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Unpacking mental models, strategies, and schemas pre-service mathematics teacher in solving maximum rectangular areas

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Abstract

A mental model is an essential element that influences the quality of problem-solving. This study explores mental models, strategies, and schemes that are active in solving the maximum rectangular area problem. The approach used in this study is a descriptive qualitative approach. A total of four pre-service mathematics teachers as subjects were selected from 108 prospective subjects involved in the study. Data collection was carried out by providing task sheets for solving the most significant rectangular area problem and semi-structured interviews. Data reduction, data presentation, and conclusion are three stages in analyzing the research data. This study indicates three mental models for pre-service mathematics teachers in solving maximum rectangular area problems: initial, adaptive, and formal. Each has different strategies and schemes that are active while solving problems. The results of this study imply that mental models influence the quality of the problem solving process until results are obtained.

Keywords: mental models, problem-solving, rectangular area, schemes, strategies

INTRODUCTION

Mental models are a central issue in cognitive science and science because they influence the problem-solving process. The influence of mental models in the problem-solving process also affects the accuracy of the resulting solutions (Gentner & Gentner, 1983; Hester et al., 2012; Ifenthaler et al., 2008; Prayekti et al., 2019). Therefore, mental models play an essential role in the problem-solving process.

Mental models are internal representations constructed by individuals during the learning process (English, 2013; Halford, 1993; Jones et al., 2011; Loarces et al., 2019; Meela & Yuenyong, 2019; Riemer & Schrader, 2019; Rouse & Morris, 1986; Stamovlasis et al., 2018). Because the mental model is in mind (Convertino et al., 2016; Mayr et al., 2016; Radvansky & Zacks, 2012), it cannot make direct observations or measurements. However, mental models can be identified through various external representations. It contains behavior written on answer sheets through written text, namely pictures, symbols, mathematical equations, graphs, and verbal results using an interview protocol containing generative questions (Gogus, 2013; Justi & Gilbert, 2000). The mental model in this study is defined as an internal representation of an object built from an individual's learning experience and then constructed into an external representation when solving problems.

Several studies on mental models have been conducted at the preschool level (Kildan et al., 2013; Sackes, 2015), elementary school (Bofferding, 2014; Turk et al., 2015), and high school (Chinnapan, 1998). Kildan et al. (2013) examined mental models by analyzing content in pictures drawn by children and producing two groups of mental models: scientific and non-scientific. Furthermore, Sackes (2015) researched mental models of the day and night cycles and produced three mental models, namely naïve, synthetic, and scientific. On the other hand, Bofferding (2014) examined integer mental models and produced five mental models: initial, transition I, synthesis, transition II, and formal. Lastly, Turk et al. (2015) researched the mental models of the seasons with the result that students construct the formation of seasons in various ways in their minds.

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Contribution to the literature

- This research adds to the limitations of research on mental models in mathematics education.
- Research shows that changing the knowledge structure (accommodation of mental models) is needed by prospective mathematics teachers during the problem solving process.
- This study shows the effect of mental models on strategies and schemes that are active during mathematical problem solving to generate solutions.

Table 1. Description of mental model process accommodation

Components of process of formation or adaptation of mental models	Description
Accommodation	- Changing representation of sides of rectangle - Changing representation to equation of area of a rectangle

Researched schemas and mental models for solving geometric problems (Calderon et al., 2019; Chinnapan, 1998; Yaagoubi et al., 2012). The result is that the quality of geometric knowledge that students develop can powerfully affect their mental models and subsequent use of that knowledge. Based on several studies that have been conducted, it is necessary to conduct research at the level of prospective mathematics teacher students because they have studied many concepts, and this will provide a wide variety of mental models that can be explored (Johnson-Laird, 1983; Utami et al., 2018; Zwaan, 2016). This variation arises because individuals experience interactions between stimuli and thoughts during the learning process, resulting in changes in mental models (Loarces et al., 2019; Meela & Yuenyong, 2019; Riemer & Schrader, 2019; Vandenbosch & Higgins, 1996).

The mental model research conducted by Chinnapan (1998) did not clearly describe the characteristics of the mental models of the subjects studied, and the different strategies implemented in solving the problem were kept secret. In fact, problem-solving strategies depend on mental models (Gentner & Gentner, 1983), and mental models depend on beliefs (Sternberg & Sternberg, 2016). In addition, strategy plays a vital role in successful problem-solving (Posamentier & Krulik, 2008). Several strategies might be implemented when someone solves a problem, such as a strategy of finding patterns, adopting them, taking pictures, calculating all possibilities, and working backward (Posamentier & Krulik, 2008; Spangenberg & Pithmajor, 2020). The instrument used by Chinnapan (1998) also provides a limited opportunity for activating schemes from various domains that can be accessed during problem-solving. The schema in this study refers to the knowledge structure containing information about the core concepts and the relationships between the concepts.

Furthermore, neither Bofferding (2014), Chinnapan (1998), as well as Greefrath et al. (2020, 2022) does not disclose and describe the process of accommodation of mental models during problem-solving. Whereas Oleson et al. (2010), Piaget (1954), and van Ments & Treur (2022) state that mental models experience an adaptation process through two forms, namely assimilation and accommodation, but according to Ifenthaler et al. (2008) and van Ments and Treur (2022) mental models only experience accommodation processes. Nonetheless, neither Ifenthaler et al. (2008) nor van Ments and Treur (2022) also do not describe how the process of mental accommodation is described when solving problems. The establishment process is changing the representation owned or forming a new representation of an object to suit the structure of the problem at hand. The description of the mental model poses used as a framework in this study is presented in **Table 1**.

The accommodation process of the mental model is triggered by a situation of disequilibrating, namely an imbalance in one's mental processes (assimilation and accommodation) when the individual is faced with a problem in one's mental state (Subanji & Nusantara, 2016).

