed a teaching and learning sequence with this function for construction engineering students. Our focus was on the didactic-mathematical knowledge about the effects of varying coefficient *b* on the graphical representation of this function that the participating teachers made evident. We constituted a focus group with 10 teachers and then qualitatively analyzed their dialogues using the mathematics teacher's didactic-mathematical knowledge and competencies model. We highlight the following results: (a) the importance of mathematical knowledge and that of the epistemic facet to interpret the effect of coefficient *b* on the graphical representation of the quadratic function and (b) the proposal of an interpretation for the graphical behavior of coefficient *b* that contributes to the teaching of the quadratic function.

Keywords: coefficient *b*, didactic-mathematical knowledge, quadratic function, university teachers

INTRODUCTION

One of the key concepts in the development of mathematics is that of function (Clement, 2001). The evolution of the "function" object has its genesis approximately 4,000 years ago of which around 3,700 years have been prior to its formalization (Kleiner, 1989). Given this historical importance, literature in mathematics education has broadly addressed the teaching and learning of this mathematical object from different perspectives and in diverse educational contexts. More specifically, in recent decades there has been an increasing interest in researching the professional knowledge, particularly, the knowledge related to the topic of functions that teachers who teach mathematics should have (see further details in the next section). This type of knowledge is made up of the mathematical and didactic knowledge since both are involved in the mathematical teaching and learning processes. In addition, it is currently widely shared that merely mathematical knowledge *per se* is not sufficient for the mathematics teacher's practice (Godino, 2009).

From the types of functions included in the educational curricula for the mathematics subject, our article focuses its interest on the quadratic function on which, in principle, two didactic-mathematical studies stand out and motivate us to delve into this topic. At the didactic level, Graf et al. (2018) indicate that research on quadratic function has shown that, usually, its approach is focused on mechanical and procedural teaching, mostly algebraic. At the mathematical level, Davis (2012) proposes an activity to understand the effects of the three coefficients on the graphical representation of a quadratic function in its general algebraic form $f(x) = ax^2 + bx + c$, using the TI-Nspire CAS graphing calculator, making evident how the students use statistical tools (regression analysis), instead of geometric ones, to describe the function that models the different vertices of the family of parabolas that are generated by varying coefficient b of the quadratic function.

In this context, it is interesting for us to deepen the study of the behavior of the quadratic function from the perspective of the didactic-mathematical knowledge

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Contribution to the literature

- In this article, we present an analysis on didactic-mathematical knowledge of university teachers from the field of construction engineering.
- The results on this knowledge are consistent with other investigations that show, on one hand, the diversity of types of knowledge that teachers have (*epistemic*, *cognitive*, *mediational knowledge*, etc.) about the teaching of the quadratic function and, on the other hand, since it is a situation based on design and not on implementation, there are some types of knowledge that are not manifested (such as *interactional* and *affective knowledge*), because they have more weight in the implementation phase.
- This article also shows that an *epistemic* type of knowledge is generated about coefficient b from the general algebraic form of the quadratic function, being consistent with other investigations that reaffirm that mathematical content knowledge *per se* is not sufficient for teaching practice.

required for its teaching. Research on this function has focused mainly on the study of its semiotic representations, didactic designs, preconceptions necessary for its approach, and its characteristics, but regarding the graphical behavior of this function, the reviewed literature shows a trend to focus on coefficients a and c of its general algebraic expression, leaving aside the interpretation of coefficient b.

Given this situation, this article reports an experience with university teachers during a program to improve the mathematics teaching in the construction engineering degree in the Chilean context, where one of the activities they carried out consisted of the design of a teaching and learning sequence with the quadratic function, where the participating teachers discussed this didactic proposal and an interpretation emerged about the graphical behavior of this function when varying the coefficient b in its general algebraic expression. In this study, we aim to answer the following research questions:

- What is the didactic-mathematical knowledge that can be inferred from the discussion of university teachers on the effects of varying coefficient b on the graphical representation of the quadratic function?
- How can the effects of varying coefficient b on the graphical representation of the quadratic function can be interpreted?

To answer the first question, we inferred the didactic-mathematical knowledge of a group of university teachers during the discussion on the planning of a teaching and learning process that incorporated the study of the effects of varying coefficient b from the general algebraic expression of the quadratic function on its graphical representation. To achieve this objective, we qualitatively analyzed a fragment of the participating teachers' group discussion using the categories provided by the mathematics teacher's didactic-mathematical knowledge and competencies (DMKC) model (Pino-Fan & Godino, 2015; Pino-Fan et al., 2015), proposed by the onto-semiotic approach (Godino et al., 2007). To answer the second question, we described and systematized the participating teachers' interpretation of the effect of varying coefficient b on the graphical representation of the quadratic function from a mathematical point of view.

The relevance of our study lies in the fact that, on one hand, we delve into a topic that, although it has been widely studied, as in the case of functions and their different types, focuses, above all, on the university teachers' didactic-mathematical knowledge in a specific context such as engineering education in Chile, using a theoretical model of knowledge that has not been used for this mathematical object in this educational context. On the other hand, our study focuses on the effect of coefficient b of the quadratic function on its graphical representation, extending other studies that have analyzed only particularized effects of the variation of b (as in the case of Davis, 2012).

LITERATURE REVIEW

As mentioned above, literature in mathematics education has broadly addressed the teaching and learning of functions. In this sense, by reviewing the investigations that directly address the quadratic function, we can distinguish some lines of research that contribute to elucidating the gaps that our study aims to address.

Regarding the conceptual difficulties in the study of this function, the Díaz et al.'s (2020) work shows how secondary education students have problems to formulate quadratic functions that solve a contextualized problem in the real world, which, by constituting an initial blockage, does not allow them to progress in solving the problem. In this same line, Ozaltun Celik and Bukova Guzel (2017) point out that it is essential to review the proposals on the quadratic function that exist in the literature and, by also considering the student previous knowledge, propose hypothetical learning trajectories when teaching this mathematical object.

