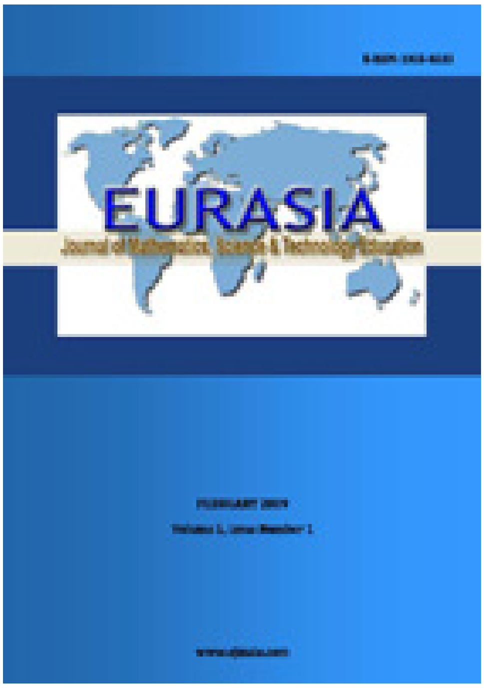
Eurasia Journal of Mathematics, Science & Technology Education
www.ejmste.com



What ‘Picture in Mind’ do Secondary Students have about Defining, Proving, and Modelling?

Victoria Sánchez, and Mercedes García
Universidad de Sevilla, SPAIN

Received 25 May 2011; accepted 14 Deember 2011

Published on 29 April 2012

APA style referencing for this article: Sanchez, V. & García , M. (2012). What ‘Picture in Mind’ do Secondary Students have about Defining, Proving, and Modelling?. *Eurasia Journal of Mathematics, Science & Technology Education*, *8*(2), 95-102.

Linking to this article: DOI: 10.12973/eurasia.2012.823a

URL: http://dx.doi.org/10.12973/eurasia.2012.823a

ISSN: 1305-8223 (electronic) 1305-8215 (paper)

Eurasia Journal of Mathematics, Science & Technology Education, 2012, **8**(2), 95-102

What 'Picture in Mind' do Secondary Students have about Defining, Proving, and Modelling?

Victoria Sánchez, and Mercedes García
Universidad de Sevilla, SPAIN

Received 25 May 2011; accepted 14 Deember 2011

The present study tries to identify the 'picture in mind' that students' have about defining, proving and modelling. We access to it through external representations of their mental representations; in particular, from the way in which these representations are expressed in written form (sentences and situations). In the study we have identified different ways of considering defining, proving and modelling: superficial, utilitarian and intrinsic, which were maintained by some students for the three elements in the different written representations. This coherence can be perceived as a characteristic of students' mathematical understanding. Our results provide an analytical scheme whose domain goes beyond the way of students' approaching to the mathematical elements considered in the study.

Keywords: Defining, mathematical understanding, modelling, proving.

INTRODUCTION

The mathematical background of first year university students is an issue of concern and debate in our country. In recent years, Spanish university mathematics teachers have been observing a lack of understanding of basic mathematical ideas in first year students, which significantly affects their access to advanced mathematical thinking. In this context, the highest stage (16-18 year-old students) of secondary education requires special attention. Its importance as a preparatory stage that guarantees the bases for tertiary studies is emphasised among the aims of this stage in the Spanish curriculum guidelines. The research reported in this article is part of a wider-ranging study aimed at exploring the understanding of 16-18 year old students with respect to some elements that form part of the core of what it means to do mathematics: defining, proving and modelling. They have been object of many studies and considered under different perspectives, coming from other disciplines or generated within the field of mathematics education. In the latter, albeit with very different approaches (in some cases closer to consideration as concepts and in others as processes), their importance in students' mathematical learning processes has been shown by several researchers over the last few decades (Hanna, & Jahnke, 1996; Vinner, 1996; Mariotti, 2006; Harel, Selden, & Selden, 2006; Blum, Galbraith, Henn, & Niss, 2007).

We fully agree with the importance of other elements such as reasoning, representing and so on. Without minimizing their great value, our study has focused on the three afore-mentioned elements. We would emphasize that, at least in Spain, students enter the highest grade of secondary education with many experiences that are bound to shape their learning about defining, proving and modelling. Nevertheless, they are not explicitly mentioned in the Spanish school curriculum, but students have been approached them in an indirect way, through other mathematics curricular topics.