Norman et al. (1983) mentions several characteristics of mental models, including: mental models are incomplete, unstable, and change from time to time. Changes in mental models indicate changes in mental structure caused by conceptual changes (Duit & Treagust, 2003; Stark, 2003). During conceptual change, individuals often develop alternative conceptions to build a formal model, which is considered the goal, and seek to incorporate new information into their conceptual structure to provide insight into how they think about concepts (Stafylidou & Vosniadou, 2004). This conceptual shift is rooted in Piaget's notion of accommodation (Duit & Treagust, 2003). So, accommodating this mental model is likely to occur, especially because mental models are dynamic (van Ments & Treur, 2022). Based on this literature review, several prospective mathematics teacher students were given a pre-test with the command "draw various rectangular models." Most of the rectangular images produced by them are shown in **Figure 1**.

Based on the results of this test, researchers are interested in exploring why the resulting rectangular images are only shaped like that (similar and only

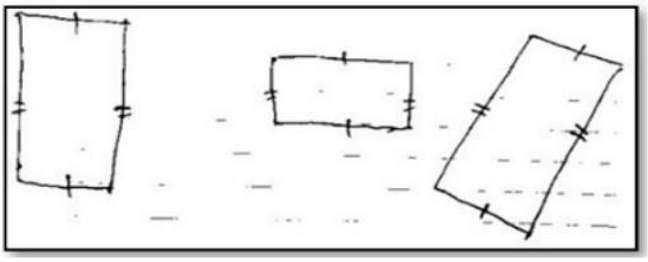

Figure 1. Rectangle mental model (Source: Authors' own elaboration)

rotated), and no one draws squares, even though squares are genetically rectangular. It shows that the resulting rectangular model is firmly embedded in the mind and originates from long-term memory, and this is how mental models work (Baguley & Payne, 2000).

Research related to mental models has been carried out on several aspects, including analogical problem-solving (Gentner & Stevens, 1983), deductive and inductive reasoning (Johnson-Laird & Byrne, 1993), probabilistic inference (Kahneman & Tversky, 1982), and mental models of the derivative (Greefrath et al., 2022), However, research on mental models in problem solving, especially finding the maximum rectangular area involving the concept of derivatives, has not been done much. Finding the maximum rectangular area is a mathematical problem because it is challenging for individuals but cannot be solved directly with a specific algorithm (Hoosain, 2004) because constraints limit the problem, namely the size of the given circumference.

Based on the literature review and analysis of the initial tests, it shows that there has been no research yet related to mental models and changing mental models when solving problems, precisely the problem of finding the maximum rectangular area, what strategies are used, and what schemes are active during the problem-solving process. This study aims to explore mental models, what strategies are used, and what schemes are active in solving the enormous broad problem. The results of this study provide information on the importance of mental models in solving problems and become the basis for designing learning based on mental model construction (Lin, 2017; Moutinho et al., 2017).

METHODOLOGY

This study employed a qualitative approach to explore and provide an accurate description of the characteristics of situations or phenomena that occur in the field (Johnson & Christensen, 2020). The subjects of this study were 108 prospective mathematics teacher students from the second and third years who had taken calculus and analysis courses to support solving the problems presented. Data collection was carried out using the purposive sampling technique at four universities in Lampung (L), Malang (M), West Nusa Tenggara (N), and Semarang (S) using the most significant area problem-solving task sheet and semi-

If perimeter of a rectangle is 20 *cm*, what is the largest area of the rectangle?

Figure 2. Worksheet for the maximum rectangular area problem (Source: Authors' own elaboration)

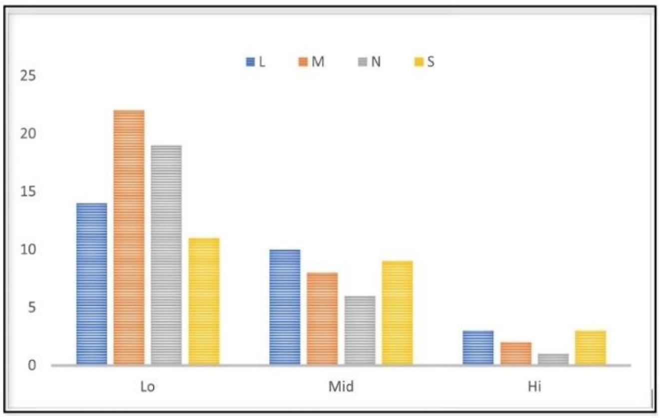

Figure 3. Data categories in solving the largest rectangular area problem (Source: Authors' own elaboration)

structured interviews. This research was conducted in four different areas to ensure that the data obtained was saturated and to maintain consistency of data produced. **Figure 2** shows task sheet done by participants.

The research data is in the form of 108 task answers, which are then categorized into three categories, as shown in **Figure 3**. The data categories are based on different representations of the sides of the rectangle, the strategies used, and the schemes that are active in solving problems. Category 1 is coded as *Lo*, category 2 is coded as *Mid*, and category 3 is coded as *Hi*.

Category 1 (*Lo*) represents the sides of a rectangle as initials and uses a strategy of calculating all possibilities. Furthermore, category 2 (*Mid*) represents the sides of a rectangle as initials but changes its representation to a variable and uses an adoption strategy. Lastly, category 3 (*Hi*) represents the sides of a rectangle as a variable and uses an adoption strategy. Each prospective mathematics teacher student from each data category was selected as a subject with consideration of the subject's way of thinking and understanding in problem-solving. **Table 2** shows the number of prospective mathematics teacher students who are prospective subjects and subjects in this study.

The four research subjects in **Table 2** are subject 1 (S1), subject 2 (S2), subject 3 (S3), and subject 4 (S4) (pseudonym). Next, each subject was interviewed semi-structured for about 25 minutes, which was done immediately after the subjects completed the task.

