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Eurasia Journal of Mathematics, Science and Technology Education

Volume 1, Number 1, November 2005

WORKING WITH FUNCTIONS WITHOUT UNDERSTANDING: AN ASSESSMENT OF THE PERCEPTIONS OF BASOTHO COLLEGE MATHEMATICS SPECIALISTS ON THE IDEA OF' FUNCTION

Mokaeane Victor Polaki

ABSTRACT. It is a well-known fact that the idea of function plays a unifying role in the development of mathematical concepts. Yet research has shown that many students do not understand it adequately even though they have experienced a great deal of success in performing a plethora of operations on function, and on using functions to solve various types of problems. This paper will report about an assessment of the perceptions of Basotho college mathematics specialists on the notion of function. Four hundred and ninety one (491) mathematics specialists enrolled at the National University of Lesotho (Years 1 - 4) in the 2002/2003 academic year responded to the questionnaire that challenged them, amongst other things, to (a) define a function, (b) give an example of a function, and (b) distinguish between functional and non-functional situations embedded in a variety of contexts. In addition to the difficulties observed in their attempt to define a function and to provide an example of a function, results suggests that, for the majority of those who responded to the questionnaire, the idea of function seemed to be limited to common or prototypical linear and quadratic situations that could be expressed either in symbolic or graphical forms. Additionally, arbitrary correspondences and functional situations that were presented implicitly were not identified as functions by the majority of the students. This paper discusses instructional, curricular, and research implications of the findings.

KEYWORDS. Concept, Assessment, Function, Mathematics.

INTRODUCTION

The idea of function plays an important role in the development of mathematical concepts in that it cuts across a range of mathematics content domains including those of algebra and geometry (National Council of Teachers of Mathematics (NCTM), 2000). However, research on students' understanding functions (e.g. Tall, 1996; Markovits et al. 1988) has shown that it is one of the least understood topics. A common definition of function is that of a correspondence that associates with each element in the first set a unique element in the second set. Some of the research (e.g. Vinner, 1992, Clement, 2001) has examined the extent to which one's concept image of function is consistent with the modern mathematical definition of function. According to Vinner, a person's concept image consists of all the mental pictures and perceptions that he or

ISSN: 1305-8223

she constructs as a result of having interacted with the concept over an extended period. Research on the relationship between one concept image and definition has revealed some serious discrepancies. For instance, Clement (2001) observes that documented students' concept images of function include (a) tendency to regard a function as something that can be defined in terms of a simple rule, (b) relation whose graph is continuous, and (c) a relation that is one-to-one. The foregoing is clearly a very narrow conception of function, given that some functions can neither be represented in the form of a symbolic rule nor in the form of a graph. Moreover, some functions are not continuous, and others are onto.

Although most of the research work on students' understanding of function conducted in English-speaking cultures of the world (e.g. Markovits et al., 1988; Tall, 1996) has accumulated a useful body of knowledge pertaining to students' difficulties, conceptions, and definitions of function, little similar work has been done in non-English-speaking cultures of the world such as that of Lesotho in Southern Africa. Furthermore, to improve students' understanding of function, there is a need to develop detailed accounts of how they develop increasingly sophisticated ideas associated with function in an instructional setting. Accordingly, as a preliminary survey designed to collect baseline information, this study explored Basotho university mathematics specialists' understanding of function. More specifically, this study sought to explore students' ability to: (a) define a function, (b) provide an example of function, and (c) distinguish between functional and non-functional situations presented in symbolic and graphical forms, and (d) distinguish between functional and non-functional situations that are defined either implicitly or as arbitrary correspondences. It was hoped that the information thus generated, would, amongst other things, provide a basis for developing and testing instructional programmes that are capable of moving students from lower to higher levels of understanding the notion of function.

Theoretical Considerations

This paper is grounded on the assumption mathematical understanding is a complex and multi-faceted phenomenon. Consistent with this line of thinking, Kaput (1989) identifies two sources of conceptual understanding in mathematics: (a) referential extension which refers to the ability to make translations between mathematical representations, and to make translations between mathematical and non-mathematical situations, and (b) consolidation which refers to the ability to operate within a system, recognizing the pattern and syntax of the system, and building conceptual entities via reifying actions and procedures. In unpacking referential extension in the context of the function concept, O'Callaghan (1998) identifies and describes three essential components of understanding functions: (a) modeling, (b) interpreting, and (c) translating. Whereas modeling entails ability to represent a mathematical situation using a picture, symbol, graph or table, interpreting involves ability to draw conclusions about functions from different representations. Finally, reifying entails construction of a mental object of the idea of function from what was essentially seen as a process or procedure. In the case of functions, process or procedure refers to various operations with functions such as drawing graphs, differentiating functions, and doing analysis of functions. Accordingly, the tasks used in investigating Basotho university mathematics students' understanding of function sought to evoke responses that would reveal these various aspects of understanding the concept of function. The researcher's hypothesis was that given that most of these college mathematics specialists who took part in this study had, on average, attained a reasonable degree of proficiency in performing such operations as differentiation, integration and the proof of the continuity of function, they would be equally successful in understanding the object they have demonstrated so much success in manipulating it.