This importance has led us to explore in a previous research the way in which defining, proving and modelling are presented in Spanish school texts, considered these textbooks like a context for 'seeing' the

Correspondence to: Victoria Sanchez, Professor of Mathematics Education, Departamento de Didáctica de las Matemáticas, Universidad de Sevilla, Pirotecnia, 41013 Sevilla, SPAIN
E-mail: *vsanchez@us.es*

ISSN: 1305-8223

State of the literature

- Defining, proving and modelling are important elements in the construction of mathematical knowledge. Their importance in students' mathematical learning processes has been shown by several researchers.
- In mathematics education research, their joint study has been usually restricted to some of these elements - defining and proving, modelling and proving, among other possibilities.
- Students' mental representations can be a way to access to students' ideas related to those elements. Researchers have distinguished between internal and external representations., considering that external representations can be a powerful tool with which to approach students' understanding.

Contribution of this paper to the literature

- The joint study of defining, proving and modelling, from the point of view of their consideration as mathematical content, helps to inform us on the 'picture in mind' that students can have about them.
- The identification of distinct ways of considering them through different external representations (textual and specific) illustrates how different students can 'see' different features of mathematical topics.
- The coherence in the adoption of common vision of defining, proving and modelling through those representations could be perceived as a characteristic of mathematical understanding.

meanings that, somehow, are going to take part in what the students learn (Sánchez, García, Escudero, Gavilán, Trigueros, & Sánchez-Matamoros, 2008). In addition, we have tried to study the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve those processes (García, Sánchez, & Escudero, 2009).

In this context, we are interested in exploring the 'picture in the mind' that students can have of defining, proving and modelling. Particularly, in this work, we wanted to see if there is something in common in the students' way of considering them.

Theoretical background

Two different dimensions form our theoretical background. On the one hand, we consider "defining" (D), "proving" (P) and "modelling" (M) as important elements in the construction of students' mathematical knowledge. On the other hand, we consider the mental representations as a way to access to students' ideas related to them. These two dimensions provide us a frame of reference for our study and will be developed below.

The first dimension

In our research, among many other characteristics, "defining" prescribes the meaning of a word or phrase in terms of a list of required properties. This prescription has characteristics that could be imperative (not contradictory, not ambiguous, invariant under the change of representation, hierarchic nature) or optional (for example, minimality) (van Dormolen, & Zaslavsky, 2003; Zaslavsky, & Shir, 2005).

Focusing on proving, the contributions of different authors (Balacheff, 1987; Moore 1994; Hanna, 2000; García, & Llinares, 2001; Knuth, 2002; Weber, 2002) led us to consider among its characteristics the existence of both a premise / terms of reference / proposition and a sequence of logical inferences, which are accepted as valid characteristics by the mathematical community in the sense of 'not erroneous'.

Finally, according to Blum, & Niss (1991), we use modelling "to mean the entire process leading from the original real problem situation to a mathematical model" (Blum, & Niss, 1991, p. 39). For us, mathematical modelling is characterized as a non-linear process that considers the movement from real-world situations to mathematics, working the mathematics, and translating the results back into the real-world context (Lesh, & Doerr, 2003; Lesh, & Harel, 2003).

While most research has considered defining, proving and modelling separately, several authors have considered jointly some of them. Harel, Selden, & Selden, (2006) have collected some works in which the dialectical interplay between defining and proving is emphasised. Boero, & Morzelli, (2009) have focused on the use of algebraic language in modelling and proving, pointing out in their educational implications the relevance of this language as an important tool for the understanding of modelling and proving.

The second dimension

We assume that understanding implies the mental representation of the object/idea understood. The distinction between internal and external representations is extremely important and necessary in our study. With respect to internal representations, researchers in mathematics education have used knowledge structures to describe mental structures that people have for storing and encoding information. With respect to external representations, Izsák, (2003) suggests that: "When discussing external representations, researchers have referred to artefacts that human create for thinking or conveying information about some context distinct from those artefacts" (p. 194). Our underlying assumption is that access to students' ideas related to D, P and M might arise through external representations of mental representations; in particular, it could be informed from the way in which they are expressed in written form. For us, this written form can be considered in two contexts: the sentences the students write when trying to express their idea of them (which we consider textual representations) or the situations in which they find their meanings (which we consider specific representations). To generate specific representations, students must have knowledge for relating and selecting attributes from those situations (Izsák, 2003).