Regarding the teaching and learning of this function, the Burns-Childers and Vidakovic's (2018) work suggests that, in order to improve the understanding of the concept of function in calculus courses, it is important for the teacher to review the properties of the

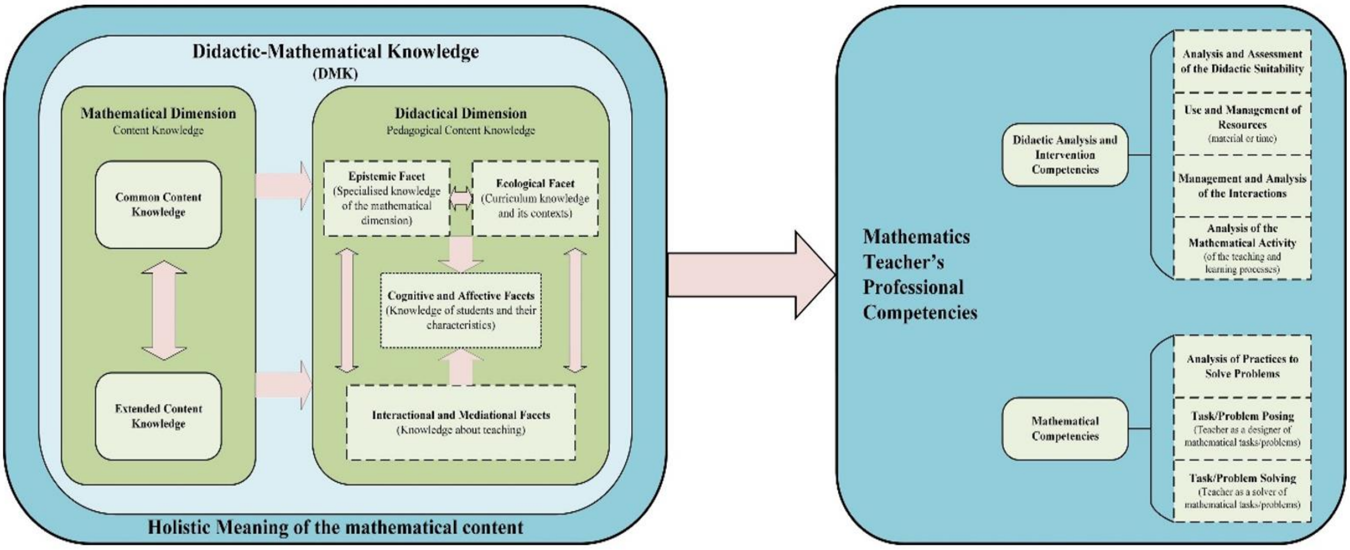

Figure 1. Mathematics teacher's DMKC Model (adapted from Pino-Fan et al., 2023, p. 1413)

quadratic function, emphasizing the idea of vertex and its critical points, using different representations. Following this line, Ramírez et al. (2022) observe the different representations of a quadratic function and conclude that it is usual–for students with higher capabilities–to promote the use of various representations and algebraic symbols; at the same time, they indicate that it is expected that students who work with quadratic functions opt for the handling of verbal and symbolic representations, this mainly due to the ease that such registers provide to explain some idea. Finally, Ledezma et al. (2022) focus on the knowledge about mathematical modelling that can be inferred from the reflection of a prospective teacher on a didactic unit with the linear function for secondary education students in the Spanish context.

In the Chilean context, which is, where our study is located, the didactic-mathematical knowledge about functions has also been deepened. For example, Pino-Fan et al. (2019) analyze the intended meaning of functions in the Chilean curriculum for the mathematics subject, reflecting of the importance of approaching this mathematical object from an informal perspective, using different representations, and building the formal definition on the basis of concepts familiar to students that contribute to their later mathematical development. In other works by these authors (Parra-Urrea & Pino-Fan, 2022; Pino-Fan & Parra-Urrea, 2021), criteria to systematize reflection on the assessment of teaching and learning processes with functions are also proposed.

Although we can derive important findings from these studies about the knowledge necessary for teaching and learning functions, the reviewed literature shows that there is not a wide range of interpretations for the behavior of coefficient b in the general algebraic expression of the quadratic function, which is presumably due to the fact that the value of this coefficient is the most difficult to understand and teach (in terms of Graf et al., 2018), so delving into this topic becomes even more relevant.

MATHEMATICS TEACHER'S DIDACTIC-MATHEMATICAL KNOWLEDGE & COMPETENCIES MODEL

In the literature of mathematics education, different researchers have generated models about the knowledge and competencies that a mathematics teacher must have to manage the student learning (see Hill et al., 2008; Kunter et al., 2013; Neubrand, 2018; Petrou & Goulding, 2011; Rowland, 2013; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008; Shulman, 1987). The purpose of these models is that they are necessary to organize educational programs for teachers, both at the initial and continuous levels, and thus evaluate their effectiveness (Godino, 2009). In the onto-semiotic approach, DMKC model (Pino-Fan & Godino, 2015; Pino-Fan et al., 2015) is proposed (see **Figure 1**), which is the didactic-theoretical reference considered for our study.

Although DMKC model considers both the knowledge and competencies that a teacher must have to design, implement, reflect on, and improve mathematical teaching and learning processes, in this article, we only consider the part of the model that refers to the teacher's knowledge, due to the needs of our study. In DMKC model, the mathematics teacher's knowledge is organized in three large dimensions: the mathematical, the didactic, and the meta-didactic-mathematical dimensions, which we describe below.