Table 2. Research subjects

Data category	Subject candidates	Subject
Lo	66	1
Mid	33	2
Hi	9	1
Total	**108**	**4**

Table 3. Mental models for solving maximum rectangular area problems

	Mental models			
	Initials	Adaptive		Formal
Subject	S1	S2 & S3		S4
Strategy	Accounting for all possibilities	Adopt		Adopt
Active schema	Rectangular Circle rectangle Area of rectangle Number combinations Largest area	Subject 2 (S2) Rectangular Circle rectangle Area of rectangle Quadratic equations Quadratic function Graph of quadratic functions Axis of symmetry Maximum value Largest area	Subject 3 (S3) Rectangular Circle the rectangle Area of the rectangle Quadratic equations Quadratic function Derivative Extreme point Stationary point Maximum value Largest area	Rectangular Circle rectangle Area of rectangle Arithmetic mean (AM) Geometric mean (GM) Arithmetic-geometric mean inequality (AM-GM) Maximum area

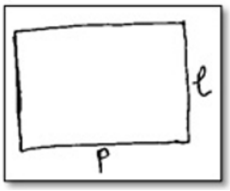

Figure 4. Subject's answer 1 (Source: Authors' own elaboration)

This interview was used as triangulation to obtain valid data, as stated by Creswell and Creswell (2017). Furthermore, the interview protocol with generative questions was also used to investigate the subject's mental model (Chi, 2006). During the interview, all the subject's responses were recorded on video with the help of a tape recorder and then transcribed. Next, the researcher validates the data by triangulating the method, comparing the data from assignment answers and video recording transcripts. Additionally, the data is reviewed by a mathematics expert lecturer and crosschecked with the research members.

Data analysis in this study went through 3 stages: data reduction, data presentation, and drawing conclusions (Miles et al., 2014). At the data reduction stage, selecting, categorizing, and focusing on exploring mental models is carried out. The selection of data is made by choosing the main things and only focusing on the essential things from the data while ignoring some of the answers of prospective mathematics teacher students who do not play a significant role. Data is then presented on the reduction results, and conclusions are drawn.

RESULTS

The results showed that three mental models were developed to solve the maximum rectangular area problem, namely initial, adaptive, and formal mental models. In addition, it was also found that there were differences in the strategies used and the active schemes. Adaptive mental models refer to mental models that experience a process of accommodation adaptation. It differs from the characteristics of mental models found by several other researchers, such as Bofferding (2014) and Sackes (2015). Mental models, strategies, and schemes that were active while solving an enormous broad problem generated by the four selected subjects are presented in **Table 3**.

Exploration of mental models and accommodation processes of mental models for the four subjects selected as representatives of each group are as follows.

Initial Mental Models

Subject 1 (S1) represents a rectangle as an object with four sides and has different sizes of adjacent sides. This can be seen in test answers produced by S1 in **Figure 4**. Interview identifies S1 representations related to rectangles based on the answers in **Figure 4**.

P: What do you think of the "rectangle" object?

S1: I envision a flat area such as a house door. A rectangle has four sides, where the opposite sides are the same length, and <u>the adjacent sides are different</u>.

Based on student answers in **Figure 4** and results of interview, consistently, S1 represents a rectangle as a flat plane with four sides, where the opposite sides are of the same length. In contrast, adjacent sides are different (interview underlined). Based on this representation, representation of rectangle by S1 is very dependent on the sides. Representation of S1 related to the sides of the rectangle is in the form of initials or abbreviations, where the length is abbreviated with the letter p and the width is abbreviated with l. S1 then activates perimeter rectangle scheme, as shown in **Figure 5**.

Activating the rectangle perimeter scheme generates a mathematical formula $p + l = 10\ cm$ (**Figure 5**). S1 then activates the rectangular area scheme and uses the strategy of calculating all possible rectangular areas that can be calculated from satisfying number combinations

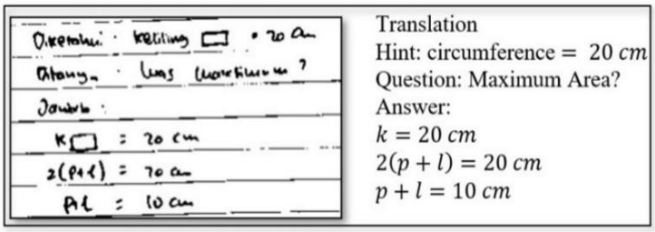

Figure 5. Subject's answer 1 (Source: Authors' own elaboration)

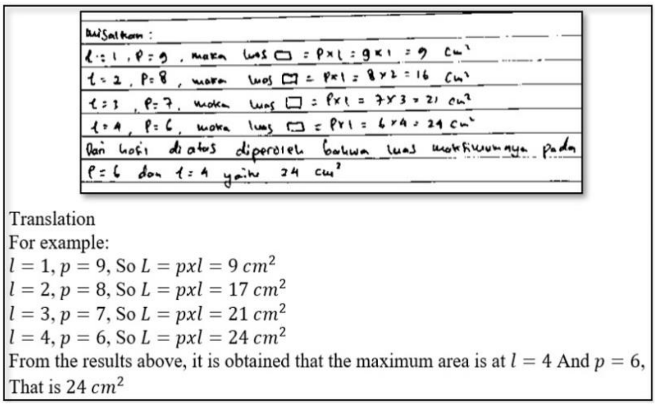

Figure 6. Subject's answer 1 (Source: Authors' own elaboration)

$p + l = 10\ cm$. The following answer from S1 is presented in **Figure 6**.

The following is an interview to reveal why the sides of the rectangle were determined and how the belief in the largest area was obtained.

> P: Why do you exemplify *l* and *p*?
>
> S1: Because one of the values of *l* and *p* is unknown, whereas to calculate the area I need the size of *l* and *p*.
>
> P: Why do not you find?
>
> S1: Is it possible?
>
> P: Are you sure with your answer that the largest area is 24 cm²?
>
> S1: Certain.