We recognize that when a student expresses his/her idea of D, P and M, he or she may or may not be accurately representing his/her understanding. Nevertheless, we consider that external representations (and, in particular, the different written forms in which a person makes his/her comprehension explicit) can be a powerful research tool with which to approach students' understanding (Duval, 2002; Pitta-Pantazi, Gray, & Christou, 2004).

In particular, the study reported here tries to identify features that inform us about the way in which D, P and M, considered in our case important elements in the construction of mathematical knowledge, are viewed by some students. We hypothesised that these features could provide insight about the 'picture in mind' that those students have about them.

We wish to indicate that we do not try to access directly to something as complex as the students' understanding of D, P and M, but we try to identify clues that lead us to an identification of features or characteristics of that understanding. In this article we try to approach these features or characteristics on the basis of textual and specific representations.

RESEARCH METHODOLOGY

Participants

Ninety-eight students (aged 16-18 years) participated in this part of the study. The students belonged to three public Secondary schools (students 12 to 18 years of age) in three cities of the South of Spain, with a human population of approximately one million, two hundred thousand and thirty thousand respectively. These schools did not differ with respect the socioeconomic background of the children. According to the parental level of education and employment, we may consider it as a medium socioeconomic status.

In Spain, the secondary education (12 to 18 years) has two stages: Compulsory Secondary Education (12-16 years) and Non-Compulsory Secondary Education (16 to18 years). The Compulsory Secondary Education comprises of 4 courses divided into 2 cycles of 2 years each (12 to 14 years and 14 to 16 years). This stage provides to the students a common mathematical background that consists of five blocks: numbers, algebra, geometry, functions and graphs, and statistics and probability. In addition, all courses include a common content block which makes explicit reference to 'solving problems'. They are not separated blocks: numerical and algebraic techniques are used in all blocks, and any of them can be useful to make a table, generate a graphic, raise probabilistic situations and, of course, in problem solving situations. On successfully completing this education stage, pupils are awarded the certificate of Secondary Education Graduate giving access to Non-Compulsory Secondary Education (Bachillerato). The students considered here belonged to four classes of this stage (one class in the each one of the biggest cities and two in the smaller city). At the time of the study, there were no differences with respect to ethnic/racial groups in relation to these students, who were mixed across gender but diverse with respect to their mathematical academic performance. The classes represented a wide range of achievement levels based on their performances in the previous year.

The research instrument

Our data source included questionnaires and semi-structured interviews with teachers and students. Since we focus on the results of the students' questionnaire, we will detail only this research instrument. The questionnaire items were inspired by existing instruments (Healy, & Hoyles, 2000; Zaslavsky, & Shir, 2005) and the experience of teachers and researchers. It consisted of an initial presentation followed by three parts (corresponding to defining, proving and modelling).

Since our aim was exploring if there was something in common in the way of considering D, P and M, these parts were considered separately and had, in general terms, the same structure. They included two types of statements to access different aspects related to the way in which the students had constructed the different mathematical elements considered in our study, so that they allowed gathering of a variety of points of view (Healy, & Hoyles, 2000).

In the first type of statements, students were asked to provide descriptions on every element, expressing in their own words the associated meaning, and including an example they considered most suitable (see Figure 1 for the case of proving; the same structure was used for defining and modelling). The aim of this type of statements was that the student showed us through the language and examples that they proposed the aspects that they considered fundamental in the description of D, P, and M, what parts, properties, qualities or

Statement 1

a) What does proving in Mathematics mean to you?

…………

b) What do you think would be a good example of proving?

…………

Figure 1. First page of the questionnaire corresponding to proving

circumstances of these elements they considered to have sufficient value to allow them to explain said elements.

The second type of statements presented different possibilities for each element according different systems of representation and different roles, which represented different facets of them. These statements were related to two mathematical topics. They included three correct/incorrect expressions for each topic. The mathematical topics were from different mathematical domains (Algebra, Analysis and Geometry), and were practically extracted from the school textbooks. For example, with respect to defining, we selected three definitions of perpendicular bisector (mediatrix) and three of the greatest common divisor. The students had to indicate whether or not these definitions were correct; which one they preferred; and which one they thought their teacher would prefer, giving reasons for each of their answers.