Knowledge from Mathematical Dimension

The mathematical dimension of DMKC model contemplates two types of knowledge that a mathematics teacher must have (Pino-Fan & Godino, 2015). On one hand, the *common content knowledge*, which corresponds to that a teacher must have about a

Table 1. Description of facets of knowledge from didactic dimension

Facets	Description
Epistemic	Refers to specialized knowledge from mathematical dimension. In the context of our study, it means that teacher must use different representations, arguments, task/problem-solving strategies, & partial meanings of/with quadratic function.
Cognitive	Refers to knowledge about students' cognitive aspects. In the context of our study, it means that teacher must foresee (during planning) & try (during implementation) possible responses to certain tasks/problems in which difficulties, errors, cognitive conflicts, or misconceptions emerge in students with quadratic function.
Interactional	Refers to knowledge about interactions that emerge in classroom, where teacher must foresee, implement, & evaluate sequences of interaction between agents involved in mathematical teaching & learning process (teacher-student, student-student, student-resources, & teacher-resources-students).
Mediational	Refers to the knowledge about the resources and media that can enhance student learning, and the time designated for teaching. In the context of our study, the teacher must be able to assess the relevance of the use of material and technological resources for the learning of the quadratic function, as well as the time assigned to develop the teaching and learning process with this mathematical object.
Affective	Refers to the knowledge about the students' affective, emotional, and attitudinal aspects. In general terms, this knowledge allows the teacher to describe the students' experiences and sensations in a certain lesson or mathematical task/problem at a particular educational level.
Ecological	Refers to knowledge about different aspects (curricular, contextual, social, economic, etc.) that influence management of mathematical teaching & learning processes. In context of our study, teacher must be able to know curricular mathematics of level that considers study of quadratic function, links to other curricula, usefulness of such mathematical content in students' social & work life, etc.

Note. Adapted from Pino-Fan et al. (2018)

particular mathematical object (for example, about the quadratic function), which allows him/her to solve tasks or problems proposed by the curriculum or the textbooks at a certain educational level (for example, in a mathematics course during the first year of construction engineering degree), and that is shared between the teacher and the students. On the other hand, the *extended content knowledge*, which corresponds to that a teacher must have about a particular mathematical object that, taught at a specific moment (such as the quadratic function in a mathematics course during the first year of construction engineering degree), will serve as the basis for mathematical contents of higher educational levels (for example, the quadratic function for the study of differential equations in a mathematics course during the second year of construction engineering degree). In other words, the *extended content knowledge* allows the teacher to pose mathematical challenges in the classroom, to link a mathematical object to other mathematical notions, and to lay the foundation for students to study mathematical notions after the mathematical object studied at a certain moment.

Knowledge from Didactic Dimension

The authors of different mathematics teachers' knowledge models agree that, in addition to the mathematical content knowledge, the teacher must have knowledge about the various factors that influence the teaching and learning processes of such mathematical contents (Godino, 2009; Pino-Fan et al., 2015). In this sense, the didactic dimension of DMKC model contemplates six *facets* for this type of knowledge, which we describe in **Table 1**.

Knowledge from Meta-Didactic Mathematical Dimension

The meta-didactic-mathematical dimension of DMKC model contemplates the teacher's knowledge necessary to systematize the reflection on his/her own practice, which allows him/her to be able to assess implemented mathematical teaching and learning processes, to make judgements about these processes, and to elaborate redesign proposals to improve future implementations (Pino-Fan et al., 2023).

The three dimensions of knowledge of DMKC model described above are present in the different stages of a teaching and learning process (preliminary study, planning, implementation, and assessment stage) with a certain mathematical object (Pino-Fan et al., 2018).

Studies With DMKC Model

In the literature of mathematics education, research has been reported, where DMKC model is used to study the didactic-mathematical knowledge of teachers who teach mathematics in engineering degrees. For example, Arana-Pedraza et al. (2019) focus on the analysis of the curricular documents and textbooks used in a linear algebra course for the teaching of linear systems in Mexican engineering degrees; while Garcés (2021) focuses on the analysis of a teacher's practice in the classroom when teaching the derivative in a basic sciences course for engineering in Peru. Unlike these two studies, which focus on the curricular aspects and the teaching practice in the area of engineering, respectively, the study that we report allows us to continue expanding the spectrum of implementation contexts of this model and, furthermore, to address a topic that, as far as we know, has not been studied with this theoretical construct, as the case of the didactic-mathematical knowledge for the teaching of the effect of coefficient b on the graph of the quadratic function that university teachers make evident in the context of the education of prospective construction engineers in Chile.

METHODOLOGICAL ASPECTS

In this study, we followed a qualitative research methodology from an interpretative paradigm (Cohen et al., 2018), which mainly consists of the analysis of a particular episode taken from a group discussion between university teachers about the teaching of the quadratic function to students of the construction engineering degree. In this section, we describe the methodological aspects of our work.

Research Context

This research was conducted at a Chilean university, within the framework of a program to improve the mathematics teaching in the construction engineering degree. The aim of this program was to find teaching practices that ensure the necessary preparation for prospective construction engineering professionals, based on discussions with university teachers.

For this program, a team of 10 university teachers was formed, who taught in the construction engineering degree, and the following guiding questions were posed to them:

- What knowledge should the construction engineers have after completing their professional education?
- What mathematical knowledge should the construction engineers have after completing their professional education?
- What practices will the prospective construction engineers perform in their working life, where some type of mathematical knowledge emerges?

To answer these guiding questions, a discussion began with these 10 teachers in which practices typical of construction engineering were identified, such as quantization, laying of electrical systems, determination of materials, and budget design. Subsequently, the correspondence between these practices and the curricular content established by the educational institution for the teaching of mathematics was analyzed, and common or starting points were sought for the approach of such curricular contents from the perspective of construction engineering.

The episode reported in our study emerged within one of these discussions among these 10 teachers, when they were asked to design a teaching and learning sequence with the quadratic function for first-year students of the construction engineering degree. For the preparation of this teaching and learning sequence, typical phenomena of this area of study were analyzed, such as the speed necessary to achieve an adequate mixture of materials (concrete, sand, water, etc.). However, during the study of these phenomena, the central element of our study emerged, namely, the implications of the coefficients of the quadratic function and their influence on the graphical representation of said function.