Based on the answers in **Figure 6** and the results of interview, S1 consistently uses representations related to rectangles to solve the problem of finding the maximum rectangular area. This consistency can be seen in **Figure 4**, which shows that there are different symbols for sides of the rectangle and different sizes. Next, the accounting for all possibilities strategy is applied to obtain the largest expected area through a given combination of numbers representing length (p) and width (l) and satisfying $p + l = 10\ cm$, resulting in S1 obtaining several possible rectangular areas. Based on several possible areas of the rectangle, S1 then compares areas of the

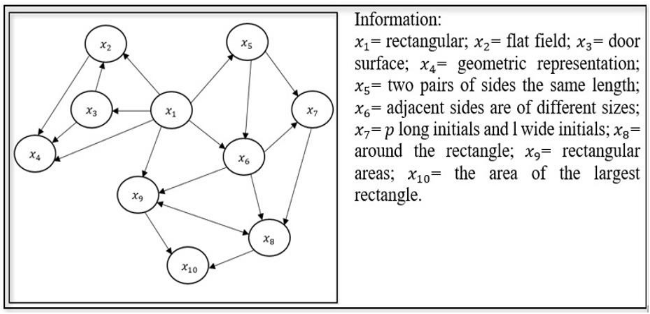

Figure 7. Architecture of initial mental model (Source: Authors' own elaboration)

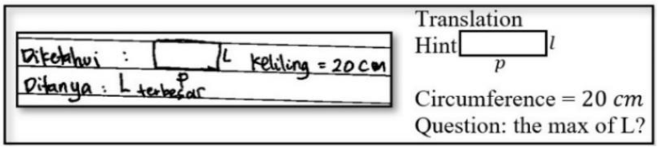

Figure 8. Subject 2's answers (Source: Authors' own elaboration)

rectangles that have been calculated. S1 then finds area of the largest rectangle among other rectangles. The architecture of S1 mental model in solving the maximum rectangular area problem is visualized in **Figure 7**.

Adaptive Mental Models

Subject 2 (S2) represents a rectangle in the form of a quadrilateral with four sides and the symbols p and l on both adjacent sides. **Figure 8** shows answers to S2 test.

Furthermore, the following is an interview to conducted to reveal the S2 mental model, which produced the answers in **Figure 8**.

> P: What do you think of the "rectangle"?
>
> S2: A quadrilateral, has two pairs of parallel sides of equal length, includes a simple closed curve, and all four angles are right angles.

Based on S2's answer in **Figure 8** and clarified through the interview, S2's representation of a rectangle is a shape with four sides with opposite sides parallel and the same length. S2 has a mental model that is more complete than S1. It is because it does not limit the sizes of adjacent sides to be different, even though the rectangular images made it appear that the adjacent sides have unequal lengths and are given different symbols. It accommodates a square as part of the rectangular. Based on this representation, researchers identified that S2's model related to rectangles could lead to a rectangular genetic model, namely a square. S2 also identifies that a rectangle is a curve that is then called back at the next stage of solving problem. S2 then activates the rectangular scheme, as shown in **Figure 9**.

The activation of the rectangular scheme performed by S2 generates a mathematical formula $l = 10 - p$. S2 activate the area of a rectangle scheme and generate the

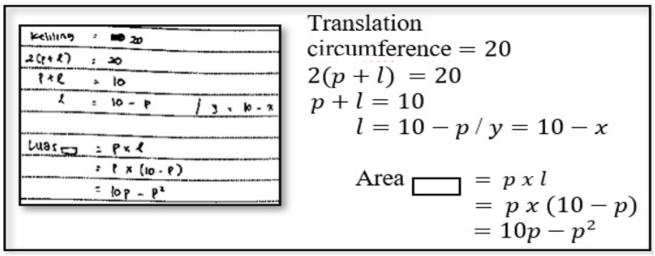

Figure 9. Subject 2's answers (Source: Authors' own elaboration)

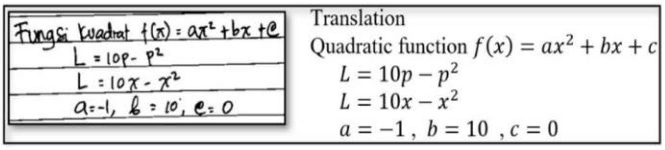

Figure 10. Subject 2's answers (Source: Authors' own elaboration)

math formula for the area of a rectangle= $10p - p^2$, where p is the initial length and l the width. Based on S2's answer in **Figure 9**, S2 tries to change his representation regarding the sides of the rectangle shown by changing the representation of the sides of the rectangle from initial to variable. This change is clarified based on the results of the following interview.

P: y=10-x. What is this?

S2: Similarity Mr.

P: What is contained in the equation?

S2: Coefficients, variables, and constants.

Based on S2's answers shown in **Figure 9** and the results of the interview, S2 has changed his representation regarding the sides of the rectangle. The sides of the rectangle were previously represented as an initial change to variables. Changing the representation regarding the sides of this rectangle is called the mental model accommodation process (Ifenthaler et al., 2008). This is in line with what was expressed by Jones et al. (2011) that mental models change over time. S2 then activates the quadratic function scheme (shown in S2's answer in **Figure 10**) and uses the adoption strategy, where the maximum value of a function is represented as the largest area.

Quadratic function, $f(x) = ax^2 + bx + c$, is an early indication of efforts to change the representation of the rectangular area equation $L = 10p - p^2$ be a function. This change was further clarified in the following interview.

P: Why did you write down $f(x) = ax^2 + bx + c$?

S2: Because $L = 10p - p^2$ in the form of a **quadratic function**.