The initial version of the questionnaire thus obtained was then sent to five expert secondary teachers. Their comments were used to modify the formulation of almost every statement. Next, the revised version of the questionnaire was piloted. For this purpose, a sample of 26 secondary students was chosen. These students were from one of the secondary schools that participated in our study, but they were not included in the final sample. According to the analysis of their answers, some items were subsequently deleted from the questionnaire, because the original formulation was ambiguous or unclear, or did not provide important information. The final version of the questionnaire was given to the 98 students.

Due to the aim of the part of the research reported here, in this report we are going to consider only the first type of statements of the questionnaire related to descriptions and examples of the considered elements.

Data analysis

Three analyses were carried out separately for the research team. From these individual analyses, the final analysis was performed.

First analysis

The aim of this first analysis was to identify characteristics in the external representations (textual and specific) of defining, proving and modelling that the students had provided.

Firstly, we analysed the sentences related to D, P and M provided by the students, identifying cases in which students had responded to the statements related to all these mathematical elements (69 students, 70.41%). We excluded the cases in which students had not replied to the questions related to some elements (incomplete responses (22 students, 22.45%) or no response (7 students, 7.14%)). In the 69 cases considered, we admitted as valid the arguments that, in some way, related D, P and M with the aspects identified in our theoretical framework. According to this consideration of the elements, we identified 55 students that provided us with 165 sentences with valid arguments (three sentences related to defining, proving and modelling respectively per student). 14 students provided invalid arguments that were excluded. For instance, responses such as that provided by student C118 (the first letter identifies the school (A, T or C), the following number the course (1 or 2) and, finally, the last numbers indicate the student) were considered invalid:

Defining signifies … I think maths could be defined as a culture that, each day, we learn any place and everywhere we are… (Student C118)

As we can see, in this expression the student does not speak about defining but on mathematics from a general point of view.

Nevertheless, the response of student T112 was accepted as valid, given that the sentence he provided shows characteristics related in some way to the proving process:

Proving is verifying if something is true (Student T112)

Secondly, we analysed the situations provided by the students related to the D, P and M. As in the previous analysis, we took into account the cases in which all the examples had been included (44 students, 44.90%), excluding the other possibilities. According to the above-mentioned general meanings, we identified 129 situations with valid arguments, coming from 43 students. Only one student was excluded in this case.

A situation considered as valid was:

An example of definition would be: an equilateral triangle is that in which all sides are equal (Student T17)

In this answer, specific characteristics of the mathematical object are included.

Nevertheless, the response of student A14 was considered to be invalid, since the situation this student provided to illustrate the process of modelling does not include any aspects that identify this process:

An example of modelling would be that everything was explained step to step (Student A14)

In a third step, we classify the 294 valid responses (165 sentences and 129 situations) with respect to the different aspects of the elements that were considered in our theoretical framework. In this classification, we took into account "qualities" of the aspects that were manifested in the representations (textual and specific). We differentiated these qualities in function of what they were standing out of defining, proving and modelling: external noticeable features (coming from the outside), internal (existing inside them), and so on. Through this classification, we obtained groups of responses with differentiated characteristics. For instance, the situation provided by the student T17 above mentioned related to definition was considered in a different group of responses that the following sentence provided by the student A21:

Defining means expressing what one is trying to explain in an easy-to-understand and simple manner.

The number of students that gave valid arguments in the textual representation with respect to the three elements was 55 of the 98 (56.12%) and the number of students that gave valid arguments related to the specific representation was 43 of the 98 (43.88%); 33 of these students overlapped.

Second analysis

We analysed separately how each student considered D, P and M in the sentences and how considered D, P and M in the situations.

For instance, the student A21 wrote the following sentences:

Defining means expressing what one is trying to explain in an easy-to-understand and simple manner.

Proving means providing an example (graphic representations, operations with explanations …).

Modelling means that it can be adapted to the real life things.

As we can see in these sentences, considered as textual representations, the aspects emphasised in each one of the processes are superficial and reveal a way of considering D, P and M that only takes into account aspects coming from the outside.

Third analysis

Finally we considered jointly how each student considered D, P and M in the sentences and the situations as a way of identifying features of students' picture in the mind. In this way, for instance, in the specific representations of student A21 above, we identified characteristics different to those identified in the case of textual representations and, consequently, they were situated in a different type.