It should be noted that the teachers did not have additional reference materials, such as textbooks, provided by the university, therefore, they prepared their lesson plans solely based on the institutional curricular programs for each course, where the contents to teach and the learning objectives to achieve were described, along with the recommended methodology, materials, and bibliography, and the evaluation methodology.

Study Participants

Table 2. Disciplinary field, professional preparation, & labels for participants of our study

Disciplinary field	Professional preparation	Labels
Mathematics	Bachelor of mathematics	T1 & T2
Education	Mathematics teacher	T3, T4, & T5
Engineering	Construction engineering	T6 & T7
Communication	Journalist	T8
Education	Kindergarten teacher	T9
Employability	Social worker	T10

The participants of our study were 10 teachers who taught a course in the construction engineering degree at a Chilean university. Table 2 presents the professional profiles of these 10 participating teachers along with the labels used to identify them (from T1 to T10). The plurality of the participating teachers' professional profiles turned out to be an enriching factor in the proposed discussion since, in terms of Hydén and Bülow (2003), to discuss a specific topic, bringing together different sectors of the selected population plays a fundamental role, as in this case, the teachers who taught a course in the construction engineering degree.

The participating teachers assumed different roles in the group discussion according to their disciplinary field. For example, the teachers from the fields of mathematics and education fostered the didactic-cognitive aspects in the discussion about the quadratic function, such as its characteristics and properties; the teachers from the field of engineering related the group's proposals to the professional practice of a construction engineer, by determining possible difficulties and restrictions specific to the area in order to design the teaching and learning sequence; and the teachers from the fields of communication and employability were in charge of the elaboration of the problems and situations raised, prioritizing equitable communication and adequate use of the language.

Figure 2. Focus group session with participating teachers (Source: The authors' archive)

Data Collection & Analysis Techniques

For this research, we adopted the *focus group* technique for data collection. This technique consists of a form of group interview in which the interaction of the participants around a topic provided by a researcher is studied (Morgan, 1988). The choice of this particular technique is justified by its usefulness to generate and evaluate data from different subgroups of a population, collect qualitative data about attitudes, values, perceptions, points of view, and opinions, among other aspects (Gibbs, 2012).

In the context of our investigation, the participants corresponded to the 10 university teachers described in **Table 2**, who were part of a 10-sessions experience, lasting five hours each, held once a week in charge of the second author of this article (participating researcher). In each session, this work team proposed a didactic sequence to address a specific content of the curricular program of the mathematics I course. In the first session, the participating teachers were presented with the improvement program proposed by the university, the work methodology to be followed, and the informed consent was requested from each teacher to participate in this program. Each of the nine remaining sessions were dedicated to dealing with a specific mathematical content, as follows:

(2) concepts of function and relationship,

(3) concept of linear function,

(4) applications of linear function,

(5) concept of quadratic function,

(6) applications of quadratic function,

(7) concept of exponential function,

(8) applications of exponential function,

(9) other types of functions, and

(10) use of functions in construction.

Since it is from the interaction between the group participants that the data emerges, which is why the group dynamics is fundamental (Denscombe, 2014), we used the video recordings of the *focus group* sessions as the main data collection instrument (see **Figure 2**). These sessions were developed in the context of the design of a teaching and learning sequence with the quadratic function. For the analysis of the video recordings, we transcribed the dialogues of the group discussions and identified their interventions using the labels in **Table 2**. Thus, the unit of analysis was made up of fragments of the dialogue between the participating teachers.

Table 3. Categories & subcategories of analysis

Analysis categories	Subcategories of analysis
Mathematical dimension (MD)	Common content knowledge (CCK) Extended content knowledge (ECK)
Didactic dimension (DD)	Epistemic facet (EF) Cognitive facet (CF) Interactional facet (IF) Mediational facet (MF) Affective facet (AF) Ecological facet (EcF)

Given that one of the objectives of our study is to infer the didactic-mathematical knowledge of university teachers about the effects of varying coefficient b on the graphical representation of the quadratic function, we selected the session (5) concept of quadratic function, and we analyzed a particular fragment, where this group of teachers discussed the behavior of the graph of the function as its coefficients a, b, and c are varied. In this context, based on the dimensions of knowledge proposed by DMKC model, we considered the mathematical and didactic dimensions as categories of analysis, and the two types of mathematical content knowledge (*common* and *extended content knowledge*) and the six *facets* from the didactic dimension (*epistemic*, *cognitive*, *interactional*, *mediational*, *affective*, and *ecological facets*) as subcategories. These categories and subcategories are presented in **Table 3**, along with the respective codes used in the data analysis.

Finally, in the fragments of dialogue (units of analysis), we identified the types of knowledge (categories/subcategories of analysis) that the participating teachers made evident. To this end, we adapted the methodology used by Ledezma et al. (2022) (who infer knowledge based on the argumentation of a prospective teacher) to the context and needs of our study. During this step, once we established the categories and subcategories of analysis, we conducted a triangulation in the following way: firstly, each author analyzed the transcription of the fragment of discussion between the participating teachers and identified the type of knowledge that could be inferred in each individual or group intervention; secondly, we compared our analyzes, achieving an agreement percentage of 84% among the four of us; finally, we discussed our differences on the types of knowledge identified and achieved a consensus, due to our experience in this type of analysis.

PRESENTATION & ANALYSIS OF RESULTS

In this section, we present and analyze the results of our study. To this end, in the first subsection, we describe and present the episode analyzed; in the second subsection, we address the didactic-mathematical knowledge; and in the third subsection, we address a mathematical analysis that supports group discussion that took place among the participating teachers.