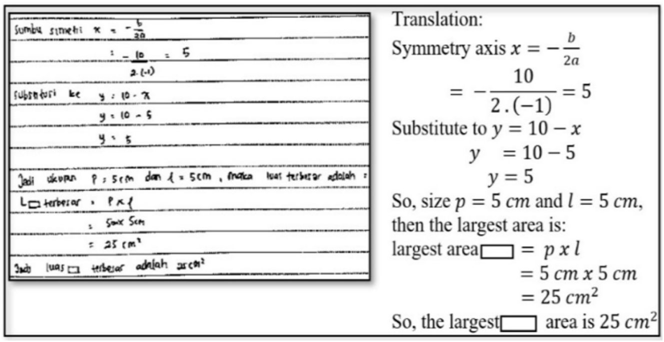

Figure 11. Subject 2's answers (Source: Authors' own elaboration)

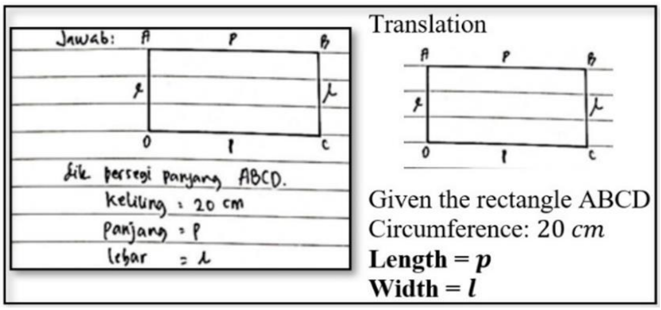

Figure 12. Subject 3's answers (Source: Authors' own elaboration)

P: Why do you imagine $L = 10p - p^2$ as a quadratic function form?

S2: L represents the area of the rectangle, then the rectangle is a curve and equation $L = 10p - p^2$ in the form of a quadratic, then I remembered that the curve is a graph of a function, so I looked $L = 10p - p^2$ as a quadratic function that has a maximum value that lies on its axis of symmetry.

The interview makes it clear that S2 transforms the representation related to the area equation of a rectangle into a function. Changing this representation is done because a quadratic function has a maximum value (interview in bold) and encourages S2 to activate the symmetry axis scheme to obtain the largest area, as shown in **Figure 11**.

Subject 3 (S3) represents a rectangle as an object with four sides with the sizes of the opposite sides parallel and the same length and has four vertices, namely A, B, C, and D. This can be seen in the test answers produced by S3 in **Figure 12**.

The following is an interview to reveal the S3 mental model, which produces answers in **Figure 12**.

P: What do you think of the "rectangle" object?

S3: I imagine a 2D shape that has four sides, where the opposite sides are parallel and the same length, a rectangle is one of the side shapes of a beam, a rectangle can be formed from two right

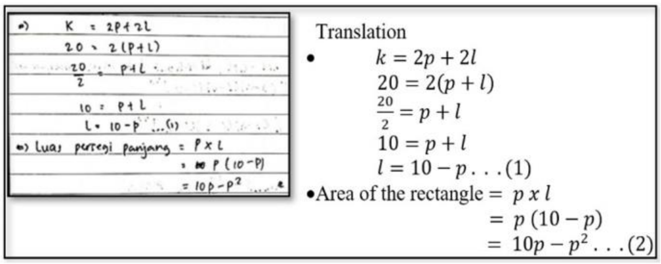

Figure 13. Subject 3's answers (Source: Authors' own elaboration)

triangles whose hypotenuses are squeezed together, and the four angles are right angles.

Based on S3's answer in **Figure 12** and clarified through the interview, S3's representation related to a rectangle can accommodate 2D shape figures: a rectangle and a square. It is because S3 does not limit the sizes of adjacent sides to be different. Based on this representation, the researchers identified the S3 model related to the rectangle that has undergone modifications because the model disclosed can lead to a rectangular genetic model, namely a square.

Furthermore, S3's representation of the sides of the rectangle is in the form of initials, where the length is abbreviated as p, and the width is abbreviated as l. The mental model of the sides of this rectangle can be seen in S's answer, which is shown in **Figure 12**. S3 then activates the perimeter and area of the rectangle scheme, as shown in **Figure 13**.

Based on the activation of the circuit scheme, S4 obtains two mathematical formulas with the symbols p and l, with $l = 10 - p$ and the area of the rectangle= $10p - p^2$. Based on S3's answer shown in **Figure 13**, S3 has changed the representation regarding the sides of the rectangle. The sides of the rectangle that were previously represented as an initial change to variables (interview in bold). Changing the mental model of the sides of the rectangle is also supported based on S3's answer shown in **Figure 14**, where from the Area of the rectangle= $10p - p^2$, the first derivative is calculated to be $L' = 10 - 2p$ (**Figure 13**).

S3 changes the existing representation or creates a new representation of the sides of a rectangle to suit the structure of the problem at hand. Changing the representation made by S3 is referred to as the accommodation process of the mental model (Ifenthaler et al., 2008). At this stage, there is a process of changing mental models, which is in line with what was stated by Jones et al. (2011), that mental models change over time. S3 then activates the first derivative scheme, extreme point, stationary point, and maximum value and uses the adoption strategy to solve the problem later, as shown in **Figure 14**.

Based on the answers in **Figure 14**, the researcher conducted the following interview to reveal the S3 representation regarding the equation $L = 10p - p^2$.