FINDINGS OF THE STUDY

From the first data analysis of the 294 sentences and situations (165 sentences plus 129 situations), considered as sources that inform us about textual and specific external representations respectively, we identified the following groups of:

A group of responses focused on describing characteristics of D, P and M, without considering their intrinsic aspects and giving a general idea of the same, which involved only the most obvious things. We called this way of considering the elements "Superficial (S)". The sentences and situations we included in this group were related to aspects that focused only on the surface of the elements. For these students, defining allowed access to the mathematical elements in a more or less clear manner. In addition, proving was considered a way of making the explanations easier and modelling as a way of describing situations. Examples were understood to be superficial and general expressions of situations that reflected or gave meaning to the elements. The following excerpts are representative of this way of considering.

The first excerpt is a sentence related to defining. The student emphasises a superficial characteristic of this element related to its role as the facilitator of the communication.

Defining means expressing what one is trying to explain in an easy-to-understand and simple manner (Student A21)

The second excerpt is an example of situation related to proving, in which the emphasis is on the idea of successive steps as a feature of this element, and not on the content of these steps.

A good example of proving would be… one with all the steps to be taken, including a drawing or graph if necessary (Student A19)

Another group of responses was related to the actions and relations that can be identified from the use of elements, which were viewed through their usefulness. We identified this way of considering D, P and M as a Utilitarian point of view (U). In this case, defining was understood as something used for identifying things, proving was used for verifying questions requiring an answer, and modelling for finding out things or valid results for real-life situations. The chosen examples were expressions of actions that were given in situations in which the elements are involved.

The following excerpts exemplify a sentence and a situation that are representative of the importance of being useful rather than other aspects of proving:

Proving means providing an example to confirm that what is being dealt with is either true or false (Student A19)

A good example of proving would be … to use Pythagoras' theorem to prove some height you have calculated by another means (Student T19)

Finally, other responses emphasised the intrinsic properties that constitute the elements, showing in some way an understanding of their meaning (Intrinsic point of view (I)). For these students, defining was something that allowed them to express mathematical ideas or concepts. Proving was considered a way of verifying a definition or a mathematical expression, and the properties of modelling allowed explaining of real world situations. These students proposed specific situations in which the elements could be clearly identified. We do not want to say that the aspects considered in these sentences and situations were totally correct properties from a formal mathematical point of view, but they were extracted from features or characteristics that we considered as valid arguments related to D, P and M. For instance, the following sentence was considered in this group because it highlights the use of mathematical language to express something (not the use of the model itself for something, which would be considered in our analysis as a utilitarian point of view)

> *Modelling means …to utilize mathematical resources to find a solution or theorem for physical phenomena or procedures* (Student T29)

These results allowed us to identify three different ways of considering D, P and M, which we have called Superficial, Utilitarian and Intrinsic. We would like to emphasize that all of these ways of considering try to characterise D, P and M. The singularity of each one comes from the fact that its focus of attention varies.

From the second analysis, we were able to identify how these different ways of considering were particularised in the textual and specific representations, as we can see in the following.

Textual representations: From the 55 students that had provided valid arguments, 34 students had a similar way of considering the three elements, and 21 students changed in function of the element (D, P or M). We can say that these 21 students saw the elements from different points of view. They were not included in the part the study reported here.

With respect to the 34 students that had a common consideration for each element, 4 students were situated in the "S" way of considering, 11 in the "U" way of considering, and 19 were situated in the "I" way of considering.

Specific representations: From the 43 students who had answered all the questions related to these representations, 38 students had a common consideration for the three elements with respect to the situations in which they found their meanings, and 5 students changed in function of the elements. As in the textual representations, we can say that these 5 students saw the elements from different points of view and they were not considered in the part the study reported here.

In this case, 7 students were situated in the "S" way of considering, 3 were situated in the "U" way of considering, and 28 students were situated in the "I".

The specific representations as a vehicle of communication of the students' ideas revealed more difficulties than the textual representations. Nevertheless, the students who were able to express their ideas of the mathematical elements through a situation had a joint vision on more occasions. Furthermore, that joint vision was linked for these students to the intrinsic properties of the elements (the "I" way of considering).