Episode Analyzed

Table 4 presents the episode analyzed, which consists of a fragment of group discussion among the participating teachers during the session (5) concept of quadratic function. To this end, we present, in the first two columns, the fragments of the dialogue between the participating teachers during their group discussion (units of analysis) and an identification number of each

Table 4. Analysis & categorization of didactic-mathematical knowledge made evident by participating teachers

No	Units of analysis	C–S*	Observations
01	T1: I think that question 4 should be asked first, before [question] 3, what do you think? That they [students] see [coefficient] *c* first?	DD–CF & EcF	T1 makes evident knowledge of CF & EcF with respect to fact that teaching of coefficient *c* in study of quadratic function is earlier than teaching of coefficient *b*, & that it could be easier to understand for students.
02	T2: Yes, let them [students] see movement of [coefficient] *c*, sliding of function [performs a vertical movement with his hand].	MD–CCK & DD–EF	T2 makes evident CCK regarding relationship between elements of algebraic & graphical representations of quadratic function. He also makes evident knowledge of EF regarding ways of representing this function.
03	T4: I am concerned about [coefficient] *b*, that is, question 3. T1: And, what about [coefficient] *b*? T3: And what does [coefficient] *b* do? T4: Well, that is it, [coefficient] *b* does not do anything [*sic*].	MD–CCK & DD–EF	T4 makes evident a weakness in CCK on quadratic function with respect to behavior of coefficient *b* in graph of this function.
04	T3: Be careful! [coefficient] *b* does do … T2: It does affect! T4: Yeah, I do know that it affects, but how do I prove it?	MD–CCK & DD–EF	T2 & T3 refer to implications that coefficient *b* has on graphical behavior of quadratic function, but their degree of knowledge about this coefficient is still not clear.
05	T2: Is not [*b*] more noticeable [coefficient], you say? [coefficient] *c* is more noticeable that [coefficient] *b*, in any case, because in *b*, if one changes it from positive to negative, it makes graph move like this [points to graph on screen], a sinusoidal movement [*sic*] from one side to other with respect to *y*-axis. That is proof.	MD–CCK & DD–EF	T2 makes evident CCK regarding relationship between coefficients of algebraic & graphical representations of quadratic function. He also makes evident knowledge of EF about treatment & conversion of semiotic registers with this function.
06	T1: That is why I think that from [coefficient] *c* is easier …	DD–CF	T1 makes evident knowledge of CF when weighing student possible difficulties or errors in interpretation of coefficients of quadratic function.
07	T3: Let us see, let us do proof [takes control of sliders & varies value of coefficient *b*]. T2: Look, did you see? That is what [coefficient] *b* does. T3: It moves it in a parabola, I mean, it seems that bottom should be a parabola [see **Figure 3**].	DD–MF	T3 considers GeoGebra software as a means of visual proof.
08	T2: In fact, it could be made as a different drawing [graph] for that.	MD–CCK, DD–EF, & MF	T2 makes evident knowledge of EF about different representations of quadratic function, using GeoGebra software as a means.

Table 4 (Continued). Analysis & categorization of didactic-mathematical knowledge made evident by participating teachers

No	Units of analysis	C–S*	Observations
09	T3: A trace? T2: No … T3: Put [a] trace on the vertex and move it? T2: When you move that point, you can see it. T3: What can be done, thinking about your idea, is to place a slider that leaves the trace for the vertex and shows that it moves like a parabola.	MD–CCK, DD–EF, CF, & MF	T2 and T3 make evident knowledge of the EF about the representations of the quadratic function; knowledge of the CF, due to the interest in foreseeing student possible difficulties and errors when working with this function; and knowledge of the MF associated with the usefulness of GeoGebra software for working with this function.
10	T4: I mean, I do see it, but it is hard for me to say it. T3: I mean, one sees it, but [students] are going to tell you "it moves there & here" [makes a parabolical movement with his hand], in fact, there it stays as an axis of symmetry, axis of symmetry of other quadratic [function] stays.	DD–CF	T3 makes evident knowledge of the CF because he foresees a student possible answer when interpreting the indicated coefficients.
11	T4: Hey! But, for example, [parabola] moves, this is a question, because I really do not know, does it move on reflection of same parabola? T2: Do you mean same [parabola]? That is, [parabola] that appears in middle. T4: I mean, if you have a parabola there & you reflect it, when you move [coefficient] *b*, does it move on that parabola?	MD–CCK & DD–EF	T2 and T4 graphically confirm the behavior of the variation of coefficient *b*, and express concern about the meaning of the movement of the parabola. They also make evident knowledge of the EF about the representations of the quadratic function.
12	T2: That is what we have to find out, that is why we set slider to see if parabola stays, I think so. T4: It would be to make proof because I do not understand. T3: Neither do I, but you can see it in drawing [graph].	MD–CCK, DD–EF, & MF	T2, T3, and T4 make evident knowledge of the EF when considering the different representations of the quadratic function; and knowledge of the MF as they consider technological resources, their value, and relevance.
13	T4: Actually, we would have to see it, because we have [slider] of [coefficient] *a* & [slider] of [coefficient] *c*. Typically, [students] ask you questions, because if you talk about [coefficients] *a* & *c*, question arises as to what happens with [coefficient] *b*. T2: Since it is natural, because you are talking about all elements that [quadratic] function has.	MD–CCK, DD–EF, & CF	T4 makes evident knowledge of the CF, since he foresees a student possible question; T2 reaffirms and indicates that the question is natural; T2 and T4 make evident the participating teachers' knowledge about the absence of an appropriate didactic interpretation for coefficient *b*.