Significance of the Study

In the only study that investigated Basotho students' understanding of function, Morobe (2000) worked with a small sample of pre-service mathematics teachers (12) at the National University of Lesotho (NUL) during the 1999/2000 academic year. The results of this study suggested, amongst other things, that the teachers held a pervasive belief that every function was linear. Additionally, they struggled somewhat in dealing with the less common functions such as piece-wise functions, constant functions, and discontinuous functions. The present study was designed to extend Morobe' work by looking at a much bigger sample of 491 mathematics specialists enrolled at the NUL in the 2002/2003 academic year. This group included prospective teachers of mathematics and those who were taking mathematics as one of their two majors. Whereas Morobe used the tasks that could easily be represented either in a symbolic, graphical or tabular forms, the current study included arbitrary correspondences and implicitly defined functional situations that could not necessarily be represented in the form of a table, symbol, or graph. More specifically, the tasks used in this study included the following representations of function (a) symbolic forms of functions, (b) graphical representations of functions, (c) arbitrary correspondences, and (d) a functional situation that was described implicitly. It is hoped that the results of this study should constitute a basis for thinking about possible intervention strategies designed to improve students' understanding of function at tertiary institutions. Accordingly, research that builds on the current study might include the design of teaching experiments that are aimed at documenting students' development of the function concept in instructional settings. These ideas, when documented, can constitute a basis for developing instructional materials and activities that support or nurture the development of students' development of a richer understanding of function.

METHODOLOGY

Sample

Mathematics students enrolled at the National University of Lesotho during the 2003/2004 academic year constituted the population of the present study. Four hundred and ninety-one (491) of these students responded to a 10-item questionnaire that challenged them to define function, give an example of a function and to distinguish between functional and non-functional situations presented different representations and contexts. Table 1 show the number of students who participated in this study. This sample included some 93 social sciences students who took a second year mathematics course as a service course (M205 group). Drawn from years one through four of the degree program, the students responded to the questionnaire during regular classroom time. All students who participated in this study had undergone some formal training on the formal definition, recognition, and interpretation of functions.

Category of Students	Number of Registered Students	Number & Percentage of Students Responding to Questionnaire Number [%]
1st Year (Math)	267	250 [94%]
2nd Year (Math)	119	105 [88%]
2nd Year Soc. Sciences (Math)	101	93 [92%]
3rd Year (Math)	26	23 [88%]
4th Year (Math)	26	20 [77%]

Table 1. Number of College Mathematics Specialists who Took Part in the Survey

Instrumentation

The instrument used in this paper was designed in such a way that it would evoke responses that would reveal participants' concept image of function. The idea was to access their concept image by asking them to (a) define function, (b) provide an example of function, (c) identify functional and non-functional situations presented in the form of graph, table, or symbols, and (d) recognize a function presented in an implicit form. Adapted from Clement (2001), some of the tasks in the questionnaire required students to respond to 10 items. The first 5 of these covered the demographic characteristics of the participants. The sixth item required students to define the mathematical concept of function, and to provide an example of a function. The seventh item required students to recognize and identify functions presented in a graphical form. The eighth item asked the students to identify functions presented in a symbolic form. The ninth item challenged the students to decide whether an arbitrary correspondence presented in a tabular form (Figure 1) was a function. The last item (Figure 2) sought to determine whether the students could recognize a functional situation that was defined implicitly, and embedded in a context that was neither a graph, table, or symbols. In each case the students were given enough space to justify their responses in writing.

If we let x = club member's name and y = amount owed, is y a function of x?

Name	Owed	Name	Owed
Sue	$17	Iris	6
John	6	Eve	12
Sam	27	Henry	14
Ellen	0	Louis	6

Figure 1. The task showing an arbitrary correspondence.

From "What do students really know about functions? By L. Clement (2001), Mathematics Teacher, 94, 9, p. 746. Copyright by L. Clement, Reprinted with permission.

A caterpillar is crawling around on a piece of paper as shown below.

a) If we wished to determine the creatures' location on the paper with respect to time, would this location be a function of time? Why or why not?
b) Can time be described as a function of its location? Explain.

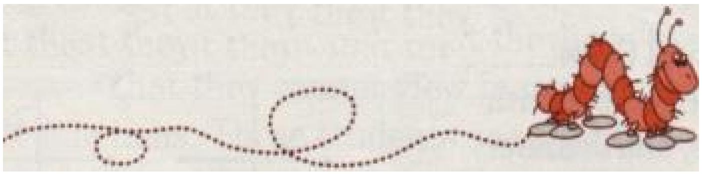

Figure 2. The task showing a functional situation defined implicitly.

From "What do students really know about functions?" By L. Clement (2001), Mathematics Teacher, 94, 9, p. 746. Copyright By L. Clement. Reprinted with permission

Procedure

The students responded to the questionnaire during regular instruction time. Prior to asking the students to respond to the questionnaire, the researcher explained that the purpose of the exercise was to study their understanding of the idea of function. Furthermore, the students were made aware that the questionnaire had nothing to do with regular testing. Finally, the researcher made sure that the students understood what each task required them to do by going through each item in the questionnaire. Completed questionnaires were collected immediately after the students had completed them. In other words, the students were not allowed to take the questionnaire home.