Finally, we wish to point out that the 55 of 98 students who participated in this study expressed their ideas with arguments that we considered valid arguments by means of textual representations; 34 of

Table 1. Students' way of considering textual and specific representations

TEXTUAL SPECIFIC	**Superficial**	**Utilitarian**	**Intrinsic**
Superficial	C119	A19, T15	-
Utilitarian	-	T113	C129
Intrinsic	A21	A218,T12,T17, T117, T211	A25, A27, A28, A217, T18, T116, T27, T29, C110, C130

Table 2. Responses of student A28 that show the characteristic of 'coherence'

	What does … in mathematics mean to you?	**What do you think would be a good example of ...?**
Defining	*It is the expression of a concept you need to use in the practice of mathematics*	*The perpendicular bisector of a segment AB is the locus of points...*
Proving	*It's when you want to see if the mentioned before is really is true*	*Commutative property, the order of the factors does not alter the product* [proof]: *2x4=8, 4x2=8*
Modelling	*Means that when there is a problem in everyday life it can be solved with mathematics*	*Manolo had 8 sweets, he gave 2 to each of his two brothers, how many has he got now?*

these 55 students maintained a common way of considering the elements. However, although they expressed their ideas through the specific representations with more difficulty (only 43 of 98 students), in this case a large majority of students (38 of 43 students) maintained a common view of D, P and M.

From the third analysis, we accessed features of a student's ideas considering student/way of considering/written external representations (textual and specific) jointly. We identified students who had a common way of considering D, P and M through the textual and/or specific expressions of the representations (21 students, 21.43%). As can be seen in the following table, 9 students (9 of 21, 42.86%) did not maintain the same way of considering the textual and specific representations for D, P and M, and 12 students (12 of 21, 57.14%) had the same consideration in the two representations. Among these 12 students, the "I" way of considering had the greater number of common cases.

This common way of considering the elements through the external representation in the two manifestations used in our study (textual and specific) could be considered a characteristic of the students' picture in the mind, that we call 'coherence'. This coherence is shown in the responses of student A28 (see Table 2).

If we consider these answers as the external representation of the student's ideas of the elements taken into consideration, we can identify an overview (linked here to significant aspects thereof) that could indicate a meta-interpretation of defining, proving, and modelling.

DISCUSSION AND FURTHER CONSIDERATIONS

For us, understanding implies not only using/applying the elements to solve a task in a more or less adequate way, it also implies that the student has to be able to express the understood idea through different representations. Our study has illustrated how textual and specific representations are tools that allow us access to the 'picture in the mind' that students have about some topics, identifying aspects of this picture. The two types of external representations used in this study have been shown as complementary in the sense that through them we can see different particularities of students' ideas that remain invariable (or not) in the three elements under consideration.

The identification of distinct ways of considering D, P and M (superficial, utilitarian and intrinsic) through different external representations illustrates how different students can 'see' different features of mathematical topics. For instance, the consideration of a mathematics topic from a 'superficial' point of view highlights characteristics that could be irrelevant under utilitarian or intrinsic approaches. Superficial, utilitarian and intrinsic constitute from our point of view an analytical scheme whose domain goes beyond the way of students' approaching to the mathematical elements considered in our study. We think that the use of this scheme with other topics and the improvement of its characterization open up routes to new lines of research.

Our way of approaching D, P and M has allowed extracting considerations of the mathematical contents from a more general point of view. We think the identification of a common vision of D, P and M through the different "ways of considering" them and the coherence in the adoption of that common vision through the two representations could be perceived as characteristics of understanding of the students in our study (of course, we do not try to generalize to other students). Therefore, we can say these two aspects broaden our knowledge of students' understanding and enlarge the results of authors such as Pitta-Pantazi et al., (2004) who point out that the different kinds of mental representations may be related to different levels of mathematical achievement. Our study provides a new variable, coherence, which can help to better characterize students' understanding and which, in some way, can also affect said mathematical achievement.

With respect to the educational implications, we think that the different ways of considering defining, proving and modelling can inform mathematics teachers about other facets of the diversity that they can find in their classrooms. This diversity, which is often considered in its affective or cognitive aspects, is enlarged with a specific diversity, more closely linked to teaching / learning of mathematics. In addition, these distinct ways of considering the elements give us an idea of different perspectives from which students can look at the mathematical contents. All of them help to describe the idea that students' have of those elements, but the aspects intervening in said description vary.