Note. C–S: Categories–subcategories & *the codes used in this column are specified in **Table 3**

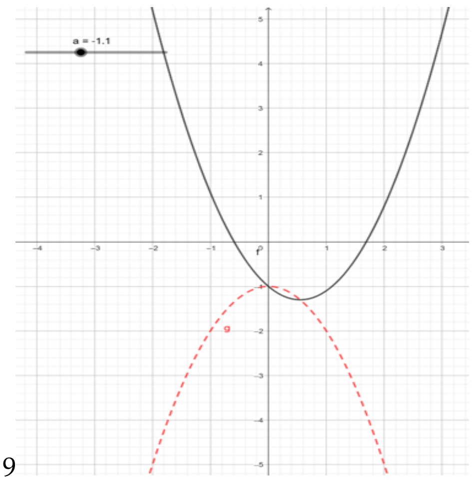

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Figure 3. Representation of movement of coefficient b in GeoGebra (GeoGebra applet available at: https://www.geogebra.org/classic/ev7h3vc2) (in fragment 07 from **Table 4**, T3 refers to red-colored parabola)

fragment (no.); in the third column, the types of knowledge that were made evident (categories–subcategories); and in the fourth column, the observations that we made to each fragment of dialogue.

This episode occurred during the process of designing a teaching and learning sequence with the quadratic function in which the participating teachers decided to give an important role to the study of the effect produced by the coefficients of this function on its graphical representation. This didactic proposal included the following five questions:

1. Graph the function $f(x) = ax^2 + bx + c$, and assign a slider to coefficients a, b, and c.
2. Move the slider assigned to coefficient a and describe the behavior of the graph of the function.
3. If we fix coefficients a and c, what effect does varying the value of coefficient b have on the graph of $f(x)$?

4. If we fix coefficients a and b, what effect does varying the value of coefficient c have on the graph of $f(x)$?
5. Move the slider assigned to coefficient c and describe the behavior of the graph of the function.

Question 3 and question 4 triggered an interesting discussion (see **Table 4**) among four of the 10 participating teachers (T1, T2, T3, and T4) on the behavior of the graph of the quadratic function when coefficient b is varied, while they were using the GeoGebra software on a computer.

Given that the 10 participating teachers had different roles in the group discussion according to their disciplinary field (as mentioned before), and that the interest of our study is focused on the mathematical and didactic aspects of teaching the quadratic function, in **Table 4** only the fragment of the group discussion in which these elements are highlighted is presented. In this fragment, the teachers from the fields of mathematics (T1 and T2) and education (T3 and T4) intervened, whose responsibility within the group was to foster the didactic-cognitive aspects of the quadratic function. On their part, the other participating teachers (T5 to T10) had other responsibilities (establishing relationships with the area of engineering; style and use of language, etc.) that are not addressed in this study, since they exceed our interest.

Participating Teachers' Knowledge from Mathematical & Didactic Dimensions

In this subsection, we address the knowledge made evident by the participating teachers, specifically, that from the mathematical and didactic dimensions proposed by DMKC model. It is important to clarify that the teachers involved in our study had no knowledge of DMKC model or other tools provided by the onto-semiotic approach (which is the theoretical framework that proposes this model). For this reason, in our study, this model is used to infer the types of knowledge that we could identify during the participating teachers' group discussion, as presented in **Table 4**.

Regarding the knowledge from the mathematical dimension of DMKC model, a first aspect to highlight from these results is the absence of evidence of the *extended content knowledge* about the quadratic function by the participating teachers. As shown in fragments of dialogue 02-05, 08, 09, and 11-13 in **Table 4**, the participating teachers made evident the *common content knowledge* about the quadratic function. This type of knowledge was manifested, mainly, when they referred to specific aspects of the quadratic function, such as its forms of representation (algebraic and graphical) and the relationship between the variation of the coefficients of the algebraic expression of the function and its graphical behavior. However, this knowledge was not projected to higher educational levels, where the quadratic function serves as a basis for learning more complex mathematical notions.

A plausible explanation for this lack of evidence of the *extended content knowledge* is that, since the focus of the teaching and learning sequence that the participating teachers were designing was specifically for a mathematics course in the first year of the construction engineering degree, the need to project this mathematical content for the following mathematics course did not emerge.

Regarding the knowledge from the didactic dimension of DMKC model, a second aspect to highlight from these results is the prioritization of the knowledge of the *epistemic facet* in the participating teachers' group discussion. This type of knowledge was manifested, mainly, when they referred to aspects of representativeness of the complexity of the quadratic function, for example, the treatment of its representation registers (when varying the coefficients of the function) and its subsequent conversion (when analyzing the graphical behavior of the function). A third aspect to highlight from these results is that the knowledge of the *cognitive* and *mediational facets* was relegated to the background. The knowledge of the *cognitive facet* was manifested when the participating teachers foresaw the potential difficulties and errors that the students could have while learning the quadratic function; while the knowledge of the *mediational facet* was manifested when the GeoGebra software was endowed with the character of visual proof to analyze the behavior of coefficient b of the quadratic function and its effects on the graphical representation of this function (see fragments of dialogue 07-09 in **Table 4**).

A plausible explanation for this prioritization of the knowledge of the *epistemic facet* is that, since the focus of the discussion was on the graphical behavior of the quadratic function when varying coefficient b in its algebraic expression, the participating teachers prioritized the mathematical aspects of teaching this concept. Hand in hand with the above and considering that this teaching and learning sequence would be designed for university students who should already have previous knowledge about the quadratic function since their secondary education, the knowledge of the *cognitive facet* may have been subordinated by this condition. In other words, the teaching and learning sequence with the quadratic function designed by the participating teachers would be an instance for levelling and consolidation of this content, rather than for the introduction of new knowledge. Therefore, it is plausible to explain this relegation to the background of the knowledge of the *cognitive facet* because of the individuals with whom the implementation would be carried out. Although the knowledge of the *mediational facet* was also relegated to the background, the utility of the GeoGebra software stood out within the participating teachers' group discussion, being considered as the central technological means within the design of the teaching and learning sequence with the quadratic function.