Data Analysis

On the basis of the researchers' own mathematical understanding of function, and on the research questions the researchers sought to pursue, students' thinking was analyzed according to five general themes: (a) definition and examples of function, (b) ability to recognize functions expressed in symbolic form, (c) ability to recognize functions presented in a graphical form, (d) ability to identify a function expressed in a tabular form but without an explicit rule linking elements of the domain and those of the range, and (e) facility at seeing and dealing with functions presented in an implicit form. Finally, a double-coding procedure (Miles & Huberman, 1994) was used to identify and categorize students' responses to each item. The researcher and another person trained to do the job independently read and coded 100 randomly sampled responses to each item. Agreement was reached on 95% of the selected cases. Disagreements were discussed until consensus was reached.

RESULTS

Students' Definitions of Function

The overall picture was that the majority of students were unable to provide a correct definition of function. Table 2 summarizes students' definitions of functions. Students from various levels of education seemed to differ in terms of the way they chose to define function. Whereas the majority of first year students [137 (55%)] defined function as a relationship that has only one image in the co-domain (range), their second year counterparts [58 (55%)] defined a function as a relationship in which the first component of the ordered pair is not repeated. The difference between the two definitions is that the latter uses ordered pairs. However, they both stress the fact that every element of the domain has exactly one image (univalence property of function). In other words, one-to-one and many-to-one relations are functional, but one-to-many relations are not functional. The majority of students in the social sciences category (those taking M205) [63 (68%)] provided a definition that seemed to emphasize the dependency property of function, with scant regard for the need for a functional situation not to have one-to-many correspondences. More specifically, they defined a function as a rule that shows how one variable depends on the other. Finally, students in the third and forth years of study tended to defined a function as a relation in which there is only one image in the co-domain in the same way as their first year counterparts.

The correct formal definition of function based on the idea of set of ordered pairs did emerge only on less than five instances. For example, Tefo in the second year of study defined a function as a "the cross product of two sets such that the first component in the ordered pair is not repeated". Some definitions stressed the fact that in a functional situation, one can have one-to-one and many-to-one situations, but not one-to-many situations. Others clearly reflected the many misconceptions that the students held with regard to function. For instance, Mpho in her forth year of study argued that a function was a "mathematical equation that has a domain and range whereby the domain is mapped to the range on a 1-1 bases. Firstly, Mpho's claim that a function is an equation underscores a possible confusion between the idea of a function as a very large abstract object and an equation a one way of modeling or representing only a limited number of functions. Secondly, her claim that the domain is mapped onto the range on a 1-1 basis mirrors a possible confusion between the univalence property of a function and the one-to-one property of some functions, with scant regard for the fact others are in fact onto. On several occasions, respondents described a function as a relation in which an input is turned into an output, showing lack of understanding of the idea of function as a special type of a relation in which each input (if we use their language) corresponds to exactly one output. It is interesting to note that less than 50% of third and forth year students dared to define a function. This suggests that their confidence with the idea of function was so low that they chose not to commit themselves.

Students' Definitions of Function	1st Year N=267	2nd Year N=119	S0c. Sciences 2nd Year N=101	3rd Year N=26	4th Year N=26	Total N=491
A function has only one image in the co-domain	137[55]	21[20]	2[2]	6[26]	6[30]	172[35]
A function shows how one variable depends on the other	0[0]	6[2]	63[68]	5[22]	5[25]	79[17]
First entry of ordered pair does not correspond to more than one second entry	0[0]	58[55]	0[0]	0[0]	0[0]	58[12]
A function is relation between variables x and y	29[12]	6[6]	12[13]	0[0]	0[0]	47[10]
A function has one image but an image can have more than one partner	10[4]	1[1]	0[0]	0[0]	0[0]	11[2]
In a function an input mapped on to an output	14[6]	0[0]	0[0]	3[13]	2[10]	19[4]
No definition	20[8]	11[10]	8[3]	0[0]	0[0]	39[8]
Numbers and letters to represent given information	0[0]	0[0]	4[4]	0[0]	0[0]	4[1]
One-to-one and onto mapping with domain and range	0[0]	1[1]	0[0]	3[13]	4[20]	8[1]
Subset of cross product of 2 sets such that first entry in the ordered pair is not repeated	0[0]	2[2]	2[2]	0[0]	0[0]	4[1]
Idiosyncratic definitions	36[14]	2[2]	4[4]	3[13]	3[13]	48[10]

Table 2. Definitions of Functions Given by College Mathematics Specialists [Number (%)]

1 Similar concerns have been raised by a number of science graduates as well. Some examples are "materials are not available…no space to store materials, models and charts…" (personal notes)

2 Time is constraint…I had to achieve all the objectives…I could not… reading process for the students is problem…discussion in some things becomes long…and planning could not be completed on time… (Immediately after lesson self-reflection Saira September 27, 2000)

Students' Examples of Function	Year 1 N= 267	Year 2 N=119	Social Sciences Year 2 N=101	Year 3 N=26	Year 4 N=26	Total N=491
Linear or quadratic functions	109[40.8]	36[30.3]	62[61.4]	10[38.5]	11[42.3]	228[46]
Arrow diagram	81[32]	6[6]	0[0]	0[0]	0[0]	87[18]
Exponential function	0[0]	0[0]	5[5]	0[0]	0[0]	0[0]
Students & ages, non can have more than one age	14[6]	3[3]	6[6]	0[0]	0[0]	23[5]
F(x) = [(1,5), (2,6), (2,7)]	15[6]	42[40]	0[0]	0[0]	0[0]	57[12]
F (x) = f (l, k)	0 [0]	0[0]	7[8]	0[0]	0[0]	7[1]
Other examples	0[0]	5[5]	0(0)	0[0]	0[0]	5[1]
No example given	42[17]	21[20]	14[15]	13[50]	11[42]	101[21]