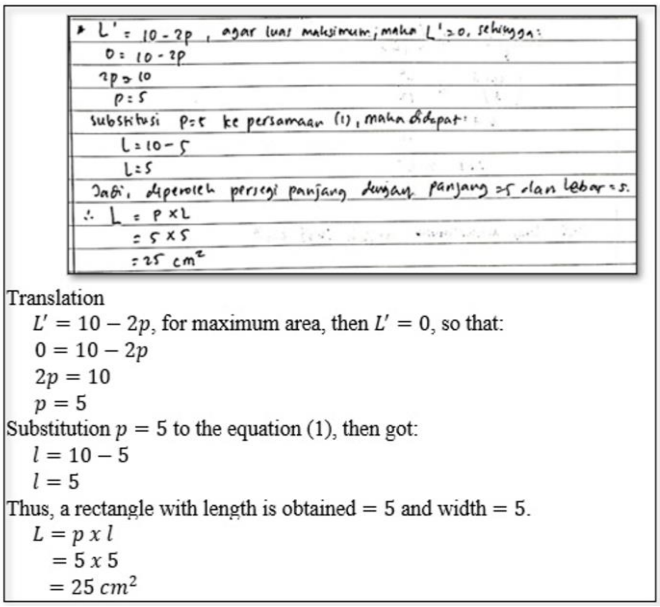

Figure 14. Subject 3's answers (Source: Authors' own elaboration)

P: Does an equation have a maximum value?

S3: Erm…, no sir

P: Then why are you looking for the maximum value of $L = 10p - p^2$?

S3: Erm ... because $L = 10p - p^2$ function sir.

Based on S3's answers listed in **Figure 14** and the results of the interview, S3 experienced a change in representation related to mathematical formulas $L = 10p - p^2$. This change is supported by the solution that was made, namely S3 looking for the maximum area of $L = 10p - p^2$, where, which has a maximum value is a function, not an equation. That is, there is a change in the representation of $L = 10p - p^2$, where $L = 10p - p^2$ previously represented as an equation and then converted into a function, and this is referred to as the mental model accommodation process (Ifenthaler et al., 2008).

Based on the method used by S2 and S3, the maximum rectangular area is represented algebraically. These results indicate that mental models are useful and functional as knowledge synthesis and adaptation when solving problems (Mikkilä-Erdmann et al., 2008). The architecture of the mental models S2 and S3 in solving the maximum rectangular area problem is visualized in **Figure 15** and **Figure 16**.

Formal Mental Models

Subject 4 (S4) did not externalize his internal representation of the rectangle in the form of a picture in his answer, but the researcher explored representation he had through this interview.

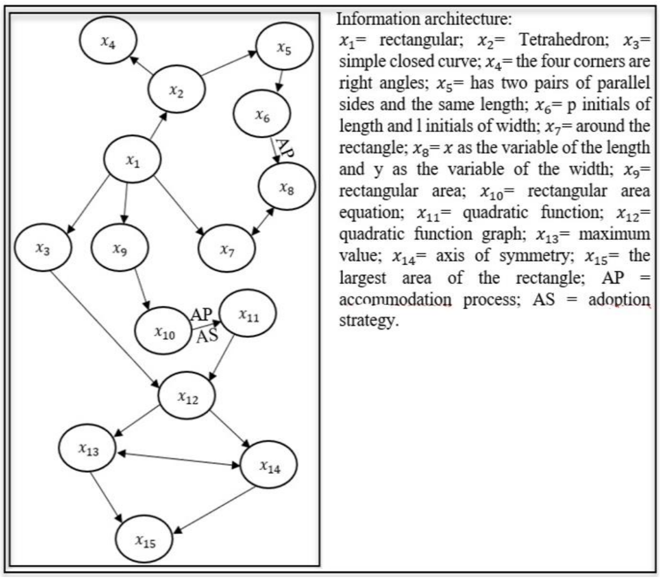

Figure 15. Adaptive mental model architecture S2 (Source: Authors' own elaboration)

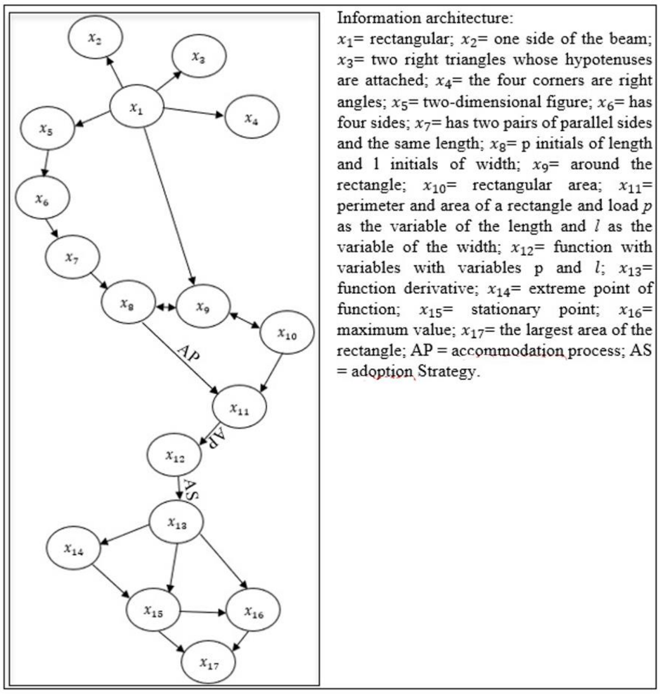

Figure 16. Adaptive mental model architecture S3 (Source: Authors' own elaboration)

P: What do you think of the "rectangle"?

S4: I imagine a quadrilateral with **two pairs of parallel sides that are the same length**, a rectangle is a parallelogram with one right angle, a square is also a rectangle, the four angles are right angles, and it has two diagonals that intersect.

Based on the interview, S4 formally reveals the characteristics of a rectangle. It hierarchically relates the relationship between a rectangle and a quadrilateral with two pairs of parallel sides, such as a parallelogram and a square (interview in **bold**). Sides of the rectangle are then represented as x and y symbols, with x being the length and y being the width. S4 then activates rectangle perimeter scheme to obtain mathematical formula $x + y = 10$, as shown in **Figure 17**.