A fourth aspect to highlight from these results is the almost null evidence of knowledge of the *ecological facet* in the participating teachers' group discussion. This type of knowledge was succinctly manifested in the fragment of dialogue that started the group discussion, where the fact that it is more common to teach the behavior of coefficient c in the graph of the quadratic function was alluded, and where we can infer a curricular-like justification. Given that knowledge of the curriculum is part of the knowledge of the *ecological facet*, we can make two observations in this regard. On one hand, in the context of our study, the participating teachers already knew the content unit, the course, and the degree to which the teaching and learning sequence that they had to design was directed to which is closely related to the knowledge of the curriculum of the educational level in which teaching is performed. On the other hand and linking up with the discussion previously addressed on the knowledge of the *cognitive facet*, the participating teachers were also clear about the students to whom the teaching and learning sequence that they had to design was directed to and, in addition, that it was about a mathematical content on which these students should already have some previous knowledge derived from their secondary education.

Finally, there were two types of knowledge from the didactic dimension of DMKC model that were not made evident in the episode of the participating teachers' group discussion, namely, those of the *interactional* and *affective facets*. However, since the episode that we report in our study did not give rise to discussions about the implementation in the classroom of the teaching and learning sequence with quadratic function designed by the participating teachers, we do not consider it as relevant in this investigation, since our interest is focused on the didactic-mathematical knowledge about the effect of coefficient b on the graphical behavior of the quadratic function, and not on the implementation of the didactic sequence.

Mathematical Analysis

After the group discussion and since the participating teachers were able to make the geometrical construction that allowed explaining the behavior of coefficient b with the GeoGebra software, the group of teachers, along with the participating researcher, began a reflection stage regarding the graphical behavior of the quadratic function when this coefficient was varied. As a result of this reflection, a mathematical analysis of the effect of coefficient b of the quadratic function on its graphical representation was performed. We describe and systematize this analysis below:

Let the quadratic function be:

$$f(x) = ax^2 + bx + c, \quad x \in \mathbb{R}, \tag{1}$$

where we will analyze how the variation of coefficient b affects Eq. (1), which is equivalent to looking at any point of the function as a function at b.

Without loss of generality, we will use the vertex of the given parabola:

$$v = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right), \tag{2}$$

to define the vectorial function $r(b)$ from Eq. (2):

$$r(b) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \quad b \in \mathbb{R}, \tag{3}$$

If Eq. (1), then,

$$\begin{aligned} f\left(-\frac{b}{2a}\right) &= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c \\ f\left(-\frac{b}{2a}\right) &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ f\left(-\frac{b}{2a}\right) &= \frac{b^2}{4a} - \frac{b^2}{2a} + c \\ f\left(-\frac{b}{2a}\right) &= \frac{b^2 - 2b^2 + 4ac}{4a} \\ f\left(-\frac{b}{2a}\right) &= \frac{-b^2 + 4ac}{4a} \end{aligned} . \tag{4}$$

Therefore,

$$r(b) = \left(-\frac{b}{2a}, \frac{-b^2+4ac}{4a}\right), \quad b \in \mathbb{R}. \tag{5}$$

In order to analyze the graphical representation of this function, it is enough to review its cartesian equation. Let Eq. (6) be:

$$x = -\frac{b}{2a}, \quad \text{where} \quad \begin{cases} x \cdot 2a = -b \\ b = -2ax \end{cases}, \tag{6}$$

where it is possible to verify that

$$\begin{aligned} \frac{-b^2+4ac}{4a} &= \frac{-(-2ax)^2+4ac}{4a} \\ \frac{-b^2+4ac}{4a} &= \frac{-4a^2x^2+4ac}{4a} \\ \frac{-b^2+4ac}{4a} &= \frac{4a(-ax^2+c)}{4a} \\ \frac{-b^2+4ac}{4a} &= -ax^2 + c \end{aligned} . \tag{7}$$

Therefore, we verify that the variations made to coefficient b in Eq. (1) generate a family of parabolas whose vertices belong to the function:

$$g(x) = -ax^2 + c, \tag{8}$$

that understood as a displacement of the original Eq. (1), could be seen as a parabolical movement associated with the change of values for its coefficient b.

DISCUSSION & CONCLUSIONS

In our study, the results presented refer to knowledge that we could infer during an episode of the participating teachers' group discussion when designing a teaching and learning sequence with the quadratic function for first-year students of the construction engineering degree. DMKC model is a tool that partially emerges from the consensus and results reached in mathematics education about the knowledge (and competencies) a teacher must handle to teach mathematics (Breda et al., 2017, 2018), and this knowledge can be inferred in the teachers' discourse and practice. For this reason, although the teachers involved in our study had no knowledge of DMKC model or other tools provided by the onto-semiotic approach, we could infer didactic-mathematical knowledge from their group discussion.

Regarding the knowledge from the mathematical dimension, the participating teachers mainly made evident the *common content knowledge* about quadratic function during the group discussion, which was expected since they are in charge of teaching the mathematics I subject in the context studied. Through the analysis of the group discussion, we were able to identify knowledge such as that referred to a function, the recognition of a function as a relationship between sets under certain conditions, and the characteristics of a quadratic function. Finally, our focus was on the mathematical knowledge about coefficient *b* of the quadratic function and the effects of its variation on the graphical representation of this function.

Regarding the *extended content knowledge*, the fragment of group discussion analyzed does not present evidence of this type of knowledge. However, it is likely that this knowledge has been manifested in other focus group sessions that are not addressed in our study, for example, during session (6) applications of quadratic function. When trying to establish relationships between mathematical knowledge and a real-world situation, common knowledge about a mathematical object must be extended to attend intra- and extra-mathematical aspects of the real situation that is proposed to be solved with mathematical tools (see a discussion on this in Ledezma et al., 2023). In this way, a projection of our study would consider the *extended content knowledge* around the other types of functions worked on during the focus group sessions, where we expect to have evidence of other types of knowledge that we did not find in the fragment of discussion analyzed.