Table 2. Definitions of Functions Given by College Mathematics Specialists [Number (%)]

Students' Examples of Function

As expected, the most salient features of students' understanding of function became more apparent when they were challenged to provide examples of function. Table 3 summarizes students' responses when challenged to provide examples of functions. The most common examples were functions that were either linear or quadratic. Others were an arrow diagram, series of ordered pairs, and polynomial functions, especially quadratic functions. This finding is not surprising given most of the examples used in the teaching and learning of algebra in the Lesotho context are either linear or quadratic. These examples suggest that the concept image of function that the students held was that or a relationship that could easily be described in terms of well-known functions such as those that were linear or polynomial. Surprisingly, a large number of third year [13 (50%)] and forth year [11(42%)] mathematics specialists could not provide an example of a function. This was despite the fact that these students had developed a sufficient facility at manipulating functions as evidenced by the fact that they had successfully completed the program for year 1 through 3 of university mathematics. In particular, they had differentiated functions, integrated functions, analyzed functions, and used functions as a basis for solving a wide spectrum of mathematical problems.

3 Syllabus is a problem…some discussions become long and we rush to complete the syllabus. (Immediately after lesson self-reflection Saira September 27, 2000)

Students' Examples of Function	Year 1 N= 267	Year 2 N=119	Social Sciences Year 2 N=101	Year 3 N=26	Year 4 N=26	Total N=491
Linear or quadratic functions	109[40.8]	36[30.3]	62[61.4]	10[38.5]	11[42.3]	228[46]
Arrow diagram	81[32]	6[6]	0[0]	0[0]	0[0]	87[18]
Exponential function	0[0]	0[0]	5[5]	0[0]	0[0]	0[0]
Students & ages, non can have more than one age	14[6]	3[3]	6[6]	0[0]	0[0]	23[5]
F(x) = [(1,5), (2,6), (2,7)]	15[6]	42[40]	0[0]	0[0]	0[0]	57[12]
F (x) = f (l, k)	0 [0]	0[0]	7[8]	0[0]	0[0]	7[1]
Other examples	0[0]	5[5]	0(0)	0[0]	0[0]	5[1]
No example given	42[17]	21[20]	14[15]	13[50]	11[42]	101[21]

Table 3. Examples of Functions by Given by College Mathematics Specialists [Number (%)]

Recognition of Symbolic Representations of Function

The conjecture that the students were more likely to recognize functions in situations that could easily be represented using familiar symbolic representations, especially those that were linear and quadratic, was further supported by their responses to a task that challenged them to identify symbolic representations of relations that were functional. Table 4 summarizes students' choices of symbolic representations of relations that they regarded as functional. As shown on Table 4 about 471 (96%) of students who participated in this study correctly identified the linear function $y = \frac{x}{2}$ as representing a functional situation. Similarly, 461 (94%) students correctly identified the quadratic function $y = x^2 - 4$ as representing a functional situation. Additionally, with the exception of first year students, the exponential function ($y = e^x$) was correctly identified as a function by all categories of students who took part in the investigation. Apparently the majority of the students had met linear, quadratic, and exponential relations identified and discussed as models of functional situations. In contrast, the numbers dropped sharply in the case of the less common relations such as the piece-wise function $\begin{cases} 1 \text{ if } x \in \text{rationals} \\ -1 \text{ otherwise} \end{cases}$

[233(47%)] and the rational function $xy = 8$ [209 (43%)]. As expected, the number of students who claimed that $x^2 + y^2 = 25$ was a function was relatively low [191 (39%)]), suggesting that the majority of the students correctly identified this circle of center (0, 0) and radius 5 units did as a non-functional situation. Perhaps the students experienced more success at classifying this because they could easily visualize it as a circle, and they recalled that a circle would always fail the vertical line test for a function. In contrast, the rational function (c) and the piece-wise functions (f) were probably more difficult to visualize.

Relations	Year 1 N =267	Year 2 N = 119	Social Sciences Year 2 N = 101	Year 3 N = 26	Year 4 N = 26	Total N=491
(a) $y = x^2 - 4$	207[78]	109[92]	98[97]	24[92]	23[88]	461[94]
(b) $y = \frac{x}{2}$	211[79]	110[92]	98[97]	26[100]	26[100]	471[96]
(c) $xy = 8$	46[17]	78[66]	43[43]	18[69]	24[92]	209[43]
(d) $x^2 + y^2 = 25$	95[36]	26[22]	45[45]	11[42]	14[54]	191[39]
(e) $y = e^x$	146[55]	98[82]	99[98]	24[92]	25[96]	392[80]
(f) $\begin{cases} 1 \text{ if } x \in \text{rationals} \\ -1 \text{ otherwise} \end{cases}$	76[28]	74[62]	55[54]	10[38]	18[69]	233[47]

Table 4. Symbolic Relations Identified by College Mathematics Students as Functions [Number (%)]

Table 4 further shows that the piece-wise function (f) seemed to have caused greater difficulties to students in the third year compared to those in the fourth year. It was also interesting to note that second year students in the social sciences experienced more success in identifying the exponential relation as a function compared to their counterparts in the pure sciences. Perhaps this results from the fact that the exponential function is often used in a wide spectrum of applications in the social sciences, and accordingly it is one of the few functional representations that the students had met several times.