In addition, the fact that some students maintain the same consideration of the three elements, regardless of the type of representation considered, can inform us about ways of understanding mathematical contents that go beyond the particularities of the elements themselves, which could be linked to a global view of these mathematical elements. We ask ourselves if these 'joint ways' of seeing the elements could inform us about different approaches of the students to mathematics. It would be necessary that, by research, more information were provided on the extent to which these ways of considering D, P and M are related with the way in which the students learn mathematics.

We have to go deep into the fact that three processes considered in our study have connotations from the point of view of school culture that are not to be found in other mathematical concepts such as derivative, function and so on. Differently of them, defining, proving and modelling do not appear commonly as topics to formalise in Spanish school curricula and their learning occurs through different contents and academic courses. We wonder if the characteristics of the students' understanding identified in this study are linked to defining, proving and modelling or whether we can generalize them to other mathematical concepts. We need to progress in this direction.

Acknowledgments

Some of the ideas expressed in this article were discussed at different 'Seminarios I+D sobre Entornos de Aprendizaje y Tutorización para la Formación del Profesorado de Matemáticas', developed in the universities of Alicante, Autónoma of Barcelona and Seville, Spain.

The research reported here was supported by the Ministerio de Ciencia e Innovación (Spain) through Grant corresponding to the project No. PSI2008-02289/PSIC.

REFERENCES

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics 18,* 147-176.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics education. *Educational Studies in Mathematics, 222*, 37-68.

Blum, W., Galbraith, P., Henn, H-W., & Niss, N. (Eds.) (2007). *Modelling and Applications in Mathematics Education: The 14th ICMI Study*. Berlin: Springer.

Boero, P., & Morselli, F. (2009). Towards a comprehensive frame for the use of algebraic language in mathematical modelling and proving. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd PME Conference (Vol. 2)* (pp. 185-192). Thessaloniki, Greece: PME.

Borasi, R. (1991). *Learning mathematics through inquiry*. Portsmouth, NH: Heinemann Educational Books, Ins.

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt, & M. Santos (Eds.), *Proceedings of the 21st PME-NA Conference (Vol. 1)* (pp. 3-26). Columbus: ERIC.

García, M., & Llinares, S. (2001). Los procesos matemáticos como contenido. El caso de la prueba matemática. In E. Castro (Ed.), *Didáctica de la matemática en la Educación Primaria* (pp. 105-122). Madrid: Síntesis.

García, M., Sánchez, V., & Escudero, I. (2009). Defining, proving and modelling: a background for the advanced mathematical thinking. Paper presented to WG 12, CERME 6, Lyon, France, January.

Gravemeijer, K., Lehrer, R., van Oers, B., & Verschaffel, L. (Eds.) (2002). *Symbolizing, Modelling and Tool Use in Mathematics Education*. Dordrecht: Kluwer Academic Pbs.

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics, 44*, 5-23.

Hanna, G., & Jahnke, H. N. (1996). Proof and Proving. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education,* cap. 23 (pp. 877-908). Dordrecht: Kluwer.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutierrez , & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 147-172). Rotterdam: Sense Publishers.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31* (4), 396-428.

Izsák, A. (2003). We want a statement that is always true: Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education*, *34* (3), 191-227.

Lesh, R., & Lehrer, R. (2003). Models and Modeling Perspectives on the Development of Students and Teachers. *Mathematical Thinking and Learning*, 5(2), 109-129

Lesh, R., & Doerr, H.M. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning, and problem solving. In R. Lesh, & H.M. Doerr (Eds.), *Beyond Constructivism: Models and Modelling Perspectives on Mathematics Problem Solving, Learning and Teaching* (pp. 3-33). NY: LEA.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutierrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 173-204). Rotterdam: Sense Publishers.

Pitta-Pantazi, D., Gray, E.M., & Christou, C. (2004). Elementary school students' mental representations of fractions. In M. J. Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference, (Vol. 4)* (pp. 41-48), Bergen, Norway.

Sánchez, V., García, M., Escudero, I., Gavilán, J.M., Trigueros, R., & Sánchez-Matamoros, G. (2008). Una aproximación a las matemáticas en el bachillerato. ¿Qué se pretende que aprendan los alumnos?. *Enseñanza de las Ciencias*, 26(2), 267-276.

Vinner, S. (1996). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (Chapter 5). Dordrecht: Kluwer Academic Pbs.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36* (4), 317-347.

◈◇◈