Regarding the knowledge from the didactic dimension, Garcés (2021) concludes that although the knowledge of the *epistemic facet* plays a fundamental role in teaching practice, it is subordinated to the knowledge of the *mediational* (time management) and *ecological facets* (compliance with the curriculum). Regarding the knowledge of the *interactional facet*, he observes that the use of different argumentative resources (to include and engage students in the mathematical teaching and learning process) is also subject to the limitations proposed by the *mediational* and *ecological* aspects, as the regulations imposed by the curriculum and the environment in which this process is implemented. Despite the fact that the results of this author differ from those reported in our study, since the subordination of the *epistemic* aspects to those of the other *facets* from the didactic dimension of DMKC model does not coincide with, it is worth highlighting the emergence of elements of the *epistemic*, *cognitive*, and *mediational facets* in the design of the teaching and learning sequence with the quadratic function as keys for the justification of the results presented in this article.

Regarding the prioritization of the knowledge of the *epistemic facet* from the didactic dimension of DMKC model that we made evident in our study, a similar situation is that reported by Ninow and Kaiber (2019) in their didactic proposal with the affine function, where they prioritized the *epistemic* and *cognitive* aspects in the designed tasks. Although the results of these authors analyze the implementation of their didactic proposal and the students' solving procedures, the situation of our study highlights that the participating teachers' knowledge about the quadratic function and its characteristics was constituted as one of the key elements for the group discussion that allowed the description and systematization of their interpretation of the effect of varying coefficient *b* on the graphical representation of this function.

Regarding the almost null evidence of knowledge of the *ecological facet* from the didactic dimension of DMKC model, this result reiterates what has been exposed in other investigations (for example, García-García, 2019) about the fact that studies addressing contexts, extra-mathematical connections, and their classifications belong to a recent field and, therefore, with many unexplored elements. In addition, this research field tends to be mostly focused on prospective and practicing secondary education mathematics teachers, but not on preschool or university education teachers. This is how, in the results of our study, there is only one element that makes it possible to make evident this *facet*, when the context and the curricular adaptation of the content were considered for the design of the teaching and learning sequence with the quadratic function in the fragment of dialogue that started the group discussion.

Another particular phenomenon that we made evident in our study was that there were two types of knowledge from the didactic dimension of DMKC model that were not made evident in the episode of the participating teachers' group discussion, namely, those of the *interactional* and *affective facets*. As justified before, since the episode that we report in our study focused on the design of the teaching and learning sequence with the quadratic function, it is consistent that there have been no dialogues about the implementation of this didactic proposal, which justified the absence of these two types of knowledge. However, there are studies (for example, Araya et al., 2021; Breda et al., 2021), where the didactic proposals of both prospective and practicing mathematics teachers are analyzed, which show little consideration of the *interactional* and *affective facets* in their reflections. This is due to the fact that they prioritize the aspects of mathematical content and didactic design before those of student interaction or involvement.

Resuming our first research question, on what is the didactic-mathematical knowledge that can be inferred from the discussion of university teachers on the effects of varying coefficient b on the graphical representation of the quadratic function, we can affirm that, regarding the mathematical dimension, the participating teachers mainly made evident the *common content knowledge* about quadratic function, but not the *extended content knowledge* since the fragment of group discussion analyzed aimed at building solid foundations on this mathematical object for the educational level, without a projection to later courses; and regarding the didactic dimension, the participating teachers mainly made evident the knowledge of the *epistemic facet*, followed by that of the *cognitive* and *mediational facets*, giving a brief sample of that of the *ecological facet*.

Resuming our second research question, on how the effects of varying coefficient b on the graphical representation of the quadratic function can be interpreted, we can affirm that, as a result, a family of functions whose graphs produce a parabolical displacement of the initial graph is generated. Making a comparison with the interpretations of coefficients a and c, the displacement produced by the variations of b is not parallel to the abscissa axis or the ordinate axis, displacing the vertex of $f(x)$ on the curve described by the function $g(x) = -ax^2 + c$. Normally, when the quadratic function is addressed in secondary education, its general algebraic expression $f(x) = ax^2 + bx + c$ is presented, where students are able to recognize the changes in concavity and the vertical displacements on the graph that the variations of coefficients a and c generate, respectively. However, coefficient b is usually avoided since there is no wide clarity about it. Therefore, the interpretation that we described and systematized in this article for the effect produced by varying coefficient b on the graphical representation of the quadratic function can be useful for a detailed and justified explanation of the implications of varying the three coefficients of this function and how they affect its graphical representation. Following Ozaltun Celik and Bukova Guzel's (2017) position, the interpretation presented in our article can serve as input to design hypothetical learning trajectories that allow students to better understand the mathematical object quadratic function. Additionally, given that this function is considered to be a necessary theoretical basis for calculus courses at higher educational levels (Burns-Childers & Vidakovic, 2018), this interpretation can contribute to that direction.

Finally, the results of our study provide a first overview of the program to improve the mathematics teaching in the construction engineering degree, where this investigation is contextualized. The first improvement points to the design of teaching and learning sequences with different mathematical objects that are taught in the mathematics courses of this university institution, which constitutes a fundamental tool for the mobilization of mathematics teachers' didactic-mathematical knowledge and the operationalization of their professional competencies. According to Mallart et al. (2018), teachers must be able not only to solve mathematical tasks or problems, but also to choose, modify, and/or pose them for educational purposes as part of their teaching practice. The second improvement points towards interdisciplinarity in the design of didactic proposals. This investigation shows that teachers who, initially, have the didactic objective of designing a sequence of tasks for the teaching of a mathematical object, ask themselves questions that require relevant mathematical knowledge and competencies, which makes evident the fact that didactic-mathematical knowledge is necessary for the teaching of mathematics since, as pointed out by the different theoretical mathematics teachers' knowledge models, mathematical content knowledge *per se* is not sufficient for teaching practice (Godino, 2009; Pino-Fan et al., 2015).

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