Recognition of Graphical Representations of Functions

Here the researcher's conjecture was that the students were more likely to experience more success at identifying functions presented in a graphical form compared to those presented in the symbolic form given that graphical representations landed themselves more readily to analysis using such learning tools as the vertical line test for a function. Table 5 summarizes students' responses when challenged to identify functions from a group of relations presented in a graphical form. Indeed, the majority of students (471[96%]) across the five groups were able to see that a graph that represented a relation of the form $y=ax^2$ represented a functional situation. As expected, a large number of participants (430[88%]) correctly identified a semi-circle that had center (0, 0) and covered the first and second quadrants (e) as representing a functional relationship. Contrary to expectations, however, the number of students who correctly classified a constant function ($y=b$) was generally low, especially amongst students in the social sciences. Interestingly, first year students did better in this exercise compared to every category of the students except those in the third year. Furthermore, graphs of the singleton point (c) and the step function (g) seemed to cause a lot of difficulties across the categories of the students who participated in this investigation. Whereas first year students outperformed every category of participants in classifying a singleton point as function, only third year students did better than them in classifying the step function as representing a functional situation. In fact only 251 (51%) of all students correctly identified this as a function. As for the step function (g), only first year students (201[75%]) and fourth year students (21[81%]) seemed to experience considerable success at recognizing this as representing a functional relationship. The foregoing were made in spite of the fact that the vertical line test could easily have been used as basis for reaching the correct conclusion that both the singleton point and the step function represented functional relationships.

Graphs of Relations	Year 1 N =267	Year 2 N= 119	Social Sciences Year 2 N = 101	Year 3 N = 26	Year 4 N = 26	Total N=491
(a) Parabola of the form $y=ax^2$	220(82)	106(89)	97(96)	24(92)	24(92)	471[96]
(b) Relation of the type $x=ay^2$	62(23)	60(50)	73(72)	11(42)	14(54)	220[45]
(c) Singleton point	165(62)	44(40)	23(23)	10(38)	9(35)	251[51]
(d) Function of the form $y=b$, where is b is a constant	194(73)	75(63)	35(35)	11(42)	21(81)	336[68]
(e) Semi-circle with center (0,0), covering 1st and 2nd quadrants	208(78)	88(74)	90(89)	23(88)	21(81)	430[88]
(f) Semi-circle with center (0,0), covering the 1st and 4th quadrants	50(19)	41(34)	69(68)	12(46)	15(58)	187[38]
(g) Step function	201(75)	49(41)	36(36)	11(42)	21(81)	318[65]

Table 5. Graphs of Relations Identified by College Mathematics Students as Functions [Number (%)]

[5] According to the textbook a radial segment is a line which joins the centre to circumference and radius is the distance between the centre and circumference.

As for identifying non-functional situations, results suggest that only 220 (45%) incorrectly identified a relation of the form $x = ay^2$ as a function. In other words 55% of the students correctly figured out that this did not represent a function. In this case first year students apparently experienced more success at classifying this relation as evidenced by the low the number of those who claimed this was a function. Similarly, only 187 students incorrectly identified a semi-circle covering the first and 4th quadrants as representing a function, which means that 62% correctly noted that this semi-circle did not represent a function. Once again more first year students correctly identified this as not representing a functional situation compared to students in the second, third and fourth years of study. In this case the researcher's conjecture that the students would experience greater success at classifying relations represented in a graphical form compared to those represented in a symbolic form was not supported. On the contrary, the students seemed to classify graphical representations of relations rather differently, showing greater facility with those they had apparently met before.

Recognition of an Arbitrary Correspondence as a Functional Situation

In order to explore students' ability to decide whether a functional relationship presented as an arbitrary correspondence was indeed a function, the students were confronted with an item showing the status of club members' dues (Clements, 2001) (see Figure 1). They were then told that x equals club member's name and y equals amount owed. They were then challenged to decide whether y was a function of x, and to justify their decision. This situation was an arbitrary correspondence in the sense that there was no specific rule that seemed to associate a club member' name to the amount owed as in the case of other forms of functions. In addition, this relationship could neither be presented in a symbolic or graphical form in the same way that one could represent linear, polynomial or exponential function. Although the majority of participants did not provide responses, the item shown in Figure 1 was able to generate a range of responses that revealed some interesting aspects of students' thinking about the idea of function. Table 6 summarizes students' responses to this item. It must be noted that compared to item 7 (graphical representations of functions) and 8 (symbolic representations of functions), this item was a bit unusual in the sense that the students had not met similar problems in their day-to-day mathematics lessons. Furthermore, it caused a lot of conceptual challenges as evidenced by the low number of students who were able to respond to it.