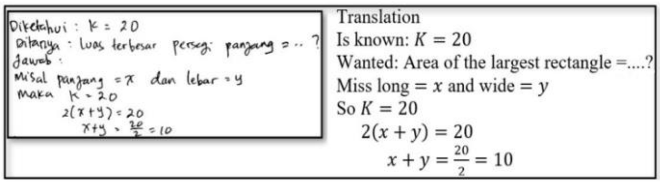

Figure 17. Subject's answer 4 (Source: Authors' own elaboration)

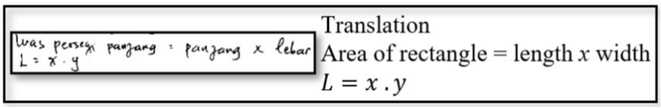

Figure 18. Subject 4's answers (Source: Authors' own elaboration)

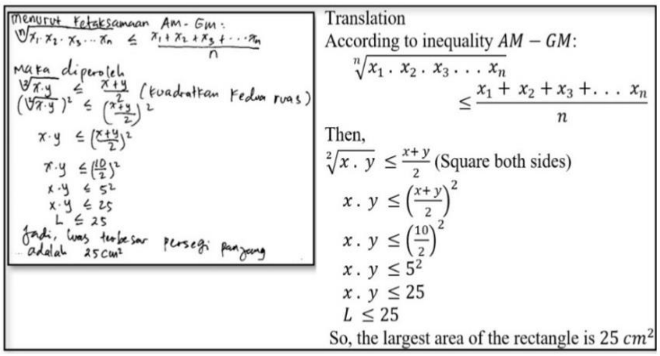

Figure 19. Subject 4's answers (Source: Authors' own elaboration)

Based on S4's answer in **Figure 17**, S4 uses formal symbols such as x and y to represent the length and width of a rectangle. S4 mental model related to the sides of a rectangular line is in line with the experts because the sides of a rectangle are represented in the form of variables that are abstract and formal in nature. The sides of a rectangle are represented by the variable symbols x and y, where x is the length and y is the width. S4 then activates the rectangular area scheme in the form of an equation containing the variables x and y, as shown in **Figure 18**.

S4 then proceeded to solve it by activating the inequality of the arithmetic mean (AM) and the geometric mean (GM) and using the adoption strategy to obtain the maximum rectangular area, as shown in **Figure 19**. Based on S4's answer shown in **Figure 19**, S4 has a representation by the representation held by the expert because it can model abstract and formal rectangular sides, which are then used to obtain the maximum rectangular area through the implementation of an adoption strategy.

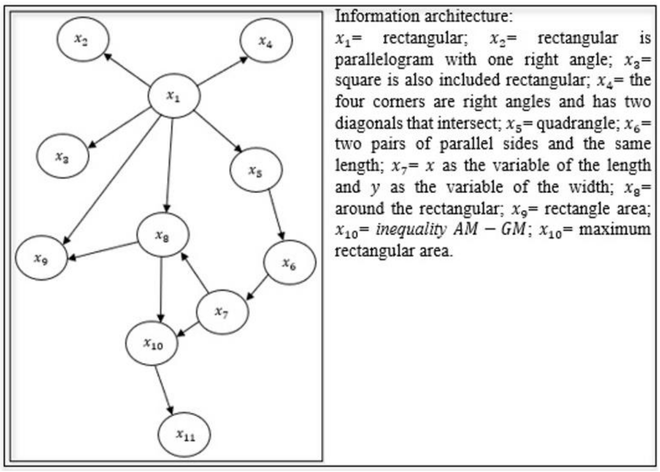

Figure 20. Architecture of formal mental model (Source: Authors' own elaboration)

Next, the researcher examines more deeply the activation of the AM-GM inequality scheme to solve the maximum rectangular area problem through the following interview.

> P: Why did you use the AM-GM inequality to find the largest area?
>
> S4: Due to the AM-GM inequality one can claim a maximum bound on a value.
>
> P: What is the meaning of "≤"as you wrote in your answer? Please explain your answer.
>
> S4: ≤ indicates that the area of the rectangle will be less than or at most equal to 25.

Based on the interview, the arithmetic and geometric mean inequality (AM-GM) scheme are activated to provide a boundary for the area of a rectangle. In line with the activation of the schema, Hasemann (1986) and Norman et al. (1976) explained that the schema represents a structural framework in which information is put together in a deduced whole. Based on the AM-GM inequality relationship, algebraic manipulation, and then linking to the equations $L = x.y$ and $x + y = 10$ obtained before is undoubtedly not easy. Based on the method, the maximum rectangular area is represented algebraically by S4. The S4 mental model architecture in solving the maximum rectangular area problem is visualized in **Figure 20**.

Based on these results, in their initial stages, they examined the different possible shapes of rectangles, experimented with different sizes of the sides of the rectangles, and observed how the area changed as the lengths of the rectangles changed. These stages allow them to form initial mental models and make connections between various aspects of the problem (Bofferding, 2014; Utami et al., 2018). As they continue to explore, they begin to recognize patterns or regularities in the relationship between the sizes of the sides of the rectangles and the resulting areas. They will notice that the possible area is maximized when the rectangle is a square. Next, they identified the sides of the rectangle as a particular variable, created an equation, and then changed the representation to a function. This stage involves the development of adaptive mental models (van Ments & Treur, 2022). Furthermore, teacher candidates use formal mathematical reasoning to explain their observations and strategies (Bofferding, 2014; Utami et al., 2018). They derived a mathematical formula or equation to represent the relationship between the sides of a rectangle and its area. They begin to understand the concept of optimization using techniques of calculus or algebra and perform analysis to find the correct size of the sides of a rectangle for maximum area.

This study's findings indicate differences in mental models between prospective mathematics teachers, which shows their cognitive development. It aligns with the cognitive science theories developed by Piaget (1954). Piaget (1954) stated that individuals build mental models or schemes to organize and interpret their experiences. The initial mental model is the basis individuals develop when facing new concepts or problems. This initial model is based on the individual's prior knowledge, experience, and intuitive understanding (Bofferding, 2014). Adaptive mental models refer to the ability to modify or adapt existing mental models in response to new information or experiences. They involve a process of restructuring or revising initial mental models to accommodate new knowledge and improve Van's problem-solving strategies (van Ments & Treur, 2022). Formal mental models refer to a more structured and systematic representation of knowledge aligned with predefined concepts, procedures, and strategies for solving mathematical problems (Bofferding, 2014; Utami et al., 2018).