Students' Analyses of an Arbitrary Correspondence	Year 1 N =267	Year II N = 119	Social Sciences Students N =101	Year 3 N = 26	Year 4 N = 26	Total N= 491
Yes! Amount is owed to one member	118(47)	22(21)	36(39)	7(30)	0(0)	183[37]
No! Y does not depend on x	0(0)	5(5)	28(30)	3(13)	2(10)	38[8]
Yes! Amount reflects character of club member	0(0)	0(0)	6(6)	0(0)	0(0)	6 [1]
Yes! No explanation	31(12)	30(29)	0(0)	2(9)	4(20)	67[14]
No! No explanation	7(3)	22(21)	0(0)	2(9)	11(55)	42[9]
No! Different names have same amount	32(13)	11(11)	0(0)	6(26)	0(0)	49[10]
Yes! After drawing an arrow diagram	9(4)	0(0)	0(0)	0(0)	0(0)	9[2]
Other Responses	6(2)	2(0)	14(15)	0(0)	0(0)	22[4]
No Response	42(17)	17(16)	9(10)	3(13)	3(15)	74[15]

Table 6. College Mathematics Students' Analyses of an Arbitrary Correspondence [Number (%)]

The most common incorrect response across the four groups (183[37%]) was that the task shown in (Figure 1) represented a function because, as Thabang argued, "The amount owed was owed to one member". This response was indeed incorrect for, as shown on Figure 1, John and Iris actually owe the same amount. Furthermore, 49(10%) students argued that item 9 did not represent a function because 2 members owed the same amount. As Molete explained it, "This is not a function because Iris, Louis, and John owe the same amount and this means we have more than one image." Other students justified similar responses by drawing an arrow diagram that showed amount owed (y) as constituting the domain (corresponds to objects) and club member's names (x) as co-domain (corresponds to image). It should be noted that, in this case, an arrow diagram was used as basis for reaching an incorrect decision. The foregoing perceptions could have resulted from a misunderstanding of the phrase "y is a function of x". Whereas a correct interpretation of "y is a function of x" is that y depends on x, an incorrect interpretation that surfaced in this case was that "y is a function of x" means x depends on y.

Furthermore, students' responses to this item also suggest that, because their concept image of function was that of a relation that could be expressed using either a formula or graph, they had difficulty in recognizing and accepting a functional relationship that was not be expressed in any of the usual representations. Common incorrect responses included expressing discomfort with the fact that there seemed to be no explicit relationship between a club member's name and the amount owed. For example, Matseliso argued a similar point thus: "No, y is not a function of x. There is no way the member's name and amount owed are related". Clearly, Matseliso is perturbed by the fact that there is no explicit relationship between a member's name

[6] Based on concepts such as learning with reasoning, encouraging students' participation in activity and thinking and organising the classroom for cooperative learning

and the amount owed. Other students seemed to express a similar sentiment, but were more forthright about what they objected to compared to Matseliso. For instance, Thabo said Figure 1 did not represent a functional relationship for as he explained it, "Club member's names are not represented by digits". Consistent with this type of thinking, Lineo said: "Y is not a function of x because the member's name does not depend on the amount owed. We cannot construct an equation that relates y to x". Thus these students were not inclined to accept an arbitrary correspondence as a function. More specifically, they seemed to be looking a numerical relationship that could easily be represented symbolically. Apparently, their experiences with functions had excluded arbitrary correspondences that could be functional or non-functional. More importantly, they were not aware that symbols and graphs are merely models or representations of abstract objects called functions, and that some of these objects can neither be represented graphically nor symbolically.

As in the case of graphical representations of functions, traces of correct classification of the relation shown in Figure 1 as functional seemed to emerge from the first year category of participants. Correct responses included mentioning the fact that one member owed one amount. As shown in Table 6, some 9 (4%) first year students drew an arrow diagram, the first column of which showed names of club members (x) and the second column of which depicted the amounts owed (y) before reaching the valid conclusion that Figure 1 represented a functional relationship. Apparently, the arrow diagram did enable the students to recognize that whereas the relation consisted of one-to-one and many-to-one correspondences, one-to-many correspondences did not exist. In other words, they did realize that the situation shown in Figure 1 did satisfy the univalence property of function. Thus the arrow diagram was correctly used in this case as a tool of analysis that enabled the students to decide whether the arbitrary correspondence described in Figure 1 indeed represented a functional relationship. It should be noted that the students could, for the task shown in Figure 1, easily landed itself to analysis using an arrow diagram. The researcher further sought to find out how the students would deal with a functional situation that could neither be represented using a symbol, graph or arrow diagram.

Recognition of a Function Defined Implicitly

In the last item adapted from Clements (2001), the students were shown the picture of a crawling caterpillar that first moved forward (not in a straight line) for a few minutes and then turned around before continuing for a few minutes (see Figure 2). Then the creature turned around again before continuing. Thus the path of the caterpillar consisted of several loops. The students were asked to say whether location would be a function of time if one wished to determine the caterpillar's location on paper at a particular time. Table 7 summarizes students' responses to item 10.

Students' Response	Year 1 N=267	Year 2 N=119	Social science Year 2 N=101	Year 3 N=26	Year 4 N=26	Total N=491
(a) Yes! Location would be a function of time because location depends on time, and time keeps on changing.	49(20)	41(39)	71(76)	12(52)	6(30)	179[36]
(a) No! Location is not a function of time because the creature crosses one place more than once.	106(42)	30(29)	5(5)	8(35)	5(25)	154[31]
(a) No! Location is not a function of time because the caterpillar keeps on changing speed.	6(2)	0(0)	4(4)	0(0)	1(5)	11[2]
(a) No! Location is not a function of time because vertical line test fails.	17(2)	3(3)	0(0)	0(0)	1(5)	21[4]
(a) Other responses	13(5)	13(12)	12(13)	5(22)	7(35)	50[10]
(a) No response given	35(14)	19(18)	4(4)	0(0)	0(0)	58[12]
(b) Yes! Time can be a function of location (no explanation)	23(9)	20(19)	39(42)	0(0)	5(25)	87[18]
(b) No! Time cannot be a function of location because one location can have different times	31(12)	20(19)	22(24)	12(52)	6(30)	91[19]

Table 7. College Mathematics Responses to the Item on the Caterpillar Moving in Loops [Number (%)]

The task on the crawling caterpillar confronted the students with a lot of difficulties as evidenced by the low number of those who provided responses. The most common incorrect response was that location would not be a function of time because the creature seemed to have been at the same place at the same time. Thabiso echoed a similar sentiment in arguing as follows: "This is not a function because as we look at our time, the creature crosses twice at a certain point of time. This means that a certain distance has different time intervals which do not agree with the function rule of having only one image for each object". Once again the confusion here seems to reside in students' lack of understanding of the question:" Would location be a function of time?" More specifically, they were apparently unable to identify the dependent and the independent variable. In the foregoing question location is the dependent variable and time is the independent variable. Although different times can correspond to one location, each time will have exactly one location. Therefore location is indeed a function of time since the univalence of property of function is satisfied. Apparently those who reasoned like Thabiso took location as the independent variable and time as the dependent variable.

Another common misconception was the tendency to regard the path of a crawling caterpillar as a model or graph of the relationship between location and time. Consequently, some students erroneously applied the vertical line test to reach the incorrect conclusion that location was not a function of time. For example Tumo argued that location was not a function of time for as he explained it, "The vertical line cuts the graph twice at the same points". This response not only reflects a mechanical understanding of the use of the vertical line test as a tool for testing whether a relationship was functional but it also mirrors failure to identify related variables and to correctly interpret their behavior. As in the case of an arbitrary correspondence (Figure 1), these responses also showed lack of understanding of the phrase "is a function of." For these students, location was mapped onto time, and consistent with this way of looking at things, the same location would be mapped onto different times, violating the univalence aspect of the definition of function.

The most common correct response was that location would indeed be a function of time because time kept on changing even though, in some occasions, the distance covered did not change. As Mahlape explained it, "location would be a function of time because the caterpillar can be at any location at different times. Meaning that for different locations there can never be the same time." This explanation suggests that Mahlape has not only correctly identified related variables but also understand their behavior. Interestingly, students in the social sciences seemed to experience more success at answering this question correctly [71 (76%)], with first year students showing the least success. With regard to the second part of the same task where the students were challenged to say whether time would be a function of location, only 91 (19%) of all students correctly concluded that time would not be a function of location because one location would be mapped to more than one reading of time. As in the case of the first part of the task, a greater proportion of third year students [12(52%)] reached a correct conclusion, and the smallest proportion of first year students [31 (12%] provided similar responses.

CONCLUDING REMARKS

The purpose of this exploratory study was to look at Basotho university mathematics students' understanding of the notion of function. Although the absence of interviews with a sub-sample of those who responded to the questionnaire calls for caution in drawing conclusions, students' work as they responded to the questionnaire and justified their responses in writing has revealed some interesting aspects of their thinking with regard to the idea of function. In particular, the results suggest that the majority of the students generally had enormous difficulty in providing a correct definition of the notion of function. The definitions were often incomplete, with the students mentioning only one aspect of the definition of function. Whereas mathematics majors tended to stress the univalence aspect of the definition of function, their social sciences counterparts emphasized the correspondence or dependence aspect of the definition of function with scant regard for the univalence property of function. The fact that even third and fourth year mathematics specialists could not provide a correct definition of function when challenged to do so, suggests that interactions with function as been more operational than structural (Tall, 1996). That is, they have, amongst other things, successfully evaluated functions, differentiated functions, integrated functions, and analyzed the continuity of functions without adequately understanding the nature of the object they have been handling. There is a need therefore to restructure the university mathematics curriculum so that it provides a balanced combination of the operational and structural aspects of the idea of function.

The examples of functions that the students provided seemed to illuminate their concept image of function. In line with past research in this knowledge domain (e.g. Markovits et.al. 1988; Vinner, 1992), students' concept image of function was limited to a few prototypical situations, especially linear and polynomial functions. It was found that the majority of the students provided either linear or quadratic functions as examples of function. Given that most of the elementary algebra introduced in the secondary and high schools is essentially the study of linear and polynomial functions, students' examples of functions reflects the depth and breadth of the algebra they have studied from the high school through to the university. At the university level, it is possible that professors of mathematics genuinely choose linear and polynomials functions as easier examples of functions in order to help the students understand this apparently illusive concept. Consequently, the students end up internalizing linear and polynomial functions as prototypes of functions. In other words, when challenged to give an example of a function, concept images that is immediately evoked are that of a linear or quadratic functions. Apparently, this process continues throughout the fours years of learning mathematics even though college mathematics should constitute a context for extending and deepening students' understanding of the idea of function. To remedy this situation, those involved in the teaching and learning of functions at the school and tertiary levels might use examples and non-examples of function that deepen and expand rather than limit students' understanding of functions. This can be attained if the activities that high school and university mathematics students experience do include exposure to linear, polynomial, exponential, rational, trigonometric, and other types of functional situations in a technologically-rich learning environment. It is a well-known fact technological devices such as graphing calculators can enable students to model and visualize complicated functions that are impossible to sketch by free hand.