DISCUSSION

This study aims to explore mental models, strategies, and schemes that are active in solving the maximum rectangular area problem. Based on the research results, the rectangular representation built by subject 1 (S1) is simple, concrete, incomplete, and limited (Norman et al., 1983) due to the emphasis that the two adjacent sides of the rectangle sides must have different, same size. It contrasts subject 2 (S2) and subject 3 (S3), which only emphasize that parallel sides have the same length. Although the rectangular images presented by S2 (**Figure 8**) and S3 (**Figure 12**) seem inconsistent, they are supported in interview 4 (S2) and interview 8 (S3). It is in line with Bofferding (2014) that a person's mental model is sometimes inconsistent. The mental models S2 and S3 are inconsistent when representing the rectangle sides and the rectangle area equation. Whereas subject 4 (S4) added "a", which refers to things that are more general, abstract in nature, and not tied to the shape of a rectangle, but S4 still includes another characteristic of a rectangle, namely having four sides with two pairs of equal parallel sides. In addition, S4 can also explain hierarchically between parallelograms, rectangles, and squares. The difference in the representation of the subject's rectangle is natural because mental models represent constructs developed in understanding an object during the problem-solving process (Gogus, 2013).

The results of the data analysis revealed three essentials characteristics of the strategies used by the three groups in solving the biggest broad problem. First, S1 uses a strategy of calculating all possibilities, which is done by adopting a clear path, even though this is quite tedious. It is different from S2 and S3, who choose to use an adoption strategy by adopting and activating schemes outside of problems that can help them solve these problems. S4 also did the same, namely using the adoption strategy to find the largest area. Specifically, for the problem-solving task given, the adoption strategy used by S4 is more effective and expert than S2 and S4 because it does not require a long process of finding the largest area. However, choosing what schemes need to be activated is more challenging. Meanwhile, the adoption strategy by S2 and S3 is more effective than the strategy of calculating all possibilities. It is because calculating all possibilities strategy has a greater chance of error due to lack of accuracy. For example, as the results obtained that $24\,cm^2$ believed (Sternberg & Sternberg, 2016) to be the largest area even though the combination is more precise than size $p = 5.5$ and $l = 4.5$, then a larger area is obtained $24.75\,cm^2$. So, for prospective mathematics teachers, calculating all possibilities is not recommended because it is not effective; it takes a long time to combine possible numbers, especially if given a circumference that allows many combinations; this will be difficult to solve.

Furthermore, subjects from all three groups constructed a mental model of the largest area of the rectangle by activating schemata that they had access to while trying to solve problems (Chinnapan, 1998). S2 and S3 activate more schemes than S1 and S4. However, even though S2 and S3 activate more than S4, the scheme activated by S4 is of higher quality because it has modeled the sides of rectangles as abstract and formal variables and is also able to use analysis strategies by utilizing the arithmetic and geometric mean inequality scheme (AM- GMs).

There are few, and whether the quality of the active schemes during problem-solving is strongly suspected depending on the abilities possessed by each subject, as revealed by Chinnapan (1998). Finally, based on the representation of the rectangle and the sides of the rectangle, the strategy used, and the active scheme, the subject obtains the largest area of the rectangle modeled in three models: the largest area among the areas of the rectangle that has been obtained; the maximum area achieved at a certain length and width; and no area of a rectangle is larger than the area of the largest rectangle. Structural analysis of the research results shows that the mental model of an object built by the subject strongly affects the quality of problem-solving carried out at a later stage (Chinnapan, 1998).

For future mathematics teacher candidates with initial mental models, it is necessary to provide examples of mathematics relevant to their experience to see the relationship between mathematical concepts and the real-world (Loarces et al., 2019; Utami et al., 2018). Meanwhile, future mathematics teacher candidates who have an adaptive mental model need to be given exercises that encourage them to use a variety of problem-solving strategies they have and try new approaches when they face more complex challenges (Posamentier & Krulik, 2008). Furthermore, for future mathematics teacher candidates with a formal mental model, it is necessary to develop ways of applying relevant mathematical rules and formulas to solve concrete problems (Posamentier & Krulik, 2008; Spangenberg & Pithmajor, 2020).

CONCLUSIONS

There were three mental models for pre-service mathematics teachers in solving maximum rectangular area problems, namely beginner (initial), adaptive, and formal, each of which had different active strategies and schemes during problem-solving. Mathematics teacher candidates with beginner mental models (initials) tend to solve the largest area problem by calculating all possible areas of a rectangle by combining numbers representing length and width. Meanwhile, pre-service mathematics teachers with adaptive mental models tend to change mental models to adapt to the structure of the problems they face and use adoption strategies to obtain the expected solutions. Mathematics teacher candidates with a formal mental model have a model that is more in line with experts; that is, it is abstract in nature. They can use adoption strategies and activate sophisticated schemes to solve their problems. The mental model has a dynamic character, which means that the mental model changes from time to time because of learning, and changes in this mental model impact problem-solving skills. Therefore, mental models have a central role in solving problems. In addition to having an impact on the quality and determining the accuracy of the resulting solutions, mental models also trigger the implementation of strategies and active schemes during the problem-solving process.

Pre-service mathematics teachers with novice mental models (initials) need to be given a trigger in the form of learning based on mental model construction and group discussion so that their mental model changes for the better. This needs to be done because mental models can influence (or vice versa) the strategies used and schemes that are active in solving problems until the results are obtained. This study needs to improve in identifying initial abilities and differences in learning methods and models they follow in class. Therefore, this study provides an opportunity to conduct further research considering these two things.

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Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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