Consistent with their choice of examples of function functions, students' ability to identify functional and non-functional situations from a group of relations presented in symbolic and graphical forms seemed to be constrained by the breadth and depth of their past experiences with functions. In other words, their classification of symbolic and graphical representations of functional and non- functional situations was limited to some prototypes of functions and non-examples of functions. For example, they experienced little difficulty in correctly identifying linear and quadratic relations as functions. Consistent with their past learning experiences, students in the social sciences were the most successful in recognizing that the exponential function was indeed a function. Additionally, many showed not much difficulty in seeing the equation of circle with the origin as the center, and radius 5 units did not represent a functional situation. Surprisingly, many had problems recognizing that the graphs of a singleton point and that of step function represented functional situations even though they could have easily used the vertical line test. It is possible that linear and quadratic functions are often used as examples of functions, and a circle is usually used as a counter-example of a model of a functional situation. In contrast, the students experienced great difficulties in identifying the piece-wise function and the rational relations as functions. Once again mathematics teachers and educators might do well to ensure that examples and non-examples of functional situations extend beyond the familiar linear relations, quadratic relations and other models of relations. It is clear that with the use of traditional pencil and paper, it may not be possible to expose students to as many examples and non-examples of functions as is necessary. In most institutions of higher learning, students do not only learn how to sketch and draw functions, they also employ electronic devices (e.g. graphing calculators) to draw and to learn about the behavior of some functions for which it would be impossible to draw or sketch using paper and pencil alone. Morobe (2000) recorded some positive changes after exposing a small group of prospective teachers of mathematics to a series of instructional sessions in which the graphing calculator was an essential component. Greater changes in students' conceptual understanding of functions can be attained if students are exposed to at least three types of learning experiences: (a) lecture, (b) pencil-and-paper tutorial, and (c) tutorial using either a computer or graphing calculator as a learning resource.

When challenged to decide whether a table that depicted an arbitrary (Figure 1) correspondence was a function, the majority of students had great difficulty in providing responses. Similarly, a very small number of the students responded to the item on the crawling caterpillar (Figure 2). Apparently, some were perturbed by the fact that there seemed to be no explicit rule or equation that connected the variables in the table. Others openly expressed their frustration with the fact one the variables (names of club members) was not represented by digits. These responses underscore a serious gap in students' understanding of function, namely, that even arbitrary correspondences can be functional or non-functional situations. Furthermore, many had apparently not come across a functional situation that was not defined using conventional forms of representations as described in task on the crawling caterpillar. More seriously, there exists confusion between the idea of a function and an equation. Whereas a function is an abstract object, an equation is a model or symbolic that can be used to represent some but not all functional situations. Similarly, a table and a graph constitute alternative ways of representing or modeling functional and non-functional situations. Thus it is essential mathematics teachers and educators to design learning situations that will enable students to conceptually draw a distinction between a mathematical concept of function and its symbolic, graphical, and tabular representations. Moreover, it is important to stress the fact that these representations may not be used to show all existing functions. It is essential that instruction on functions exposes students to a wide spectrum of functional and non-functional situations, including arbitrary correspondences (Figure 1) and those that are defined implicitly (Figure2).

There was also some confusion with the use of the phrase "is a function of". In particular, some students argued that the arbitrary correspondence shown in Figure 1 was not a functional situation for as they argued, John and Iris owed the same amount (6). For this category of students, the confusion seemed to reside in meaning of the phrase "y is a function of x".

Apparently they regarded this to mean that "x depends on y" rather than "y depends on x". This misconception resurfaced again when students were challenged to respond to the task on the crawling caterpillar. Some students argued that location would not be a function of time for as they explained it, "the caterpillar would be at the same location at different times". Clearly, these students had misinterpreted the phrase, "location is a function of time" to mean time depends on location rather than location depends on time. Thus students' access to the meaning simple expressions such as "y is function x" should not be take granted. On the contrary, mathematics educators and teachers should expend more time to ensure that these are well understood. Interestingly, some students were able to correctly decide that the arbitrary correspondence shown in Figure 1 was a functional situation by drawing an arrow diagram that clearly suggested that the table satisfied the univalence requirement for a function. In this case an arrow diagram was used as a representational tool that made a functional relationship more apparent. Once again effort should be made to draw a distraction between an abstract object of function and an arrow diagram as a model or a representational tool, and that some functions may not be represented in the form of an arrow diagram.

As this was an explanatory study, further research in this area might be aimed at documenting, in greater detail, how students acquire increasingly complex ideas of function in an instructional setting. Such teaching experiments necessarily have to be preceded by collection of baseline information by way of a questionnaire coupled with clinical interviews that cover a wider spectrum of constructs, including the idea of a function as an abstract entity that can be represented in several ways, arbitrary correspondences, equations, arrow diagrams, tables, graphs and those defined implicitly. When available, the data generated from these teaching experiments should not only contribute to theory-building on the development of functional concepts but it should also serve as a basis for developing appropriate instructional materials, including books and manuals. More importantly, it should enable curriculum developers to review the school mathematics curriculum in such a way that the idea of function becomes a unifying theme. Additionally, the information generated from the teaching experiment should produce important ideas about how to best design instructional situations that support rather than limit students' understanding of function.

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Polaki, Mokaeane Victor

Department of Science Education

Faculty of Education

National University of Lesotho

P. O. Roma 180

Lesotho, **SOUTHERN AFRICA**

E-mail: mv.polaki@nul.ls